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# SUFFICIENT EQUIVALENCES IN A SAMPLING SPACE

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**Abstract**. For a sampling design of an ordered sample on a finite population we define the induced sampling space, and the notion of consistency of a data with an unknown parameter $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_N)$, which is some characteristic of the units in the population. We examine the sufficient statistics for $\mathbf{Y}$. The sufficiency of a statistic is connected with the kernel of the statistic. We introduce sufficient equivalences on the sampling space and give a complete description of a maximal elements in the set of all sufficient equivalences.

## 1. Sampling design of an ordered sample

Let $B = \{b_1, b_2, \dots, b_N\}$ be a finite set (called population) and let $U(B)$ be the free semigroup generated by $B$. Any finite sequence $\sigma = (b_1, b_2, \dots, b_n)$ with elements (units) in $B$ is called an ordered sample on $B$. The number $n = L(\sigma)$ is called the length, the set $\{b_1, b_2, \dots, b_n\} = C(\sigma)$ the content of $\sigma$ and the number $n(\sigma) = |C(\sigma)|$ the effective size of $\sigma$. If $b \in C(\sigma)$ we will write $b \in \sigma$ and say that the unit $b$ is in the sample $\sigma$. The set of all ordered samples on $B$ is $U(B)$, which will be denoted only by $U$ if $B$ is a given population. To simplify the notation the first $N$ positive integers will be used as notations for $B$, and we will use the word sample instead of ordered sample.

**Definition 1.1.** *Sampling design is an ordered triple* $S = (B, U, p)$, *where* $p : U \longrightarrow \mathbb{R}$ *is a mapping from* $U$ *to the set of real numbers* ($\mathbb{R}$) *such that*

$$i)\ p(\sigma) \geq 0, \forall \sigma \in U;$$

$$ii)\ \sum_{\sigma \in U} p(\sigma) = 1$$

In $ii)$ we have in mind that $U$ is an infinite countable set. The subset $U_p = \{\sigma | p(\sigma) > 0\}$ of $U$ is called the support of the design $S$. The mapping $p$ is called a *probability function* of the design $S$, and $p(\sigma)$ is said to be the probability of $\sigma$ in the design $S$.

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**1.1.⁰** *The probability function of the design, p, induces a probability measure on the set of all subsets of U, B(U), defined by*

$$P(A) = \sum_{\sigma \in A} p(\sigma), \quad \forall A \subseteq U. \ \square$$

In this way, to any design $S = (B, U, p)$ corresponds a probability space $(U, B(U), P)$. From the fact that $U$ is countable set, and the $\sigma$-algebra is $B(U)$, we have that any real function $X : U \to \mathbb{R}$ is a random variable on $(U, B(U), P)$.

## 2. Sampling space induced by a sampling design

For a given population $B = \{1, 2, \dots, N\}$ and a real function $\mathbf{Y} : B \to \mathbb{R}$, we denote $\mathbf{Y}(i)$ by $Y_i$, for $i \in B$. The variable $\mathbf{Y}$ is a characteristic of the population. Let the vector $\mathbf{Y} = (Y_1, Y_2, \dots, Y_N)$ be the unknown *parameter* of the population. Let $S(B, U, p)$ be a given sampling design, $\mathbf{Y} \in \mathbb{R}^N$, and let the components of $\mathbf{Y}$ be fixed, i.e. if a unit from $B$ is available, then the corresponding component of $\mathbf{Y}$ is completely determined. So, the randomness in the model is introduced only by the definition of the sampling design. Since $\mathbf{Y} \in \mathbb{R}^N$ can be considered as a mapping $\mathbf{Y} : i \to Y_i$ from $B$ to $\mathbb{R}$ and $\sigma \in U$ as a mapping $\sigma : i \to \sigma(i)$ from $\mathbb{N}_n = \{1, 2, \dots, n\}$ to $B$, where $n = L(\sigma)$, we have that $\mathbf{Y}\sigma : i \to Y_{\sigma(i)}$ is a mapping from $\mathbb{N}_n$ to $\mathbb{R}$. Then $\mathbf{y} = \mathbf{Y}\sigma \in \mathbb{R}^{L(\sigma)}$.

One of the main notions in the theory of sampling design is the notion of *data* obtained by a given sample.

We define the sets $\triangle'$ and $\triangle'^{+}$ by

$$\triangle' = \{(\sigma, \mathbf{Y}\sigma) | \sigma \in U, \ \mathbf{Y} \in \mathbb{R}^N\} \text{ and } \triangle'^{+} = \{(\sigma, \mathbf{Y}\sigma) | \sigma \in U_p, \ \mathbf{Y} \in \mathbb{R}^N\}$$

The pair $(\sigma, \mathbf{y}) = ((s_i, s_2, \dots, s_n), \ (y_1, y_2, \dots, y_n))$ can be observed as a sequence of pairs $((s_1, y_1), \dots, (s_n, y_n))$.

Using the previous definitions the following proposition holds:

**2.1⁰**

$$\begin{aligned} \triangle' &= \{(\sigma, \mathbf{y}) | \sigma \in U, \mathbf{y} \in \mathbb{R}^{L(\sigma)}, \exists \mathbf{Y} \in \mathbb{R}^N \textit{ for which } \mathbf{y} = \mathbf{Y}\sigma\} \\ &= \{(\sigma, \mathbf{y}) | \sigma \in U, \mathbf{y} \in \mathbb{R}^{L(\sigma)}, \ \ker \sigma \subseteq \ \ker \mathbf{y}\}, \end{aligned}$$

*where in the second notation we consider $\sigma$ and $\mathbf{y}$ as $\sigma : \mathbb{N}_n \to B$ and $\mathbf{y} : \mathbb{N}_n \to \mathbb{R}$.* $\square$

The set $\triangle'$ is a subset of the set $\triangle = \{(\sigma, \mathbf{y}) | \sigma \in U, \mathbf{y} \in \mathbb{R}^{L(\sigma)}\}$. We will call the set $\triangle$ *a sampling space.* We also define $\triangle^{+} = \{(\sigma, \mathbf{y}) | \sigma \in U_p, \mathbf{y} \in \mathbb{R}^{L(\sigma)}\}$.

**Definition 2.1.** *We say that the data $\mathbf{d} = (\sigma, \mathbf{y}) \in \triangle$ is consistent with the parameter $\mathbf{Y} \in \mathbb{R}^N$ if and only if (iff) $\mathbf{y} = \mathbf{Y}\sigma$ and we write $\mathbf{d} \sim \mathbf{Y}$.*

**Definition 2.2.** *For $\forall \mathbf{Y} \in \mathbb{R}^N$ we define a mapping $p_{\mathbf{Y}} : \triangle \to \mathbb{R}$ by:*

$$p_{\mathbf{Y}}((\sigma, \mathbf{y})) = \begin{cases} p(\sigma) & \textit{if } \mathbf{y} = \mathbf{Y}\sigma \\ 0 & \textit{otherwise} \end{cases}$$

Then the following proposition holds:

**2.2⁰** *The triple* $(\triangle, B(\triangle), P_{\mathbf{Y}})$ *is a probability space, where* $\forall A \subseteq \triangle$,

$$P_{\mathbf{Y}}(A) =\cdot \sum_{(\sigma,\mathbf{y})\in A} p_{\mathbf{Y}}((\sigma,\mathbf{y})). \ \square$$

## 3. Sufficient statistic

Let $\Omega \subseteq \mathbb{R}^m$ and $f : \triangle \to \Omega$ be a statistic.

**Definition 3.1.** *The statistic* $f : \triangle \to \Omega$ *is called a sufficient statistic for the parameter* $\mathbf{Y} \in \mathbb{R}^N$ *if for given* $\mathbf{w}_0 \in \Omega$ *and* $\mathbf{Y}', \mathbf{Y}'', \in \mathbb{R}^N$ *the following equation holds*

$$P_{\mathbf{Y}'}(\mathbf{D} = \mathbf{d}|f(\mathbf{D}) = \mathbf{w}_0) = P_{\mathbf{Y}''}(\mathbf{D} = \mathbf{d}|f(\mathbf{D}) = \mathbf{w}_0).$$

From the definition of a conditional probability we have

$$P_{\mathbf{Y}}(\mathbf{D} = \mathbf{d}|f(\mathbf{d} = w_0) = \frac{P_{\mathbf{Y}}(\{\mathbf{d}\} \cap f^{-1}(\{w_0\}))}{P_{\mathbf{Y}}(f^{-1}(\{w_0\}))} \tag{1}$$

when there exists $\mathbf{d}^* = (\sigma^*, \mathbf{y}^*) \in f^{-1}(\{\mathbf{w}_0\})$ for which $\mathbf{Y}\sigma^* = \mathbf{y}^*$ and $p(\sigma^*) = p_{\mathbf{Y}}(\mathbf{d}*) > 0$. Otherwise, $P_{\mathbf{Y}}(\mathbf{D} = \mathbf{d}|f(\mathbf{d}) = \mathbf{w}_0) = 0$ when $f(\mathbf{d}) \neq \mathbf{w}_0$ or $p_{\mathbf{Y}}(\mathbf{d}) = 0$ for each $\mathbf{d}$ such that $f(\mathbf{d}) = \mathbf{w}_0$.

**Theorem 3.1.** *If* $f : \triangle \to \Omega$ *is a statistic, the following statements are equivalent:*

*i)* $f$ *is sufficient for* $\mathbf{Y}$,

*ii) If* $\mathbf{d}', \mathbf{d}'' \in \triangle'^{+}, \mathbf{Y}'.\mathbf{Y}'' \in \mathbb{R}^n$ *are such that* $f(\mathbf{d}') = f(\mathbf{d}''), \mathbf{d}' \sim \mathbf{Y}', \mathbf{d}'' \sim \mathbf{Y}''$, *then* $\mathbf{d}' \sim \mathbf{Y}'', \mathbf{d}'' \sim \mathbf{Y}'$,

*iii) If* $\mathbf{Y}', \mathbf{Y}'' \in \mathbb{R}^N$ , $\mathbf{w}_0 \in \Omega$ *are such that* $P_{\mathbf{Y}'}(f^{-1}(\{\mathbf{w}_0\})) > 0$, $P_{\mathbf{Y}''}(f^{-1}(\{\mathbf{w}_0\})) > 0$, *then* $p_{\mathbf{Y}'}(\mathbf{d}) = p_{\mathbf{Y}''}(\mathbf{d})$ *for each* $\mathbf{d} \in f^{-1}(\{\mathbf{w}_0\})$,

*iv) Under the conditions as in iii),* $p_{\mathbf{Y}'}(\mathbf{d}) = p_{\mathbf{Y}''}(\mathbf{d})$, *for* $\mathbf{d} \in f^{-1}(\{\mathbf{w}_0\}) \cap \triangle'^{+}$.

*Proof.* (1) Let $f$ be a sufficient for $\mathbf{Y}$ and the conditions in ii) hold. Then from $f(\mathbf{d}') = f(\mathbf{d}'') = \mathbf{w}_0$, $p_{\mathbf{Y}'}(\mathbf{d}') > 0$ and $p_{\mathbf{Y}''}(\mathbf{d}'') > 0$ it follows that

$$P_{\mathbf{Y}'}(f^{-1}(\{\mathbf{w}_0\})) > 0, \ P_{\mathbf{Y}''}(f^{-1}(\{\mathbf{w}_0\})) > 0, \text{ and}$$
$$P_{\mathbf{Y}'}(\mathbf{D} = \mathbf{d}'|f(\mathbf{D}) = \mathbf{w}_0) = P_{\mathbf{Y}''}(\mathbf{D} = \mathbf{d}''|f(\mathbf{D}) = \mathbf{w}_0) \text{ i.e.}$$

$$\frac{p_{\mathbf{Y}'}(\mathbf{d}')}{P_{\mathbf{Y}'}(f^{-1}(\{\mathbf{w}_0\}))} = \frac{p_{\mathbf{Y}''}(\mathbf{d}')}{P_{\mathbf{Y}''}(f^{-1}(\{\mathbf{w}_0\}))}$$

Since $\mathbf{d}' \sim \mathbf{Y}'$ it follows that $p_{\mathbf{Y}'}(\mathbf{d}') > 0$, which means that $p_{\mathbf{Y}''}(\mathbf{d}') > 0$, i. e. $\mathbf{d}' \sim \mathbf{Y}''$. By symmetry it follows that $\mathbf{d}'' \sim \mathbf{Y}'$. So we have that i)⇒ii).

(2) Let ii) holds and the conditions from iii) are satisfied. Let $\mathbf{d} \in \triangle$ be such that $\mathbf{d} \in f^{-1}(\{\mathbf{w}_0\})$, or $f(\mathbf{d}) = \mathbf{w}_0$. If $\mathbf{d} \in \triangle\backslash\triangle'^{+}$, then $p_{\mathbf{Y}'}(\mathbf{d}) = 0 = p_{\mathbf{Y}''}(\mathbf{d})$. So let $\mathbf{d} \in f^{-1}(\{\mathbf{w}_0\}) \cap \triangle'^{+}$. If $p_{\mathbf{Y}'}(\mathbf{d}) = 0$ and $p_{\mathbf{Y}''}(\mathbf{d}) = 0$ then they are equal. So let $p_{\mathbf{Y}'}(\mathbf{d}) > 0$. Then $\mathbf{d} \sim \mathbf{Y}'$. Since $P_{\mathbf{Y}''}(f^{-1}(\{\mathbf{w}_0\})) > 0$, there exists $\mathbf{d}'' \in f^{-1}(\{\mathbf{w}_0\}) \cap \triangle'^{+}$ for which $p_{\mathbf{Y}''}(\mathbf{d}'') > 0$, so $\mathbf{d}'' \sim \mathbf{Y}''$. The above discussion

shows that for $\mathbf{d} = \mathbf{d}', \mathbf{d}''$ and $\mathbf{Y}', \mathbf{Y}'' \in \mathbb{R}^N$, the conditions from ii) holds, which implies that $\mathbf{d} \sim \mathbf{Y}''$. Then the definition of $p_{\mathbf{Y}}$ implies that $p_{\mathbf{Y}''}(\mathbf{d}) = p_{\mathbf{Y}'}(\mathbf{d}) > 0$. So we proved that ii)⇒iii).

(3) It is obvious that iii)⇔ iv), so we have to prove that iii)⇒i).

(4) Let iii) holds and let $\mathbf{d}_0 \in \Delta$, $\mathbf{w}_0 \in \Omega$ and $\mathbf{Y}'$, $\mathbf{Y}'' \in \mathbb{R}^N$ are such that $P_{\mathbf{Y}'}(f^{-1}(\{\mathbf{w}_0\})) > 0$, and $P_{\mathbf{Y}''}(f^{-1}(\{\mathbf{w}_0\})) > 0$. Then, for each $\mathbf{d} \in (f^{-1}(\{\mathbf{w}_0\})$ according to iii), we have that $p_{\mathbf{Y}''}(\mathbf{d}) = p_{\mathbf{Y}'}(\mathbf{d})$. We have to prove that if

$\lambda = P_{\mathbf{Y}'}(f^{-1}(\{\mathbf{w}_0\})) > 0$, and $\mu = P_{\mathbf{Y}''}(f^{-1}(\{\mathbf{w}_0\})) > 0$, then $\lambda p_{\mathbf{Y}''}(\mathbf{d}_0) = \mu p_{\mathbf{Y}'}(\mathbf{d}_0)$.

If $\mathbf{d}_0 \notin f^{-1}(\{\mathbf{w}_0\})$, then $p_{\mathbf{Y}''}(\mathbf{d}_0) = p_{\mathbf{Y}'}(\mathbf{d}_0) = 0$ and the equality holds. So, let $\mathbf{d}_0 \in f^{-1}(\{\mathbf{w}_0\})$. Then iii) implies that $p_{\mathbf{Y}''}(\mathbf{d}_0) = p_{\mathbf{Y}'}(\mathbf{d}_0)$.

So $\lambda = \sum_{\mathbf{d}\in f^{-1}(\{\mathbf{w}_0\})} p_{\mathbf{Y}'}(\mathbf{d}) = \sum_{\mathbf{d}\in f^{-1}(\{\mathbf{w}_0\})} p_{\mathbf{Y}''}(\mathbf{d}) = \mu$ □

**Corollary 3.1.1.** *If a statistic $f : \Delta \to \Omega$ is injection, then $f$ is sufficient for* $\mathbf{Y}$.

**Theorem 3.2.** *Let $f : \Delta \to \Omega_1$, $g : \Delta \to \Omega_2$ be such that* $\ker f \subseteq \ker g$. *If $g$ is sufficient for $\mathbf{Y}$ then $f$ is sufficient for $\mathbf{Y}$.*

*Proof.* Let $g$ be sufficient for $\mathbf{Y}$. By the theorem 3.1 it is sufficient to show that $f$ satisfies the condition ii). Let $\mathbf{d}' = (\sigma', \mathbf{y}')$, $\mathbf{d}'' = (\sigma'', \mathbf{y}'') \in \Delta'^{+}$ and $\mathbf{Y}'$, $\mathbf{Y}'' \in \mathbb{R}^N$, are such that $f(\mathbf{d}') = f(\mathbf{d}'')$, $\mathbf{d}' \sim \mathbf{Y}'$, $\mathbf{d}'' \sim \mathbf{Y}''$. Then, $\ker f \subseteq \ker g$ implies that $g(\mathbf{d}') = g(\mathbf{d}'')$. Since $g$ is sufficient , ii) holds for $g$, which means that $\mathbf{d}' \sim \mathbf{Y}''$, $\mathbf{d}'' \sim \mathbf{Y}'$. So, we proved that condition ii) from Theorem 3.1 is satisfied for $f$, which implies that $f$ is sufficient for $\mathbf{Y}$. □

**Corollary 3.2.1.** *Let $f : \Delta \to \Omega_1$, $g : \Delta \to \Omega_2$ be such that* $\ker f = \ker g$. *Then $f$ is sufficient for $\mathbf{Y}$ iff $g$ is sufficient for $\mathbf{Y}$.*

**Theorem 3.3.** *Let $f : \Delta \to \Omega_1$, $g : \Delta \to \Omega_2$ be statistics such that* $\ker f_{\Delta'^{+}} \subseteq \ker g_{\Delta'^{+}}$ . *If $g$ is sufficient for $\mathbf{Y}$, then $f$ is sufficient for $\mathbf{Y}$.*

## 4. Sufficient equivalence in $\Delta$

In the previous section we have seen that for a statistic $f : \Delta \to \Omega$, its sufficiency depends of the kernel of $f$, which is an equivalence relation on $\Delta$. Let $\ker f = \alpha$. If we take $\Omega$ to be the factor set $\Delta_{/\alpha}$ and define $\bar{f}(d) = d^{\alpha}$ we have an onto mapping from $\Delta$ to $\Delta_{/\alpha}$ and $\ker f = \ker \bar{f}$ . From the previous discussion it follows that if $f$ is sufficient so is $\bar{f}$ .

This construction is possible for any equivalence $\omega$ on $\Delta$. Let nat $\omega : \Delta \to \Delta_{/\omega}$ be the natural mapping, i.e. nat $\omega(d) = d^{\omega}$. The previous discussion suggests definition of a sufficient equivalence on $\Delta$.

**Definition 4.1.** *We say that an equivalence $\omega$ on $\Delta$ is sufficient iff the natural mapping nat $\omega : \Delta \to \Delta_{/\omega}$ is a sufficient statistic.*

**Theorem 4.1.** *An equivalence $\omega$ on $\Delta$ is sufficient iff the following holds:*

*If $\mathbf{d}', \mathbf{d}''$ $\Delta'^{+}$ and $\mathbf{Y}', \mathbf{Y}'' \in \mathbb{R}^n$ are such that $\mathbf{d}' \sim \mathbf{Y}'$, $\mathbf{d}'' \sim \mathbf{Y}''$ and $\mathbf{d}'^{\omega} = \mathbf{d}''^{\omega}$, then $\mathbf{d}' \sim \mathbf{Y}''$, $\mathbf{d}'' \sim \mathbf{Y}'$.* □

Let $Eq(\triangle)$ be the set of all equivalences on $\triangle$, and $SEq(\triangle)$ be the set of all sufficient equivalences on $\triangle$.

In the family of all equivalences on $\triangle$, the smallest element is $\theta$, and the largest element is $\varepsilon$, where $\mathbf{d}^{\theta} = \{\mathbf{d}\}$ and $\mathbf{d}^{\varepsilon} = \triangle$, for $\mathbf{d} \in \triangle$.

As a direct interpretation of the results in previous section we get the following results.

4.2⁰ *For any design $S$, $\theta$ is sufficient equivalence.*

*Proof:* For given $\mathbf{d}' = (\sigma', \mathbf{y}')$ and $\mathbf{d}'' = (\sigma'', \mathbf{y}'')$ in $\triangle$, $P_{\mathbf{Y}}(\mathbf{d} = \mathbf{d}' \mid nat\ \omega = \{\mathbf{d}''\})$ exists iff $0 < P_{\mathbf{Y}}((nat\ \omega)^{-1}\{\mathbf{d}''\}) = p_{\mathbf{Y}}(\mathbf{d}'')$. The latest is true iff $p(\sigma'') > 0$ and $\mathbf{Y}\sigma'' = \mathbf{y}''$. Then

$$P_{\mathbf{Y}}(\mathbf{d} = \mathbf{d}' \mid nat\ \omega(\mathbf{d}) = \{\mathbf{d}''\}) = \begin{cases} 0 & \text{if } \mathbf{d}' \neq \mathbf{d}'' \\ 1 & \text{if } \mathbf{d}' = \mathbf{d}'' \end{cases}$$

which means that $\theta$ is sufficient equivalence. □

4.3⁰ *The equivalence $\varepsilon$ is not sufficient.*

*Proof.* Since, for each $\mathbf{d} \in \triangle$, $\mathbf{d}^{\omega} = \triangle$, it follows that

$$P_{\mathbf{Y}}(\mathbf{d} = (\sigma^0, \mathbf{y}^0) \mid nat\ \omega(\mathbf{d}) = \triangle) = p_{\mathbf{Y}}(\sigma^0, \mathbf{y}^0) = \begin{cases} p(\sigma^0) & \text{for } \mathbf{Y}\sigma^0 = \mathbf{y}^0 \\ 0 & \text{otherwise} \end{cases}$$

To show that the condition for sufficiency is not satisfied, we have to show that there are $(\sigma^0, \mathbf{y}^0) \in \triangle$ and $\mathbf{Y}', \mathbf{Y}'' \in \mathbf{R}^n$ such that $\mathbf{Y}'\sigma^0 = \mathbf{y}^0$, $\mathbf{Y}''\sigma^0 \neq \mathbf{y}^0$ and $p(\sigma^0) > 0$. If we choose $\mathbf{d}^0 = (\sigma^0, \mathbf{y}^0)$ such that $\ker \sigma^0 \subseteq \ker \mathbf{y}^0$ and $p(\sigma^0) > 0$, it is enough to choose $\mathbf{Y}'$ such that for each $i \in \mathbb{N}_n$ and $\mathbf{Y}''$ such that $\mathbf{Y}''_{\sigma^0_{(i)}} \neq \mathbf{y}^0_i$ for at least one $i \in \mathbb{N}_n$. □

**Theorem** 4.4 *Let $\alpha, \beta \in Eq(\triangle)$ be such that $\alpha \subseteq \beta$. If $\beta \in SEq(\triangle)$, then $\alpha \in SEq(\triangle)$.* □

**Theorem** 4.5 *If $\alpha, \beta \in Eq(\triangle)$ are such that $\alpha_{/\triangle'^+} \subseteq \beta_{/\triangle'^+}$ then:*

*$\alpha \in SEq(\triangle)$ if and only if $\beta \in SEq(\triangle)$.* □

It is well known that the set $Eq(\triangle)$ is a complete lattice, (i.e., for each subset $\Gamma \subseteq Eq(\triangle)$, $\sup \Gamma$ and $\inf \Gamma$ are elements in $Eq(\triangle)$). In this lattice $\inf \Gamma$ is the intersection of the relations on $\Gamma$, and $\sup \Gamma$ is the transitive product of the relations in $\Gamma$. As a consequence of Theorem 4.5 we obtain the following

**Corollary** 4.5.1 *If $\Gamma \subseteq SEq(\triangle)$ then $\inf \Gamma \in SEq(\triangle)$. (More generally, if $\Gamma \cap SEq(\triangle) \neq \emptyset$ then $\inf \Gamma \in SEq(\triangle)$).* □

It is natural to ask the question if an analogous result holds for $\sup \Gamma$. To consider this question we have to define the notion of chain in $Eq(\triangle)$. Namely, we say that $\Gamma$ is *a chain* in $Eq(\triangle)$ iff $\Gamma$ is a subset of $Eq(\triangle)$ such that for each $\alpha, \beta \in \Gamma$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. In a similar way we define a chain in $SEq(\triangle)$.

**Theorem** 4.6 *If* $\Gamma$ *is a chain in* $SEq(\triangle)$ *then* $\sup\Gamma$, *denoted by* $\gamma$, *is a union of the relations in* $\Gamma$ *and* $\gamma \in SEq(\triangle)$.

*Proof.* We will show that $\sup\Gamma$ is $\gamma$, a union of the relations in $\Gamma$. Let $\gamma = \bigcup_{\beta\in\Gamma} \beta$.

1) $\gamma$ is reflexive iff at least one relation of $\Gamma$ is reflexive;

2) Since all elements in $\Gamma$ are symmetric, so is $\gamma$;

3) Let $(\mathbf{d}_1, \mathbf{d}_2)$, $(\mathbf{d}_2, \mathbf{d}_3) \in \gamma$. Then, there exist $\alpha, \beta \in \Gamma$, such that $(\mathbf{d}_1, \mathbf{d}_2) \in \alpha$ and $(\mathbf{d}_2, \mathbf{d}_3) \in \beta$. But, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. Let $\alpha \subseteq \beta$. Then $(\mathbf{d}_1, \mathbf{d}_2)$, $(\mathbf{d}_2, \mathbf{d}_3) \in \beta$, and since $\beta$ is transitive, $(\mathbf{d}_1, \mathbf{d}_3) \in \beta$. So $(\mathbf{d}_1, \mathbf{d}_3) \in \gamma$, which means that g is transitive.

From 1), 2) and 3) it follows that $\gamma$ is an equivalence and for each $\mathbf{d} \in \triangle$,

$$\mathbf{d}^{\gamma} = \bigcup_{\beta\in\Gamma} \beta = \mathbf{d}^{\beta} \tag{2}$$

Using the last equation and Theorem 4.1 we will show that $\gamma \in SEq(\triangle)$.

Let $\mathbf{d}', \mathbf{d}'' \in \triangle$ and $\mathbf{Y}', \mathbf{Y}'' \in \mathbb{R}^N$ be such that $\mathbf{d}' \sim \mathbf{Y}'$, $\mathbf{d}'' \sim \mathbf{Y}''$ and $\mathbf{d}'^{\gamma} = \mathbf{d}''^{\gamma}$. From $\mathbf{d}'^{\gamma} = \mathbf{d}''^{\gamma}$ it follows that $\mathbf{d}'\gamma\mathbf{d}''$, which means that there exists $\beta \in \Gamma$, such that $\mathbf{d}'\beta\mathbf{d}''$, i.e., $\mathbf{d}'^{\beta} = \mathbf{d}''^{\beta}$. Since $\beta \in SEq(\triangle)$, the equation (2) and Theorem 4.1 imply that $\mathbf{d}' \sim \mathbf{Y}''$ $\mathbf{d}'' \sim \mathbf{Y}'$, and (again by Theorem 4.1) that $\gamma \in SEq(\triangle)$. □

As a consequence of theorem 4.7 and the lemma of Zorn, we get the following result:

**Theorem** 4.7 *For any sufficient equivalence* $\alpha$ *there is a maximal sufficient equivalence* $\gamma$, *such that* $\alpha \subseteq \gamma$. *In other words, for* $\forall\alpha \in SEq(\triangle)$ *there is a maximal element* $\gamma \in SEq(\triangle)$ *such that* $\alpha \subseteq \gamma$. □

At the end we will define a sufficient equivalence $\kappa$ which will enable a complete description of $SEq(\triangle)$, and by that the class of all sufficient statistics.

For each $\mathbf{d} = (\sigma, \mathbf{y}) \in \triangle$ define a set $A(\sigma, \mathbf{y}) = A(\mathbf{d})$ by

$x \in A(\sigma, \mathbf{y})$ iff $x = (s_i, y_i)$ for some $i \in \mathbb{N}_n$,

where $(\sigma, \mathbf{y}) = ((s_1, \ldots, s_n), (y_1, \ldots, y_n))$, i.e. $\mathbf{d} = ((s_1, y_1), \ldots, (s_n, y_n))$.

Let $[\triangle] = \{A(\mathbf{d}) | \mathbf{d} \in \triangle\}$. Define a statistic $k : \triangle \to [\triangle]$ by

$$k(\mathbf{d}) = A(\mathbf{d}). \tag{3}$$

**Theorem** 4.8 *The statistic* $k$ *is sufficient statistic.*

*Proof.* Suppose that the conditions from theorem 4.1 (*ii*) hold, i.e.: $\mathbf{d}' = (\sigma, \mathbf{y})$, $\mathbf{d}'' = (\tau, \mathbf{z})$ and $\mathbf{Y}', \mathbf{Y}'' \in \mathbb{R}^N$ are such that $k(\mathbf{d}') = k(\mathbf{d}'')$, $\mathbf{Y}'\sigma = \mathbf{y}$ and $\mathbf{Y}''\tau = \mathbf{z}$. We have to show that $\mathbf{Y}''\sigma = \mathbf{y}$ and $\mathbf{Y}'\tau = \mathbf{z}$.

From the equation $A(\mathbf{d}') = A(\mathbf{d}'')$ it follows that for each $i \in \mathbb{N}_n$ there is $i \in \mathbb{N}_m$ such that $\sigma_i = \tau_j$ and $y_i = z_j$. Then from $\mathbf{Y}'\sigma = \mathbf{y}$ and $\mathbf{Y}''\tau = \mathbf{z}$, it follows that $\mathbf{Y}''_{\tau_j} = z_j = y_i = \mathbf{Y}'_{\sigma_i}$. So, $\mathbf{Y}''_{\sigma_i} = \mathbf{Y}''_{\tau_j} = z_j = y_i$ , which means that $\mathbf{Y}''\sigma = \mathbf{y}$. By symmetry it follows that $\mathbf{Y}'\tau = \mathbf{z}$. □

**Theorem** 4.9 *A statistic $f : \triangle \to \Omega$ is sufficient for the parameter* **Y** *iff :*

$$\ker f_{\triangle'^+} \subseteq \ker k_{\triangle'^+}. \tag{4}$$

*Proof.* If the relation (4) holds, theorem 3.3 implies that $f$ is sufficient statistic.

Suppose that (4) is not satisfied. This means that there are $\mathbf{d}' = (\sigma, \mathbf{y})$; $\mathbf{d}'' = (\tau, \mathbf{z}) \in \triangle'^+$ such that $f(\mathbf{d}') = f(\mathbf{d}'')$, but $k(\mathbf{d}') \neq k(\mathbf{d}'')$. From the inequality $k(\mathbf{d}') \neq k(\mathbf{d}'')$, by symmetry, we can suppose that there is $i \in \mathbb{N}_n$, such that $(\sigma_i, y_i) \neq (\tau_j, z_j)$ for each $j \in \mathbb{N}_m$. Two cases are possible: 1) $\sigma_i \in \tau$ and 2) $\sigma_i \notin \tau$.

Let $\mathbf{Y}', \mathbf{Y}'' \in \mathbf{R}^N$ be such that $\mathbf{d}' \sim \mathbf{Y}'$ and $\mathbf{d}'' \sim \mathbf{Y}''$, i.e. let $\mathbf{Y}'\sigma = \mathbf{y}$ and $\mathbf{Y}''\tau = \mathbf{z}$.

In case 1), there is $j \in \mathbb{N}_m$ such that $\sigma_i = \tau_j$, but since $(\sigma_i, y_i) \neq (\tau_j, z_j)$, $y_i \neq z_j$. But then $y_i \neq z_j = \mathbf{Y}''_{\tau_j} = \mathbf{Y}''_{\sigma_i}$ . This means that $\mathbf{Y}''\sigma \neq \mathbf{y}$, or $\mathbf{d}$ is not consistent with $\mathbf{Y}''$.

In case 2), $\sigma_i \neq \tau_j$ for each $j \in \mathbb{N}_m$ . Let $\mathbf{Y}^* \in \mathbb{R}^N$ be such that $\mathbf{Y}^*_k = \mathbf{Y}''_k$ for all $k$ except for $k = \sigma_i$, namely$\mathbf{Y}^*_{\sigma_i} \neq y_i$ and all other components of $\mathbf{Y}^*$ and $\mathbf{Y}''$ are equal. Then $\mathbf{Y}^*\tau = \mathbf{z}$, but $\mathbf{Y}^*\sigma \neq \mathbf{y}$, i.e. $\mathbf{d}''$ is consistent with $\mathbf{Y}^*$, but $\mathbf{d}'$ is not consistent with $\mathbf{Y}^*$.

In both cases theorem 4.1 *(ii)* implies that $f$ is not sufficient statistic for **Y**. □

The Theorem 4.9 can be formulated in the following manner, where $\kappa = \ker k$.

**Theorem** 4.9′ *The equivalence $\alpha \in Eq(\triangle)$ is sufficient iff $\alpha_{\triangle'^+}$.* □

Let us note that the theorems 4.7 and 4.8 are consequences of theorem 4.9 or 4.9′. At the end we will describe the maximal elements in $SEq(\triangle)$.

**Theorem** 4.10 *If $\omega$ is maximal sufficient equivalence, then $\omega_{\triangle'^+} = \kappa_{\triangle'^+}$.*

*Proof.* Let $\omega$ be maximal and sufficient equivalence and let $\mathbf{d}', \mathbf{d}'' \in \triangle'^+$ be such that $\mathbf{d}'\kappa\mathbf{d}''$, but $(\mathbf{d}', \mathbf{d}'') \notin \omega$. Let $\tilde{\omega}$be the equivalence generated by $\omega \cup (\mathbf{d}', \mathbf{d}'')$. Then $\mathbf{d}'^{\tilde{\omega}} \supseteq \mathbf{d}'^{\omega} \cup \mathbf{d}''^{\tilde{\omega}}$, and $\tilde{\omega}$is defined by:

$$\text{if } \mathbf{d} \notin \mathbf{d}'^{\omega} \cup \mathbf{d}''^{\omega} \text{ then } \mathbf{d}^{\tilde{\omega}} = \mathbf{d}^{\omega} \text{ and}$$
$$\text{if } \mathbf{d} \in \mathbf{d}'^{\omega} \cup \mathbf{d}''^{\omega} \text{ then } \mathbf{d}^{\tilde{\omega}} = \mathbf{d}'^{\omega} \cup \mathbf{d}''^{\omega}.$$

By the definition of $\tilde{\omega}$ it follows that $\omega \subset \tilde{\omega}$ and that $\tilde{\omega}_{\triangle'^+} \subseteq \kappa_{\triangle'^+}$ Theorem 4.9′ implies that $\tilde{\omega}$ is sufficient equivalence for **Y**. But this together with $\omega \subset \tilde{\omega}$ contradicts the fact that $\omega$ is a maximal equivalence. □

The condition $\omega_{\triangle'^+} = \kappa_{\triangle'^+}$ is not a sufficient condition for maximallity of $\omega$. This is shown by the following example. Let $\rho$ be the equivalence defined by :

$$\text{for } \mathbf{d} \in \triangle'^+, \ \mathbf{d}^\rho = \mathbf{d}''^\kappa \cap \triangle'^+ \ \text{ and for } \ \mathbf{d} \notin \triangle'^+, \ \mathbf{d}^\rho = \triangle \backslash \triangle'^+.$$

Then $\rho \in SEq(\triangle)$ and $\rho_{\triangle'^+} = \kappa_{\triangle'^+}$ . But $\rho$ is not a maximal element, since it is a proper subset of $\bar{\rho} \in SEq(\triangle)$ defined as follows. Fix $\mathbf{d}' \in \triangle'^+$ and let $\mathbf{d}'^{\bar{\rho}} = (\triangle \backslash \triangle'^+) \cup \mathbf{d}'^\rho$, and for all other $\mathbf{d} \in \triangle$, let $\mathbf{d}'^{\bar{\rho}} = \mathbf{d}'^\rho$ . Since, $\bar{\rho}_{\triangle'^+} = \kappa_{\triangle'^+}$, $\bar{\rho} \in SEq(\triangle)$ and $\rho \subset \bar{\rho}$.

The previous discussion gives in fact the description of maximal elements of $SEq(\triangle)$, given by following theorem:

**Theorem** 4.11 *An equivalence $\omega \in SEq(\triangle)$ is a maximal element in $SEq(\triangle)$ if and only if the following conditions hold*

*i)* $\omega_{\triangle'^+} = \kappa_{\triangle'^+}$ *and*

*ii) for each* $\mathbf{d} \in \triangle$, *there is* $\mathbf{d}' \in \triangle'^+$, *such that* $\mathbf{d}^\omega = \mathbf{d}'^\omega$. □

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