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THE SPECIAL ROLE OF THE g -FUNCTIONS

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Abstract. The class of g -functions by the g -generator of the system of pseudo-operations, apply a special role on functional equations and their solutions. More properties may be found in this class and by some elementary g -functions are given further studies to the entropy of $\oplus$ - (decomposable) measure.

1. INTRODUCTION

The function corresponding to a function f introduced by the g –calculus (called g –functions and denote by f_g in general and then g -function for special case) are derived as solutions of some functionals equations using several results of Aczél [1]. To the creation of function are shown the role of the consistent sistem of pseudo-arithmetical operations generating by generator g, by obtain directly the rational function [2], [3] but g -Transform is a further development of g -calculus [5], [6], [9], [12], [17]. That is why are introduce some elementary function as solutions of corresponding functional equations [2], [6], [20]. A wide class of some elementary modified function (f_g) is investegated [2] and some rules for crossings into different parameterized classes of functional equations is obtained.

The study of entropy and further the g -entropy for $\oplus$-decomposable probability measure are encouraged more by the role of g -function and found links by g -Transform [2], [3], [16]. Reasonable, is raised the issue of the modification of the measure by g -Transform and the some relation between m_g and P_g are given.

2. MODIFICATION OF FUNCTIONS BY g -TRANSFORM

2.1. MODIFICATION OF FUNCTION AND SOME IMPORTANT g -FUNCTIONS

Two binary operation $(\oplus,\odot)$ on $[0,+\infty]$ are respectively, pseudo-addition and pseudo-multiplication corresponding to the pseudo-addition $\oplus$ (introduced first on

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$[0,+\infty]$ interval and then to the whole extended real line $[-\infty,+\infty]$), if and only if there is a generator g (a continuous monotone strictly increasing unbounded odd function) $g:]-\infty,+\infty]\to]-\infty,+\infty]$, such that $g(0)=0_{\oplus}$, $g(1)=1_{\odot}, g(+\infty)=+\infty$ so for all $x,y\in$ $[-\infty,+\infty]$ it is $x\oplus y=g^{-1}(g(x)+g(y))$ and $x\odot y=g^{-1}(g(x)\cdot g(y))$, with the convention $0\cdot(+\infty)=0$ [2], [4], [7], [6], [9], [16].

If the generator g is increasing (decreasing), then the pseudo-operation $\oplus$, through it`s generator g induces the usual order (opposite to the usual order), on the interval $[-\infty,+\infty]$ in the following way: $x\preceq y$ if and only if $g(x)\le g(y)$. We will work with the real function f, which is continuous on $]a,b[$ and $]a,b[\subseteq]-\infty,+\infty[$.

The pseudo-arithmetical operation $(\ominus,\oslash)$ are introduce on $[-\infty,+\infty]$ in [9], [16] as pseudo-operations consistent with the pseudo-addition by formulas:

$$x\ominus y = g^{-1}(g(x)-g(y)) \text{ and } x\oslash y = g^{-1}(g(x)/g(y)).$$

Definition 2.1 ([2]). Let f be a function on $]a,b[\subseteq]-\infty,+\infty[$ and the function g be a generator of the consistent system of pseudo-arithmetical operations $\{\oplus,\odot,\ominus,\oslash\}$. The function f_g given by $f_g(x)=g^{-1}(f(g(x)))$ for every $x\in(g^{-1}(a),g^{-1}(b))$ is said to be g *-function* corresponding to the function f.

Definition 2.2 ([2]). Let f be a function on $]a,b[\subseteq]-\infty,+\infty[$ and the function g be a generator of the consistent system of pseudo-arithmetical operations $\{\oplus,\odot,\ominus,\oslash\}$. The function f_g given by $f_g(x,y)=g^{-1}(f(g(x),g(y)))$, for every $x,y\in(g^{-1}(a),g^{-1}(b))$ is said to be g *-function* corresponding to the function *f.*

Definition 2.3 ([2]). A continuous function f_g such that is a solution of the functional equations $f_g(x)\oplus f_g(y)=f_g(x\odot y)$ and $f_g(g^{-1}(a))=1$, where $a>0, a\ne 1$ will be called *g-logarithmic* function and denoted by $f_{g-a,\log}$.

Definition 2.4 ([2]). A continuous function f_g such that is a solution of the functional equations $f_g(x)\odot f_g(y)=f_g(x\oplus y)$ and $f_g(1)=g^{-1}(a)$, where $a>0, a\ne 1$ will be called *g-exponencial* functions and denoted by $f_{g-a,\exp}$.

Definition 2.5 ([2]). A continuous function f_g such that is a solution of the functional equations $f_g(x)\odot f_g(y)=f_g(x\odot y)$ where $r>0, x\in(0,+\infty)$ will be called *g-power* functions and denoted by $f_{g-r,\text{power}}$.

This function is given by $f_{g-r,\text{power}}(x) = g^{-1}((g(x))^r)$, $r > 0$ where $x \in [0,+\infty[$.

Theorem 2.6 ([2]). For every $x \in]0,+\infty[$ it holds $f_{g-a,\log}(x) = g^{-1}(\log_a g(x))$.

Theorem 2.7 ([2]). For every $x \in]-\infty,+\infty[$ it holds $f_{g-a,\exp}(x) = g^{-1}(a^{g(x)})$.

Theorem 2.8 ([2]). The $f_{g-a,\exp}$ is an inverse function of a $f_{g-a,\log}$.

By the definition 2.3 and 2.4 the *g*-logarithmic and *g*-exponencial function respectively are given by the formulas:

$$f_{g-a,\log}(x) = g^{-1}(\log_a(g(x))) \text{ and } f_{g-a,\exp}(x) = g^{-1}(a^{g(x)})$$

so

$$f_{g-a,\exp}(f_{g-a,\log}(x)) = g^{-1}(a^{g(f_{g-a,\log}(x))}) = g^{-1}(a^{g(g^{-1}(\log_a g(x)))})$$
$$= g^{-1}(a^{\log_a g(x)}) = g^{-1}(g(x)) = x.$$

Also, conversely we can write:

$$f_{g-a,\log}(f_{g-a,\exp}(x)) = g^{-1}(\log_a g(f_{g-a,\exp}(x))) = g^{-1}(\log_a g(g^{-1}(a^{g(x)})))$$
$$= g^{-1}(\log_a a^{g(x)}) = g^{-1}(g(x)) = x.$$

From the two equations we have:

$$f_{g-a,\exp}(f_{g-a,\log}(x)) = f_{g-a,\log}(f_{g-a,\exp}(x)) = x.$$

By Theorem 4 and Corollary 1 in [2] can generalize the conditions of theorem 4 for some values $\alpha,\lambda \in]-\infty,+\infty[$ and $\alpha \neq \lambda \neq 1$. Easily are controllable the following assertions.

Theorem 2.9. Let *g* be be a generator of the consistent system of pseudo-arithmetical operations $\{\oplus,\odot,\ominus,\oslash\}$. Let *f* and *h* be continuous function on $]a,b[\subseteq]-\infty,+\infty[$ and $\alpha,\lambda \in]-\infty,+\infty[$ are constants. Then for every $x \in]g^{-1}(a), g^{-1}(b)[$ we have:

1. $(\alpha + f)_g = g^{-1}(\alpha) \oplus f_g = \oplus\,(g^{-1}(\alpha), f_g)$
2. $(\alpha \cdot f)_g = g^{-1}(\alpha) \odot f_g = \odot\,(g^{-1}(\alpha), f_g)$
3. $(\alpha \cdot f + \lambda \cdot h)_g = (g^{-1}(\alpha) \odot f_g) \oplus (g^{-1}(\lambda) \odot h_g) = (\alpha \cdot f)_g \oplus (\lambda \cdot h)_g$
4. $[\alpha \cdot (f + h)]_g = g^{-1}(\alpha) \odot (f_g \oplus h_g) = \odot(g^{-1}(\alpha), (f_g \oplus h_g))$
5. $(\alpha \cdot f - \lambda \cdot h)_g = (g^{-1}(\alpha) \odot f_g) \ominus (g^{-1}(\lambda) \odot h_g)$
6. $[\alpha \cdot (f - h)]_g = g^{-1}(\alpha) \odot (f_g \ominus h_g) = \odot(g^{-1}(\alpha), (f_g \ominus h_g))$

7. $(\frac{\alpha\cdot f}{\lambda\cdot h})_g = (g^{-1}(\alpha)\odot f_g)\oslash(g^{-1}(\lambda)\odot h_g)$, $\alpha\neq\lambda\neq 1,\ \lambda\neq 0$

8. $(f\cdot h)_g = f_g\odot h_g = \odot(f_g, h_g)$

9. $(f^n)_g = \odot_{i=1}^n f_g$

10. $(\sum_{i=1}^{n} f)_g = \oplus_{i=1}^n f_g$

11. $(\sum_{i=1}^{n} f_i)_g = \oplus_{i=1}^n (f_i)_g$

For certain values of α,λ $(\alpha=\lambda=1$ or $\alpha=\lambda\neq 1)$ we have again the conditions of Theorem 4, [2]:

$\alpha=\lambda=1$, $(f+h)_g = f_g\oplus h_g = \oplus(f_g,h_g)$

$\alpha=\lambda\neq 0$, $(\frac{f}{h})_g = f_g\oslash h_g$; $(\frac{1}{h})_g = g^{-1}(1)\oslash h_g = 1\oslash h_g$, (if g-normed)

$\alpha=1$, $(1+f)_g = g^{-1}(1)\oplus f_g = 1\oplus f_g$, (if g-normed)

$\alpha=1$, $(1-f)_g = g^{-1}(1)\ominus f_g = 1\ominus f_g$, (if g-normed)

3. Linear and pseudo-linear functional equations

3.1. Functional equations and their parameterized by g-Transform

$$(c\cdot f)_g(x)=\begin{cases} f_g(x), & \text{if } c=1, g-\text{normed},\\ c\cdot f(x) & \text{if } g-\text{id},\\ g^{-1}(c)\odot f_g(x), & \text{for } c-\text{other}\end{cases}$$

$$f_g(x)=1\odot f_g(x)=g^{-1}(1)\odot f_g(x) \overset{(g-TR),g-\text{normed},c=1}{\Leftrightarrow} g^{-1}(c)\odot f_g(x)$$

$$-f_g(x)=(-f)_g(x)=g^{-1}(-1)\odot f_g(x) \overset{(g-TR),g-\text{normed},c=-1}{\Leftrightarrow} g^{-1}(c)\odot f_g(x)$$

Table 1: Parameterized Linear and Pseudo-Linear Functional Equation by g-Transform

Cl	Linear Functional Equation $(\boldsymbol{L.F.Eq.-f})$ and pseudo-linear Functional Equation $(\boldsymbol{L.F.Eq.-f_g})$	Parameterized of $(\boldsymbol{L.F.Eq.-f})$ and $(\boldsymbol{L.F.Eq.-f_g})$ by g-Transform and their solutions
I	$c\cdot f(x+y)=c\cdot f(x)+c\cdot f(y)$	$(L.F.Eq.-c\cdot f)^{((+,+),\cdot,c,a)}$
II	$f(x+y)=f(x)+f(y)$	$(L.F.Eq.-f)^{((+,+),\cdot,1,a)}$
III	$f_g(x\oplus y)=f_g(x)\oplus f_g(y)$	$(PL.F.Eq.-f_g)^{((\oplus,\oplus),\odot,1,g^{-1}(a))}$
IV	$(c\cdot f)_g(x\oplus y)=(c\cdot f)_g(x)\oplus(c\cdot f)_g(y)$	$(PL.F.Eq.-(c\cdot f)_g)^{((\oplus,\oplus),\odot,g^{-1}(c),g^{-1}(a))}$

3.2. SOME APPLICATIONS FOR $f(x)=\log_a x$

$$h_g(x)=(\log_a x^{-1})_g=(\log_a \tfrac{1}{x})_g=(c\cdot f)_g(x)=(-f)_g(x)$$
$$=g^{-1}(\log_a(g(g^{-1}(\tfrac{g(g^{-1}(1))}{g(x)}))))=g^{-1}(\log_a g(1\ominus x))=f_{g-a,\log}(1\ominus x)$$

$$h_g(x)=(\log_a x^c)_g=(c\cdot f)_g(x)=\begin{cases} f_{g-a,\log}(x), & \text{if } c=1, g-\text{normed}\\ f_{g-a,\log}(1\ominus x), & \text{if } c=-1, g-\text{normed}\\ g^{-1}(c)\odot f_{g-a,\log}(x), & \text{for } c-\text{other}\end{cases}$$

$$h_g(x)=(\log_a x^{cx})_g=c\cdot(x\cdot f)_g(x)=\begin{cases} x\odot f_{g-a,\log}(x), & \text{if } c=1, g-\text{normed}\\ x\odot f_{g-a,\log}(1\ominus x), & \text{if } c=-1, g-\text{normed}\\ g^{-1}(c)\odot x\odot f_{g-a,\log}(x), & \text{or } c-\text{other}.\end{cases}$$

3.3. RELATIONS BETWEEN CLASSES OF $(L.F.Eq.)$ AND $(PL.F.Eq.)$ BY g-TRANSFORM AND PARAMETERS

- Relations between *f-solutions* and f_g *-pseudo-solutions* by *g-Transform* ([2]):

$$\begin{pmatrix} f(x)=a\cdot x\\ f(x)=\log_a x\\ f(x)=a^x\\ f(x)=x^r\end{pmatrix} \underset{(g-TR),g(x)=x}{\overset{(g-TR)}{\substack{\Rightarrow\\ \Leftarrow}}} \begin{pmatrix} f_g(x)=g^{-1}(a)\odot x\\ f_{g-a,\log}(x)=g^{-1}(\log_a g(x))\\ f_{g-a,\ \exp}(x)=g^{-1}(a^{g(x)})\\ f_{g-r,power}(x)=g^{-1}((g(x))^r)\end{pmatrix}$$

- Relations between $(L.F.Eq.)$ and $(PL.F.Eq.)$ by g-Transform

$$\begin{pmatrix} c\cdot f(x+y)=c\cdot f(x)+c\cdot f(y)\\ c\cdot f(x\cdot y)=c\cdot f(x)+c\cdot f(y)\\ c\cdot f(x+y)=c\cdot f(x)\cdot f(y)\\ c\cdot f(x\cdot y)=c\cdot f(x)\cdot f(y)\end{pmatrix} \underset{(g-TR),g(x)=x}{\overset{(g-TR)}{\substack{\Rightarrow\\ \Leftarrow}}} \begin{pmatrix} (c\cdot f)_g(x\oplus y)=(c\cdot f)_g(x)\oplus(c\cdot f)_g(y)\\ (c\cdot f)_g(x\odot y)=(c\cdot f)_g(x)\oplus(c\cdot f)_g(y)\\ (c\cdot f)_g(x\oplus y)=g^{-1}(c)\odot(f_g(x)\odot f_g(y))\\ (c\cdot f)_g(x\odot y)=g^{-1}(c)\odot(f_g(x)\odot f_g(y))\end{pmatrix}$$

$$\begin{pmatrix} (c\cdot f)_g(x\oplus y)=(c\cdot f)_g(x)\oplus(c\cdot f)_g(y)\\ (c\cdot f)_g(x\odot y)=(c\cdot f)_g(x)\oplus(c\cdot f)_g(y)\\ (c\cdot f)_g(x\oplus y)=g^{-1}(c)\odot(f_g(x)\odot f_g(y))\\ (c\cdot f)_g(x\odot y)=g^{-1}(c)\odot(f_g(x)\odot f_g(y))\end{pmatrix} \underset{(\odot g^{-1}(c))}{\overset{g-\text{normed},c=1}{\substack{\Rightarrow\\ \Leftarrow}}} \begin{pmatrix} f_g(x\oplus y)=f_g(x)\oplus f_g(y)\\ f_g(x\odot y)=f_g(x)\oplus f_g(y)\\ f_g(x\oplus y)=f_g(x)\odot f_g(y)\\ f_g(x\odot y)=f_g(x)\odot f_g(y)\end{pmatrix}$$

$$\begin{pmatrix} c\cdot f(x+y)=c\cdot f(x)+c\cdot f(y) \\ c\cdot f(x\cdot y)=c\cdot f(x)+c\cdot f(y) \\ c\cdot f(x+y)=c\cdot f(x)\cdot f(y) \\ c\cdot f(x\cdot y)=c\cdot f(x)\cdot f(y) \end{pmatrix} \begin{matrix} (g{-}TR),\ g{-}\text{normed}, c=1 \\ \Rightarrow \\ \Leftarrow \\ (g{-}TR),\ g(x)=x,\ (\cdot c) \end{matrix} \begin{pmatrix} f_g(x\oplus y)=f_g(x)\oplus f_g(y) \\ f_g(x\odot y)=f_g(x)\oplus f_g(y) \\ f_g(x\oplus y)=f_g(x)\odot f_g(y) \\ f_g(x\odot y)=f_g(x)\odot f_g(y) \end{pmatrix}$$

3.4. ABOUT f-SOLUTIONS OF $(L.F.EQ.)$ AND f_g - PSEUDO-SOLUTIONS OF $(PL.F.EQ)$ BY g-TRANSFORM

A wide class of some elementary modified function (f_g) is investegated by showing the important role of sistem of pseudo-arithmetical operations $\{\{\oplus,\odot,\ominus,\oslash\}$ in treating and solving of pseudo-linear problems [2], [6], [12]. Bellow are presented the parameterized Linear and Pseudo-Linear Functional Equation with $f-solution$ and $f_g-solution$ respectively.

Table 2: Parameterized Linear and Pseudo-Linear Functional Equation with f and f_g solutions respectively

Cl	Linear Functional Equations $(L.F.Eq.-f)$ and Pseudo-Linear Functional Equations $(L.F.Eq.-f_g)$ by g-Transform	f-solutions of $(L.F.Eq.)$ and f_g - pseudo-solutions of $(PL.F.Eq.)$ by g-Transform
I	$(L.F.Eq.-c\cdot f)^{((+,+),\cdot,c,a,-)}$	$(c\cdot f)(x)=c\cdot(a\cdot x)=c\cdot f_{a,lin}(x)$
	$(L.F.Eq.-f)^{((+,+),\cdot,1,a,-)}$	$f(x)=a\cdot x=f_{a,lin}(x)$
	$(PL.F.Eq.-f_g)^{((\oplus,\oplus),\odot,1,g^{-1}(a),-)}$	$f_g(x)=g^{-1}(a)\odot x=f_{g-g^{-1}(a),lin}(x)$
	$(PL.F.Eq.-(c\cdot f)_g)^{((\oplus,\oplus),\odot,g^{-1}(c),g^{-1}(a),-)}$	$(c\cdot f)_g(x)=g^{-1}(c)\odot f_{g-g^{-1}(a),lin}(x)$
II	$(L.F.Eq.-c\cdot f_{a,\log})^{((\cdot,+),\cdot,c,a,-)}$	$(c\cdot f)(x)=\log_a x^c=c\cdot\log_a x$
	$(L.F.Eq.-f_{a,\log})^{((\cdot,+),\cdot,1,a,-)}$	$f(x)=\log_a x$
	$(PL.F.Eq.-f_{g-a,\log})^{((\odot,\oplus),\odot,1,a,-)}$	$f_{g-a,\log}(x)=g^{-1}(\log_a g(x))$
	$(PL.F.Eq.-(c\cdot f)_{g-a,\log})^{((\odot,\oplus),\odot,g^{-1}(c),a,-)}$	$(c\cdot f)_{g-a,\log}(x)=g^{-1}(c)\odot f_{g-a,\log}(x)$
III	$(L.F.Eq.-c\cdot f_{a,\ \exp})^{((+,\cdot),\cdot,c,a,-)}$	$(c\cdot f)(x)=c\cdot(a^x)$
	$(L.F.Eq.-f_{a,\ \exp})^{((+,\cdot),\cdot,1,a,-)}$	$f(x)=a^x$
	$(PL.F.Eq.-f_{g-a,\ \exp})^{((\oplus,\odot),\odot,1,a,-)}$	$f_{g-a,\ \exp}(x)=g^{-1}(a^{g(x)})$
	$(PL.F.Eq.-(c\cdot f)_{g-a,\ \exp})^{((\oplus,\odot),\odot,g^{-1}(c),a,-)}$	$(c\cdot f)_{g-a,\ \exp}(x)=g^{-1}(c)\odot f_{g-a,\ \exp}(x)$
IV	$(L.F.Eq.-c\cdot f_{r,power})^{((\cdot,\cdot),\cdot,c,-,r)}$	$(c\cdot f)(x)=c\cdot(x^r)$
	$(L.F.Eq.-f_{r,power})^{((\cdot,\cdot),\cdot,1,-,r)}$	$f(x)=x^r$
	$(PL.F.Eq.-f_{g-r,power})^{((\odot,\odot),\odot,1,-,r)}$	$f_{g-r,power}(x)=g^{-1}((g(x))^r)$
	$(PL.F.Eq.-(c\cdot f)_{g-r,power})^{((\odot,\odot),\odot,g^{-1}(c),-,r)}$	$(c\cdot f)_{g-r,power}(x)=g^{-1}(c)\odot f_{g-r,power}(x)$

More about $f_g(c\cdot x)$; $f_g(c+x)$; $f_g(1-x)$; $f_g(N(x))$; $f_g(a\cdot x+b)$ etc. Implemented to each functions of classes shown above, will be presented further.

4. MODIFICATION OF MEASURE BY *g*-TRANSFORM

4.1. $((\oplus-\mathbf{P})-\boldsymbol{m})$

Let $(X,\mathcal{A},m)$ be a $\oplus$- measure space. Let *m* be a fixed $\oplus$- probability measure. Let *X* be a non-empty set and let $\mathcal{A}$ be a σ–algebra of subsets of X, [5], [8], [9], [13], [14].

Definition 4.1.1. A *set function* $m:\mathcal{A}\to[0,+\infty]$ will be called a $\oplus$- *probability measure* $((\oplus-\mathrm{P})-m)$ if for any sequence $(A_i)_{n\in\mathbb{N}}$ of pairwise disjoints sets from $\mathcal{A}$ holds:

P1. $m(\varnothing)=0$, $m(\mathcal{A})=1$

P2. $m(\bigcup_{i=1}^{\infty}A_i)=\oplus_{i=1}^{\infty}m(A_i)$

So shall write $\oplus_{i=1}^{n}a_i=a_1\oplus a_2\oplus\ldots\oplus a_n$ and $\oplus_{i=1}^{\infty}a_i=\sup_n(\oplus_{i=1}^{n}a_i)$. If $\oplus$ is an idempotent operation ($\oplus-ID$), then disjointness of sets and condition (1) can be omitted.

Definition 4.1.2 ([3]). A finite collection $\mathbf{B}=\{B_1,B_2,\ldots,B_n\}\subset\mathcal{A}$, is said to be a $\oplus-$ *measurable* partition of X iff it satisfies the following conditions:

C1. $B_i\cap B_j=\varnothing$, $i\neq j,\ i,j=1,2,\ldots,n$,

C2. $\oplus_{i=1}^{n}B_i=X$.

Definition 4.1.3. A finite collection $\mathbf{B}_g=\{g(B_1),g(B_2),\ldots,g(B_n)\}\subset\mathcal{A}$, is said to be a $g-\oplus-$measurable partition of *X* iff it satisfies the following conditions:

C1. $g(B_i)\cap g(B_j)=\varnothing$, $i\neq j,\ i,j=1,2,\ldots,n,$

C2. $\oplus_{i=1}^{n}g(B_i)=X$.

$g(B_i\cup B_j)=g(B_i)\cup g(B_j)$ because *g* is a continuous monotone strictly increasing unbounded odd function.

Remark 4.14. If finite collection $\mathbf{B}=\{B_1,B_2,\ldots,B_n\}\subset\mathcal{A}$ is a $\oplus-$ measurable partition then $\oplus_{i=1}^{n}m(B_i)=1$ because $1=m(X)=m(\bigcup_{i=1}^{n}B_i)=\oplus_{i=1}^{n}m(B_i)$.

Remark 4.1.5 ([3]). If finite collection $\mathbf{B}_g = \{g(B_1), g(B_2), \dots, g(B_n)\} \subset \mathcal{A}$ is a $g - \oplus -$ measurable partition then $\oplus_{i=1}^{n} m(g(B_i)) = 1$ because:

$$1 = m(X) = m(\bigcup_{i=1}^{n} g(B_i)) = \oplus_{i=1}^{n} m(g(B_i)).$$

Example 4.1.6. For function $f(m(A)) = m(A) \cdot \log_a(m(A))$ compute f_g.

$$f_g(m(A)) = g^{-1}(g(m(A)) \cdot \log_a(g(m(A)))) = g^{-1}(g(m(A)) \cdot g(g^{-1}(\log_a(g(m(A))))))$$
$$= g^{-1}(g(m(A)) \cdot g(f_{g-a,\log}(m(A)))) = m(A) \odot f_{g-a,\log}(m(A))$$
$$f_g(m(A)) = g^{-1}(g(m(A)) \cdot \log_a(g(m(A)))) = g^{-1}(f(g(m(A))))$$
$$= g^{-1}(f(P(A))) = g^{-1}((P(A) \cdot \log_a(P(A)))$$

So:

- $f(P(A)) = P(A) \cdot \log_a(P(A)) = g(f_g(m(A))) = g(m(A) \odot f_{g-a,\log}(m(A)))$
- $f_g(m(A)) = g^{-1}(P(A) \cdot \log_a(P(A))) = m(A) \odot f_{g-a,\log}(m(A))$
- $g(f_g(m(A))) = g(m(A) \odot f_{g-a,\log}(m(A))) = P(A) \cdot \log_a(P(A))$

4.2. RELATIONS BETWEEN *ENTROPY* AND g-*ENTROPY* BY g-*FUNCTION*

Following the example above and the definition of entropy immediately established relations between *entropy* and *g-entropy* by *g-Transform* [3], [8], [11], [15], [18], [19].

Definition 4.2.1 ([3]). Let $\boldsymbol{B} = \{B_1, B_2, \dots, B_n\} \subset \mathcal{A}$ is a $\oplus -$ measurable partition of ***X***. Then *g*-entropy is defined by

$$H_{a,m}^{(\oplus,\odot)}(B) = -\oplus_{i=1}^{n} h_g(m(B_i))$$

where

$$h_g(m(B)) = \begin{cases} 0, & \text{if } m(B_i) = 0 \\ m(B_i) \odot f_{g-a,\log}(m(B_i)) & \text{if } m(B_i) \neq 0 \end{cases}$$

and

$$f_{g-a,\log}(m(B_i)) = g^{-1}(\log_a(g(m(B_i))))$$

is the *g-logarithmic* function.

Theorem 4.2.2 ([3], [4], [13], [16]). Let a $(\oplus - \mathrm{P}) -$ decomposable measure m on the measurable space $(\boldsymbol{X}, \mathcal{A})$ be of type (NSA). Then there exist such an induced probability measure P on $\mathcal{A}$ that $m = g^{-1} \circ P$ where g is the normalized additive generator of $\oplus = \oplus_S$ $(\oplus_S - \mathrm{t} - \mathrm{conorm})$ and

$$H_{a,m}^{(\oplus,\odot)}(B) = g^{-1}(H_{a,p}^{(+,\cdot)}(B))$$

for every $\oplus -$ *measurable* partition $\boldsymbol{B}$.

The quantity $H_{a,p}^{(+,\cdot)}(\boldsymbol{B})$ is an entropy of the partition $\boldsymbol{B}$ on the probability space $(\boldsymbol{X}, \mathcal{A}, P)$**,** i.e.

$$H_{a,p}^{(+,\cdot)}(B) = -\sum_{i=1}^{n} h(P(B_i)),$$

Where

$$h(P(B)) = \begin{cases} 0, & \text{if } \ P(B_i) = 0 \\ P(B_i) \cdot \log_a P(B_i) & \text{if } \ P(B_i) \neq 0 \end{cases}$$

Proof. We have

$$\begin{aligned}
H_{a,p}^{(+,\cdot)} &= -\sum_{i=1}^{n} P(B_i) \cdot \log_a(P(B_i)) = -\sum_{i=1}^{n} h(P(B_i)) = -\sum_{i=1}^{n} g(h_g(m(B_i))) \\
&= -\sum_{i=1}^{n} (g \circ m)(B_i) \cdot \log_a((g \circ m)(B_i)) \\
&= -\sum_{i=1}^{n} g(m(B_i)) \cdot g(g^{-1}(\log_a(g(m(AB_i))))) \\
&= -\sum_{i=1}^{n} g(m(B_i)) \cdot g(f_{g-a,\log}(m(B_i))) \\
&= -\sum_{i=1}^{n} g(g^{-1}(g(m(B_i)) \cdot g(f_{g-a,\log}(m(B_i))))) \\
&= -\sum_{i=1}^{n} g(m(B_i) \odot f_{g-a,\log}(m(B_i))) \\
&= -g(\oplus_{i=1}^{n} m(B_i) \odot f_{g-a,\log}(m(B_i))) \\
&= g(-\oplus_{i=1}^{n} m(B_i) \odot f_{g-a,\log}(m(B_i))) = g(H_{a,m}^{(\oplus,\odot)}(B)).
\end{aligned}$$

The rest of theorem is proving is the text [3].

- $H_{a,m}^{(\oplus,\odot)}(B) = g^{-1}(H_{a,p}^{(+,\cdot)}(B))$ or $g(H_{a,m}^{(\oplus,\odot)}(B)) = H_{a,p}^{(+,\cdot)}(B)$
- $H_{a,m}^{(\oplus,\odot)}(A) = -\oplus_{i=1}^{n} m(A_i) \odot f_{g-a,\log}(m(A_i)) = g^{-1}(H_{a,p}^{(+,\cdot)}(A))$
- $H_{2,m}^{(\oplus,\odot)}(A) = -\oplus_{i=1}^{n} m(A_i) \odot f_{g-2,log}(m(A_i)) = g^{-1}(H_{2,p}^{(+,\cdot)}(A))$
  $= SH_{m}^{(\oplus,\odot)}(A) = SH_{p}^{(+,\cdot)}(A)$
- $SH_{m}^{(\oplus,\odot)}(A) = H_{2,m}^{(\oplus,\odot)}(A) = g^{-1}(H_{2,p}^{(+,\cdot)}(A)) = g^{-1}(SH_{p}^{((+,\cdot))}(A))$

Example 4.2.3 ([11], [18], [19], [20]). Change of base for logarithmic function ($a \to b$):

$$\log_b P(B) = \log_b a \cdot \log_a P(B)$$

$$\log_b g(m(B)) = (\log_b a)\cdot(\log_a g(m(B)))$$
$$\log_b g(m(B)) = (g(g^{-1}(\log_b a)))\cdot(g(g^{-1}(\log_a g(m(B))))$$
$$g^{-1}(\log_b g(m(B))) = g^{-1}((g(g^{-1}(\log_b a)))\cdot(g(g^{-1}(\log_a g(m(B)))))$$

- $f_{g-b,\log}(m(B)) = g^{-1}(\log_b a)\odot f_{g-a,\log}(m(B))$

Rules for crossings in different entropy during the change of bases ($a\to b$):

- $H_{b,m}^{(\oplus,\odot)}(B) = -g^{-1}(\log_b a)\odot(\oplus_{i=1}^{n} m(B_i)\odot f_{g-a,\log}(m(B_i)))$
  $= -g^{-1}(\log_b a)\odot g^{-1}(H_{a,p}^{(+,\cdot)}(B))$
- $H_{b,m}^{(\oplus,\odot)}(B) = -g^{-1}(\log_b a)\odot(\oplus_{i=1}^{n} m(B_i)\odot f_{g-a,\log}(m(B_i)))$
  $= -g^{-1}(\log_b a)\odot(H_{a,m}^{(\oplus,\odot)}(B))$
- $SH_{m}^{(\oplus,\odot)}(B) = -g^{-1}(\log_b 2)\odot(H_{2,m}^{(\oplus,\odot)}(B))$ (Shannon entropy)
- $H_{b,m}^{(\oplus,\odot)}(B) = -g^{-1}(\log_b a)\odot(H_{a,m}^{(\oplus,\odot)}(B)) = -g^{-1}(\log_b 2)\odot(H_{2,m}^{(\oplus,\odot)}(B))$
- $H_{b,m}^{(\oplus,\odot)}(B) = -g^{-1}(log_b 2)\odot SH_{m}^{(\oplus,\odot)}(B)$.

4.3. MODIFIED $\oplus-$ MEASURE BY $g-$TRANSFORM

For $(\oplus-\mathrm{P})-m$ by definition 4.1.1, based on the definition 2.2 for g-function ([2], [3], [5], [14], [16]) and on the sistem of the pseudo-arithmetical operations generated by generator $g,$ can modified the measure by g-Transform in the following form:

Definition 4.3.1. Let m be a set function $(\oplus-\mathrm{P})-m:\mathcal{A}\to[0,+\infty]$ and the function g be a generator of the consistent system of pseudo-arithmetical operations $\{\oplus,\odot,\ominus,\oslash\}$. The function m_g given by $m_g(B) = g^{-1}(m(g(B)))$, for every $B\in\{g^{-1}(B_1),\ldots,\ g^{-1}(B_n)\}$ is said to be $g-(\oplus-\mathrm{P})$ *measure* function (m_g) corresponding to the set function *m.*

Definition 4.3.2 Let P be a induced probability measure $(P = g\circ m)$ and the function g be a generator of the consistent system of pseudo-arithmetical operations $\{\oplus,\odot,\ominus,\oslash\}$. The function P_g given by $P_g(B) = g^{-1}(P(g(B)))$, for every $B\in$ $\{g^{-1}(B_1),\ldots,$ $g^{-1}(B_n)\}$ is said to be *g-induced probability measure* function (P_g) corresponding to the set function P $(P = g\circ m)$.

By *g-calculus* and definition of $(\oplus-\mathrm{P})-m$ hold:

$$m(A)\oplus m(B) = g^{-1}(g(m(A))+g(m(B))).$$

If apply for finite collection $\mathrm{B}_g \subset \mathcal{A}$ ($g-\oplus-$ *measurable* partition with conditions of definition 4.1.3) the definition of $(\oplus-\mathrm{P})-m$ its hold:

$$m(g(A))\oplus m(g(B))=m(g(A\cup B)) \text{ or } m(g(A))\oplus m(g(B))=m(g(A)\cup g(B)) .$$

Proposition 4.3.3. For finite collections B and $\mathrm{B}_g \subset \mathcal{A}$ with conditions of definition 4.1.2 and 4.1.3 respectively, $(\oplus-\mathrm{P})-m$ and induced probability measure P on $\mathcal{A}$ satisfy the following conditions by *g*-Transform:

1. $m_g(B)=g^{-1}(P_g(B))$
2. $P_g(A\cup B)=P_g(A)\oplus P_g(B)$
3. $g(m_g(A\cup B))=g(m_g(A))\oplus g(m_g(B))$

Easily can prove the truth of these equations by applied *g*-transform.

1. $$P_g(B)=g^{-1}(P(g(B)))=g^{-1}(g(m(g(B))))=g(g^{-1}(m(g(B)))) \\ =m(g(B))=g(m_g(B))$$

- $m_g(B)=g^{-1}(P_g(B))$

2. For induced probability measure P, to the equation

$$P(A\cup B)=P(A)+P(B)$$

we get *g*-Transform both sides:

$$g^{-1}(P(g(A\cup B)))=g^{-1}(P(g(A))+ P(g(B)))$$

$$g^{-1}(P(g(A\cup B)))=g^{-1}(g(g^{-1}(P(g(A))))+g(g^{-1}(P(g(B)))))$$

$$g^{-1}(P(g(A\cup B)))=g^{-1}(P(g(A)))\oplus g^{-1}(P(g(B)))$$

- $P_g(A\cup B)=P_g(A)\oplus P_g(B)$
- $P_g(A\cup B)=g^{-1}(P(g(A\cup B)))=g^{-1}(P(g(A)\cup g(B)))$

3. By the definition 4.1.1 to the equation

$$m(A\cup B)=m(A)\oplus m(B)$$

we get *g*-Transform both sides:

$$m_g(A\cup B)=g^{-1}(m(g(A\cup B)))=g^{-1}(m(g(A)\cup g(B)))$$

$$m_g(A\cup B)=g^{-1}(g(m_g(A))\oplus g(m_g(B)))$$

- $g(m_g(A\cup B))=g(m_g(A))\oplus g(m_g(B))$
- $m_g(A\cup B)=g^{-1}(g(m_g(A)))\oplus g^{-1}(g(m_g(B)))$

CONCLUSION

1. Relations between classes of $(L.F.Eq.)$ and $(PL.F.Eq.)$ by *g*-Transform

$$(L.F.Eq.-c\cdot f)^{((+,\cdot),\cdot,c,a,r)} \underset{(g-TR),g(x)=x}{\overset{(g-TR)}{\substack{\Rightarrow\\ \Leftarrow}}} (PL.F.Eq.-(c\cdot f)_g)^{((\oplus,\odot),\odot,g^{-1}(c),a,r)}$$

$$(PL.F.Eq.-(c\cdot f)_g)^{((\oplus,\odot),\odot,g^{-1}(c),a,r)} \underset{(\odot g^{-1}(c))}{\overset{g-\text{normed},c=1}{\substack{\Rightarrow\\ \Leftarrow}}} (PL.F.Eq.-f_g)^{((\oplus,\odot),\odot,1,a,r)}$$

$$(L.F.Eq.-c\cdot f)^{((+,\cdot),\cdot,c,a,r)} \underset{(g-TR),g(x)=x,\,(\cdot c)}{\overset{(g-TR),\,g-\text{normed},c=1}{\substack{\Rightarrow\\ \Leftarrow}}} (PL.F.Eq.-f_g)^{((\oplus,\odot),\odot,1,a,r)}$$

- Implemented to each functions presented above the cases for $f_g(c\cdot x)$, $f_g(c+x)$, $f_g(1-x)$, $f_g(N(x))$, $f_g(a\cdot x+b)$ etc. it will be expanded more the classes of functional equations by f_g – solutions.

2. Modified $\oplus-$ probability measure (m_g) and induced probability measure (P_g) by *g*-Transform

- $m_g(B)=g^{-1}(m(g(B)))$ and $P_g(B)=g^{-1}(P(g(B)))$
- $m_g(B)=g^{-1}(P_g(B))$.

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