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$(m+k,m)-$BANDS

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Abstract. In this paper $p-$zero $(m+k,m)-$semigroups are defined and it is proved that there are exactly $(m+1)$ $p-$zero semigroups. An $(m+k,m)-$semigroup $(Q;[\,])$ which is a direct product of all $(m+1)$ $p-$zero $(m+k,m)-$semigroups, defined previously, is called an $(m+k,m)-$band. Characterizations of $(m+k,m)-$bands are given.

1. p-ZERO $(m+k,m)-$SEMIGROUPS

First, we will introduce some notations which will be used further on:

1) The elements of Q^s, where Q^s denotes the s-th Cartesian power of Q, will be denoted by x_1^s.

2) The symbol x_i^j will denote the sequence $x_i, x_{i+1}, \ldots, x_j$ when $i \leq j$, and the empty sequence when $i > j$.

3) If $x_1 = x_2 = \cdots = x_s = x$, then x_1^s is denoted by the symbol $\overset{s}{x}$.

4) The set $\{1, 2, \ldots, s\}$ will be denoted by $\mathbb{N}_s$.

Let $Q \neq \emptyset$ and n, m are positive integers. If $[\,]$ is a map from Q^n into Q^m, then $[\,]$ is called an $(n,m)-$operation. A pair $(Q;[\,])$ where $[\,]$ is an $(n,m)-$operation is said to be an (n,m) groupoid. Every $(n,m)-$operation on Q induces a sequence $[\,]_1, [\,]_2, \ldots, [\,]_m$ of $n-$ary operations on the set Q, such that

$$((\forall i \in \mathbb{N}_m)\ \ [x_1^n]_i = y_i) \Leftrightarrow [x_1^n] = y_1^m.$$

Let $m \geq 2$, $k \geq 1$. An $(m+k,m)-$groupoid $(Q;[\,])$ is called an $(m+k,m)-$semigroup if for each $i \in \{0, 1, 2, \ldots, k\}$

$$\left[x_1^i \left[x_{i+1}^{i+m+k}\right] x_{i+m+k+1}^{m+2k}\right] = \left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right]$$

Definition 1.1. *An $(m+k,m)-$groupoid $(Q;[\ \ ])$ is said to be a projection $(m+k,m)-$groupoid if there are $1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_m \leq m+k$, such that*

$$[x_1^{m+k}] = x_{\alpha_1} x_{\alpha_2} \ldots x_{\alpha_m},$$

for any $x_1^{m+k} \in Q^{m+k}$.

Definition 1.2. *Let $0 \leq p \leq m$. An $(m+k,m)-$groupoid $(Q;[\,])$ is said to be a $p-$zero $(m+k,m)-$groupoid if $[x_1^{m+k}] = x_1^p x_{p+k+1}^{m+k}$, for any $x_1^{m+k} \in Q^{m+k}$.*

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The operation $[\,]$ for $p-$zero $(m+k,m)-$groupoid will be denoted by $[\,]^p$. In [1] $m-$zero $(m+k,m)-$groupoid is called left zero $(m+k,m)-$groupoid, and $0-$zero $(m+k,m)-$groupoid is called right zero $(m+k,m)-$groupoid. Left and right zero $(m+k,m)-$groupoids are examples of $(m+k,m)-$semigroups.

Proposition 1.3. *Any $p-$zero $(m+k,m)-$groupoid $(Q;[\,]^p)$ is an $(m+k,m)-$semigroup.*

Proof. Let $(Q;[\,]^p)$, $0 \leq p \leq m$ be a $p-$zero $(m+k,m)-$groupoid. Then:

$$\left[x_1^i \left[x_{i+1}^{i+m+k}\right]^p x_{i+m+k+1}^{m+2k}\right]^p = \left[x_1^i x_{i+1}^{i+p} x_{i+p+k+1}^{i+m+k} x_{i+m+k+1}^{m+2k}\right]^p =$$

$$= x_1^p x_{m+2k-(m-p)+1}^{m+2k} = x_1^p x_{p+2k+1}^{m+2k} = \left[x_1^p x_{p+k+1}^{m+k} x_{m+k+1}^{m+2k}\right]^p =$$

$$= \left[\left[x_1^{m+k}\right]^p x_{m+k+1}^{m+2k}\right]^p. \qquad \square$$

Remark 1.4. If $i \in \mathbb{N}_m$ is fixed, then either $[x_1^{m+k}]_i^p = x_i$ or $[x_1^{m+k}]_i^p = x_{i+k}$ holds in $p-$zero $(m+k,m)-$semigroup $(Q;[\,]^p)$.

Proposition 1.5. *If $(Q;[\,])$ is a projection $(m+k,m)-$groupoid which is also an $(m+k,m)-$semigroup, then $(Q;[\,])$ is a $p-$zero $(m+k,m)-$semigroup, for some $0 \leq p \leq m$.*

Proof. Let $\left[x_1^{m+k}\right]_i = x_j$. Then $i \leq j$ and $m-i \leq m+k-j$, i.e. $i \leq j \leq k+i$. It follows $\left[x_1^{m+k}\right]_i = x_{i+q}$, where $0 \leq q \leq k$ and

$$\left[x_1^q \left[x_{q+1}^{q+m+k}\right] x_{q+m+k+1}^{m+2k}\right]_i = \left[x_{q+1}^{q+m+k}\right]_i = x_{q+i+q} = x_{i+2q}.$$

We will consider two cases: A) $m \leq k$ and B) $m > k$.

A) Let $m \leq k$.

A1. Let $i+q > m$, i.e. let $i+q = m+t$, where $t > 0$. Then $\left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right]_i = x_{m+k+t} = x_{i+q+k}$. Since $(Q;[\,])$ is an $(m+k,m)-$semigroup, $\left[x_1^q \left[x_{q+1}^{q+m+k}\right] x_{q+m+k+1}^{m+2k}\right]_i = \left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right]_i$ hold in $(Q;[\,])$. Then $x_{i+2q} = x_{i+q+k}$. So, $i+2q = i+q+k$, i.e. $q = k$. Finally, $\left[x_1^{m+k}\right]_i = x_{i+k}$.

A2. Let $i+q \leq m \leq k$. We have $\left[x_1^k \left[x_{k+1}^{m+2k}\right]\right]_i = x_{i+q}$.
$\left[x_1^q \left[x_{q+1}^{q+m+k}\right] x_{q+m+k+1}^{m+2k}\right]_i = \left[x_1^k \left[x_{k+1}^{m+2k}\right]\right]_i$ implies that $x_{i+2q} = x_{i+q}$.
So, $i+2q = i+q$, i.e. $q = 0$. Finally, $\left[x_1^{m+k}\right]_i = x_i$.

B) Let $m > k$.

B1. Let $i+q > m$ i.e. $i+q = m+t$, where $t > 0$. We have $\left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right]_i = x_{m+k+t} = x_{i+q+k}$. Then $x_{i+2q} = x_{i+q+k}$. So, $i+2q = i+q+k$ i.e. $q = k$. Finally, $\left[x_1^{m+k}\right]_i = x_{i+k}$.

B2. Let $i+q \leq m$.

B2.1. If $i+q \leq k$, then $\left[x_1^k \left[x_{k+1}^{m+2k}\right]\right]_i = x_{i+q}$. So, $i+2q = i+q$ i.e. $q = 0$. Finally, $\left[x_1^{m+k}\right]_i = x_i$.

B2.2. Let $k < i+q \leq m$. Then, $\left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right]_i = \left[x_1^{m+k}\right]_{i+q}$ implies that $\left[x_1^{m+k}\right]_{i+q} = \left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right]_i = \left[x_1^q \left[x_{q+1}^{q+m+k}\right] x_{q+m+k+1}^{m+2k}\right]_i = x_{i+2q}$.

Let j be such that $i+(j-1)q \le m$ and $i+jq > m$. Let $i+jq = m+t'$, where $t' > 0$. Then, $[x_1^{m+k}]_{i+(j-1)q} = x_{i+jq}$, $[[x_1^{m+k}]\,x_{m+k+1}^{m+2k}]_{i+(j-1)q} = x_{m+k+t'} = x_{i+jq+k}$ and $\left[x_1^q\left[x_{q+1}^{q+m+k}\right]x_{q+m+k+1}^{m+2k}\right]_{i+(j-1)q} = \left[x_{q+1}^{q+m+k}\right]_{i+(j-1)q} = x_{q+i+jq} = x_{i+(j+1)q}$.
Thus, $i+(j+1)q = i+jq+k$ i.e. $q=k$ and $[x_1^{m+k}]_i = x_{i+k}$. □

Propositions 1.3 and 1.5 imply that there are exactly $m+1$ projection $(m+k,m)-$semigroups.

Remark 1.6. If $(Q;[\,])$ is a projection $(m+1,m)-$groupoid then from the definition it follows that $(Q;[\,])$ is an $p-$zero $(m+1,m)-$groupoid and so it is $(m+1,m)-$semigroup. The following example shows that, in general, projection $(m+k,m)-$groupoid need not be an $(m+k,m)-$semigroup. The $(4,2)-$groupoid $(Q;[\,])$ where $[\,]$ is defined by $[x_1^4] = x_2^3$, is a projection $(4,2)-$groupoid, but not a $(4,2)-$semigroup.

2. $(m+k,m)-$BANDS

Let $(A_i;\, [\,]^i)$, $i = 1, 2, ..., t$ be $(m+k,m)-$semigroups. Their direct product is an $(m+k,m)-$semigroup, where the $(m+k,m)-$operation $[\,]$ is defined by

$$[x_1^{m+k}] = y_1^m \Leftrightarrow x_i = (x_{i,1},\, x_{i,2}, ..., x_{i,t}),\ y_j = (y_{j,1}, y_{j,2}, ..., y_{j,t}),$$
$$y_{j,r} = [x_{1,j}x_{2,j}...x_{m+k,j}]^r\,,\ i \in \mathbb{N}_{m+k}, j \in \mathbb{N}_m, r \in \mathbb{N}_t.$$

Definition 2.1. *Let* $\mathbf{A}_p = (A_p;\,[\,]^p)$ *be p-zero* $(m+k,m)-$*semigroups,* $0 \le p \le m$. *The direct product of* $A_m, A_{m-1}, \dots, A_0$ *is called* $(m+k,m)-$*band.*

If $(A_m \times A_{m-1} \times ... \times A_0;\ [\,])$ is an $(m+k,m)-$band then its $(m+k,m)-$operation $[\,]$ is of the form

$$[x_1^{m+k}] = y_1^m \quad \Leftrightarrow \quad x_i = (x_{i,1}, x_{i,2}, ..., x_{i,m+1}),$$
$$y_j = (x_{j,1}, x_{j,2}, ..., x_{j,m+1-j}, x_{j+k,m+2-j}, ..., x_{j+k,m+1}), i \in \mathbb{N}_{m+k}, j \in \mathbb{N}_m.$$

Proposition 2.2. *An* $(m+k,m)-$*semigroup* $\mathbf{Q} = (Q,[\,])$ *is an* $(m+k,m)-$*band if and only if the following conditions are satisfied in* $\mathbf{Q}$*:*

(I) $[x_1^{m+k}]_i = [y_1^{i-1}x_iy_{i+1}^{i+k-1}x_{i+k}y_{i+k+1}^{m+k}]_i$ $i \in \mathbb{N}_m$;

(II) $$\left[\overset{j-1}{a}\left[\overset{i-1}{a}\,x\,\overset{k-1}{a}\,y\,\overset{m-i}{a}\right]_i\overset{k-1}{a}\,z\,\overset{m-j}{a}\right]_j = \left[\overset{i-1}{a}\,x\,\overset{k-1}{a}\left[\overset{j-1}{a}\,y\,\overset{k-1}{a}\,z\,\overset{m-j}{a}\right]_j\overset{m-i}{a}\right]_i,$$

for a fixed element of Q *and* $j \le i$;

(III) $$\left[\overset{i-1}{a}\left[\overset{j-1}{a}\,x\,\overset{k-1}{a}\,y\,\overset{m-j}{a}\right]_j\overset{k-1}{a}\,z\,\overset{m-i}{a}\right]_i = \left[\overset{i-1}{a}\,x\,\overset{k-1}{a}\,z\,\overset{m-i}{a}\right]_i,$$ *for a fixed element of* Q *and* $j \le i$;

(IV) $$\left[\overset{j-1}{a}\,x\,\overset{k-1}{a}\left[\overset{i-1}{a}\,y\,\overset{k-1}{a}\,z\,\overset{m-i}{a}\right]_i\overset{m-j}{a}\right]_j = \left[\overset{j-1}{a}\,x\,\overset{k-1}{a}\,z\,\overset{m-j}{a}\right]_j,$$ *for a fixed element of* Q *and* $j \le i$;

(V) $\left[\overset{m+k}{x}\right] = \overset{m}{x}$.

Proof. Let $\mathbf{Q}$ be an $(m+k,m)-$band. Then directly from the Definition 2.1 it follows that $\mathbf{Q}$ satisfies (I), (II), (III), (IV) and (V).

Conversely, suppose that the $(m+k,m)-$semigroup $\mathbf{Q}=(Q;[\;])$, satisfies (I), (II), (III), (IV) and (V), and a is a fixed element of Q.

(A) Let $A_m = \left\{ \left[\overset{m-1}{a}\, x\, \overset{k}{a}\right]_m \,\middle|\, x \in Q \right\}$ and let $\left[\overset{m-1}{a}\, x_i\, \overset{k}{a}\right]_m \in A_m$, $i \in \mathbb{N}_{m+k}$. Then:

$$\left[\left[\overset{m-1}{a}\, x_1\, \overset{k}{a}\right]_m \dots \left[\overset{m-1}{a}\, x_{m+k}\, \overset{k}{a}\right]_m\right]_i \overset{(I)}{=} \left[\overset{i-1}{a}\left[\overset{m-1}{a}\, x_i\, \overset{k}{a}\right]_m \overset{k-1}{a}\left[\overset{m-1}{a}\, x_{i+k}\, \overset{k}{a}\right]_m \overset{m-i}{a}\right]_i \overset{(IV)}{=}$$
$$= \left[\overset{i-1}{a}\left[\overset{m-1}{a}\, x_i\, \overset{k}{a}\right]_m \overset{k-1}{a}\, a\, \overset{m-i}{a}\right]_i \overset{(II)}{=} \left[\overset{m-1}{a}\, x_i\, \overset{k-1}{a}\left[\overset{i-1}{a}\, \overset{k-1}{a}\, a\, a\, \overset{m-i}{a}\right]_i\right]_m \overset{(V)}{=}$$
$$= \left[\overset{m-1}{a}\, x_i\, \overset{k-1}{a}\, a\right]_m = \left[\overset{m-1}{a}\, x_i\, \overset{k}{a}\right]_m .$$

Because $\left[\left[\overset{m-1}{a}\, x_1\, \overset{k}{a}\right]_m \dots \left[\overset{m-1}{a}\, x_{m+k}\, \overset{k}{a}\right]_m\right] = \left[\overset{m-1}{a}\, x_1\, \overset{k}{a}\right]_m \dots \left[\overset{m-1}{a}\, x_m\, \overset{k}{a}\right]_m$, $(A_m;[\;])$ is a left zero $(m+k,m)-$semigroup, i.e. an $m-$zero $(m+k,m)-$semigroup.

(B) Let $A_0 = \left\{ \left[\overset{k}{a}\, x\, \overset{m-1}{a}\right]_1 \,\middle|\, x \in Q \right\}$ and $\left[\overset{k}{a}\, x_i\, \overset{m-1}{a}\right]_1 \in A_0$, $i \in \mathbb{N}_{m+k}$. Then:

$$\left[\left[\overset{k}{a}\, x_1\, \overset{m-1}{a}\right]_1 \dots \left[\overset{k}{a}\, x_{m+k}\, \overset{m-1}{a}\right]_1\right]_i \overset{(I)}{=} \left[\overset{i-1}{a}\left[\overset{k}{a}\, x_i\, \overset{m-1}{a}\right]_1 \overset{k-1}{a}\left[\overset{k}{a}\, x_{i+k}\, \overset{m-1}{a}\right]_1 \overset{m-i}{a}\right]_i \overset{(III)}{=}$$
$$= \left[\overset{i-1}{a}\, \overset{k-1}{a}\, a \left[\overset{k}{a}\, x_{i+k}\, \overset{m-1}{a}\right]_1 \overset{m-i}{a}\right]_i \overset{(II)}{=} \left[\left[\overset{i-1}{a}\, \overset{k-1}{a}\, a\, a\, \overset{m-i}{a}\right]_i \overset{k-1}{a}\, x_{i+k}\, \overset{m-1}{a}\right]_1 \overset{(V)}{=}$$
$$= \left[a\, \overset{k-1}{a}\, x_{i+k}\, \overset{m-1}{a}\right]_1 = \left[\overset{k}{a}\, x_{i+k}\, \overset{m-1}{a}\right]_1 .$$

So, $(A_0;[\;])$ is a right zero $(m+k,m)-$semigroup, i.e. a $0-$zero $(m+k,m)-$semigroup.

(C) Let

$$A_p = \left\{ x \,\middle|\, x \in Q,\ x = \left[x\, \overset{m+k-1}{a}\right]_1 = \dots = \left[\overset{p-1}{a}\, x\, \overset{m+k-p}{a}\right]_p = \left[\overset{p+k}{a}\, x\, \overset{m-p-1}{a}\right]_{p+1} = \dots = \right.$$
$$\left. = \left[\overset{m+k-1}{a}\, x\right]_m \right\},\ 1 \le p \le m-1, \text{ and } x_j \in A_p,\ j \in \mathbb{N}_{m+k}.$$

(C1) For $i \le p$ we have

$$\left[x_1^{m+k}\right]_i \overset{(I)}{=} \left[\overset{i-1}{a}\, x_i\, \overset{k-1}{a}\, x_{i+k}\, \overset{m-i}{a}\right]_i = \left[\overset{i-1}{a}\, x_i\, \overset{k-1}{a}\left[\overset{i-1}{a}\, x_{i+k}\, \overset{m+k-i}{a}\right]_i \overset{m-i}{a}\right]_i \overset{(IV)}{=}$$
$$= \left[\overset{i-1}{a}\, x_i\, \overset{k-1}{a}\, a\, \overset{m-i}{a}\right]_i = \left[\overset{i-1}{a}\, x_i\, \overset{m+k-i}{a}\right]_i = x_i.$$

(C2) Let $i > p$. Then:

$$\left[x_1^{m+k}\right]_i \overset{(I)}{=} \left[\overset{i-1}{a}\, x_i \overset{k-1}{a}\, x_{i+k} \overset{m-i}{a}\right]_i = \left[\overset{i-1}{a} \left[\overset{i+k-1}{a}\, x_i \overset{m-i}{a}\right]_i \overset{k-1}{a}\, x_{i+k} \overset{m-i}{a}\right]_i \overset{(III)}{=}$$

$$= \left[\overset{i-1}{a}\, a \overset{k-1}{a}\, x_{i+k} \overset{m-i}{a}\right]_i = \left[\overset{i+k-1}{a}\, x_{i+k} \overset{m-i}{a}\right]_i = x_{i+k}.$$

We have $\left[x_1^{m+k}\right] = x_1^p x_{p+k+1}^{m+k}$. So, $(A_p; [\,])$, $1 \le p \le m-1$, is a p–zero $(m+k,m)$–semigroup.

(D) Let $\left(\left[\overset{m-1}{a}\, x_1 \overset{k}{a}\right]_m, x_2, ..., x_m, \left[\overset{k}{a}\, x_{m+1} \overset{m-1}{a}\right]_1\right) \in A_m \times A_{m-1} \times ... \times A_1 \times A_0$.

Define: $\alpha_0 = x_1$, $\alpha_i = \left[\overset{m-i}{a}\, \alpha_{i-1} \overset{k-1}{a}\, x_{i+1} \overset{i-1}{a}\right]_{m+1-i}$, $i \in \mathbb{N}_m$ and $\beta_0 = x_{m+1}$,

$\beta_i = \left[\overset{i-1}{a}\, x_{m+1-i} \overset{k-1}{a}\, \beta_{i-1} \overset{m-i}{a}\right]_i$ $i \in \mathbb{N}_m$.

Then

$$\alpha_m = \left[\alpha_{m-1} \overset{k-1}{a}\, x_{m+1} \overset{m-1}{a}\right]_1 = \left[\left[a\alpha_{m-2} \overset{k-1}{a}\, x_m \overset{m-2}{a}\right]_2 \overset{k-1}{a}\, x_{m+1} \overset{m-1}{a}\right]_1 \overset{(II)}{=}$$

$$= \left[a\alpha_{m-2} \overset{k-1}{a} \left[x_m \overset{k-1}{a}\, x_{m+1} \overset{m-1}{a}\right]_1 \overset{m-2}{a}\right]_2 = \left[a\alpha_{m-2} \overset{k-1}{a}\, \beta_1 \overset{m-2}{a}\right]_2 \overset{(II)}{=} ... \overset{(II)}{=}$$

$$= \left[\overset{i-1}{a}\, \alpha_{m-i} \overset{k-1}{a}\, \beta_{i-1} \overset{m-i}{a}\right]_i \overset{(II)}{=} ... \overset{(II)}{=} \left[\overset{m-1}{a}\, \alpha_0 \overset{k-1}{a}\, \beta_{m-1}\right]_m =$$

$$= \left[\overset{m-1}{a}\, x_1 \overset{k-1}{a}\, \beta_{m-1}\right]_m = \beta_m.$$

Let $\varphi : A_m \times A_{m-1} \times ... \times A_1 \times A_0 \to Q$ be the map defined by:

$$\varphi\left(\left[\overset{m-1}{a}\, x_1 \overset{k}{a}\right]_m, x_2, ..., x_m, \left[\overset{k}{a}\, x_{m+1} \overset{m-1}{a}\right]_1\right) = \alpha_m,$$

for any element $\left(\left[\overset{m-1}{a}\, x_1 \overset{k}{a}\right]_m, x_2, ..., x_m, \left[\overset{k}{a}\, x_{m+1} \overset{m-1}{a}\right]_1\right) \in A_m \times A_{m-1} \times ... \times$ $\times A_1 \times A_0$.

(D1) Proof that φ is a well-defined map.

Let $\left[\overset{m-1}{a}\, x_1 \overset{k}{a}\right]_m = \left[\overset{m-1}{a}\, u_1 \overset{k}{a}\right]_m$, $x_j = u_j$, $j \in \mathbb{N}_m \setminus \{1\}$, $\left[\overset{k}{a}\, x_{m+1} \overset{m-1}{a}\right]_1 =$ $= \left[\overset{k}{a}\, u_{m+1} \overset{m-1}{a}\right]_1$.

Then, $\left[\overset{m-1}{a}\, x_1 \overset{k}{a}\right]_m = \left[\overset{m-1}{a}\, u_1 \overset{k}{a}\right]_m$ implies

$$\left[\overset{m-1}{a} \left[\overset{m-1}{a}\, x_1 \overset{k}{a}\right]_m \overset{k-1}{a}\, x_1\right]_m = \left[\overset{m-1}{a} \left[\overset{m-1}{a}\, u_1 \overset{k}{a}\right]_m \overset{k-1}{a}\, x_1\right]_m \overset{(III)}{\Rightarrow}$$

$$\left[\overset{m-1}{a}\, x_1 \overset{k-1}{a}\, x_1\right]_m = \left[\overset{m-1}{a}\, u_1 \overset{k-1}{a}\, x_1\right]_m \overset{(I)}{\Rightarrow}$$

$$\left[\overset{m+k}{x_1}\right]_m = \left[\overset{m-1}{a}\, u_1\, \overset{k-1}{a}\, x_1\right]_m \overset{(V)}{\Rightarrow}$$

$$x_1 = \left[\overset{m-1}{a}\, u_1\, \overset{k-1}{a}\, x_1\right]_m .$$

$\left[\overset{k}{a}\, x_{m+1}\, \overset{m-1}{a}\right]_1 = \left[\overset{k}{a}\, u_{m+1}\, \overset{m-1}{a}\right]_1$ implies

$$\left[x_{m+1}\, \overset{k-1}{a} \left[\overset{k}{a}\, x_{m+1}\, \overset{m-1}{a}\right]_1 \overset{m-1}{a}\right]_1 = \left[x_{m+1}\, \overset{k-1}{a} \left[\overset{k}{a}\, u_{m+1}\, \overset{m-1}{a}\right]_1 \overset{m-1}{a}\right]_1 \overset{(IV)}{\Rightarrow}$$

$$\left[x_{m+1}\, \overset{k-1}{a}\, x_{m+1}\, \overset{m-1}{a}\right]_1 = \left[x_{m+1}\, \overset{k-1}{a}\, u_{m+1}\, \overset{m-1}{a}\right]_1 \overset{(I)}{\Rightarrow}$$

$$\left[\overset{m+k}{x_{m+1}}\right]_1 = \left[x_{m+1}\, \overset{k-1}{a}\, u_{m+1}\, \overset{m-1}{a}\right]_1 \overset{(V)}{\Rightarrow}$$

$$x_{m+1} = \left[x_{m+1}\, \overset{k-1}{a}\, u_{m+1}\, \overset{m-1}{a}\right]_1 .$$

$$\alpha_0 = x_1 = \left[\overset{m-1}{a}\, u_1\, \overset{k-1}{a}\, x_1\right]_m ;\ \alpha_0' = u_1$$

$$\alpha_1 = \left[\overset{m-1}{a}\, \alpha_0\, \overset{k-1}{a}\, x_2\right]_m = \left[\overset{m-1}{a} \left[\overset{m-1}{a}\, u_1\, \overset{k-1}{a}\, x_1\right]_m \overset{k-1}{a}\, x_2\right]_m \overset{(III)}{=} \left[\overset{m-1}{a}\, u_1\, \overset{k-1}{a}\, x_2\right]_m =$$

$$= \left[\overset{m-1}{a}\, \alpha_0'\, \overset{k-1}{a}\, u_2\right]_m = \alpha_1'$$

$\vdots$

$$\alpha_i = \left[\overset{m-i}{a}\, \alpha_{i-1}\, \overset{k-1}{a}\, x_{i+1}\, \overset{i-1}{a}\right]_{m+1-i} = \left[\overset{m-i}{a}\, \alpha_{i-1}'\, \overset{k-1}{a}\, u_{i+1}\, \overset{i-1}{a}\right]_{m+1-i} = \alpha_i',$$

$1 \leq i \leq m-1$.

Then:

$$\alpha_m = \left[\alpha_{m-1}\, \overset{k-1}{a}\, x_{m+1}\, \overset{m-1}{a}\right]_1 = \left[\alpha_{m-1}'\, \overset{k-1}{a} \left[x_{m+1}\, \overset{k-1}{a}\, u_{m+1}\, \overset{m-1}{a}\right]_1 \overset{m-1}{a}\right]_1 \overset{(IV)}{=}$$

$$= \left[\alpha_{m-1}'\, \overset{k-1}{a}\, u_{m+1}\, \overset{m-1}{a}\right]_1 = \alpha_m' .$$

So, $\varphi\left(\left[\overset{m-1}{a}\, x_1\, \overset{k}{a}\right]_m, x_2, ..., x_m, \left[\overset{k}{a}\, x_{m+1}\, \overset{m-1}{a}\right]_1\right) =$

$$= \varphi\left(\left[\overset{m-1}{a}\, u_1\, \overset{k}{a}\right]_m, u_2, ..., u_m, \left[\overset{k}{a}\, u_{m+1}\, \overset{m-1}{a}\right]_1\right).$$

(D2) Proof that φ is an injection.

Let

$$\varphi\left(\left[\overset{m-1}{a}\, x_1\, \overset{k}{a}\right]_m, x_2, ..., x_m, \left[\overset{k}{a}\, x_{m+1}\, \overset{m-1}{a}\right]_1\right) =$$

$$= \varphi\left(\left[\overset{m-1}{a}\, u_1\, \overset{k}{a}\right]_m, u_2, \dots, u_m, \left[\overset{k}{a}\, u_{m+1}\, \overset{m-1}{a}\right]_1\right) \text{ i.e. } \alpha_m = \alpha'_m.$$

Since $\alpha_m = \alpha'_m$ it follows

$$\left[x_{m+1}\overset{k-1}{a}\left[\alpha_{m-1}\overset{k-1}{a}\, x_{m+1}\overset{m-1}{a}\right]_1\overset{m-1}{a}\right]_1 = \left[x_{m+1}\overset{k-1}{a}\left[\alpha'_{m-1}\overset{k-1}{a}\, u_{m+1}\overset{m-1}{a}\right]_1\overset{m-1}{a}\right]_1 \overset{(IV)}{\Rightarrow}$$

$$\left[x_{m+1}\overset{k-1}{a}\, x_{m+1}\overset{m-1}{a}\right]_1 = \left[x_{m+1}\overset{k-1}{a}\, u_{m+1}\overset{m-1}{a}\right]_1 \overset{(I)}{\Rightarrow}$$

$$\left[\overset{m+k}{x_{m+1}}\right]_1 = \left[x_{m+1}\overset{k-1}{a}\, u_{m+1}\overset{m-1}{a}\right]_1 \overset{(V)}{\Rightarrow}$$

$$x_{m+1} = \left[x_{m+1}\overset{k-1}{a}\, u_{m+1}\overset{m-1}{a}\right]_1 .$$

Using this:

$$\left[\overset{k}{a}\, x_{m+1}\overset{m-1}{a}\right]_1 = \left[\overset{k}{a}\left[x_{m+1}\overset{k-1}{a}\, u_{m+1}\overset{m-1}{a}\right]_1\overset{m-1}{a}\right]_1 \overset{(IV)}{=} \left[a\overset{k-1}{a}\, u_{m+1}\overset{m-1}{a}\right]_1 = \left[\overset{k}{a}\, u_{m+1}\overset{m-1}{a}\right]_1 .$$

Since $\beta_m = \beta'_m$ it follows

$$\left[\overset{m-1}{a}\left[\overset{m-1}{a}\, x_1\overset{k-1}{a}\,\beta_{m-1}\right]_m\overset{k-1}{a}\, x_1\right]_m = \left[\overset{m-1}{a}\left[\overset{m-1}{a}\, u_1\overset{k-1}{a}\,\beta'_{m-1}\right]_m\overset{k-1}{a}\, x_1\right]_m \overset{(III)}{\Rightarrow}$$

$$\left[\overset{m-1}{a}\, x_1\overset{k-1}{a}\, x_1\right]_m = \left[\overset{m-1}{a}\, u_1\overset{k-1}{a}\, x_1\right]_m \overset{(I)}{\Rightarrow}$$

$$\left[\overset{m+k}{x_1}\right]_m = \left[\overset{m-1}{a}\, u_1\overset{k-1}{a}\, x_1\right]_m \overset{(V)}{\Rightarrow}$$

$$x_1 = \left[\overset{m-1}{a}\, u_1\overset{k-1}{a}\, x_1\right]_m .$$

Using this:

$$\left[\overset{m-1}{a}\, x_1\overset{k}{a}\right]_m = \left[\overset{m-1}{a}\left[\overset{m-1}{a}\, u_1\overset{k-1}{a}\, x_1\right]_m\overset{k}{a}\right]_m \overset{(III)}{=} \left[\overset{m-1}{a}\, u_1\overset{k-1}{a}\, a\right]_m = \left[\overset{m-1}{a}\, u_1\overset{k}{a}\right]_m .$$

Let $2 \le i \le m$. Since

$$\alpha_m = \left[\overset{m-i}{a}\,\alpha_{i-1}\overset{k-1}{a}\,\beta_{m-i}\overset{i-1}{a}\right]_{m+1-i} = \left[\overset{m-i}{a}\,\alpha'_{i-1}\overset{k-1}{a}\,\beta'_{m-i}\overset{i-1}{a}\right]_{m+1-i} = \alpha'_m,$$

we have

$$\left[\overset{m-i}{a}\left[\overset{m-i}{a}\,\alpha_{i-1}\overset{k-1}{a}\,\beta_{m-i}\overset{i-1}{a}\right]_{m+1-i}\overset{k-1}{a}\,\alpha_{i-1}\overset{i-1}{a}\right]_{m+1-i} =$$

$$= \left[\overset{m-i}{a}\left[\overset{m-i}{a}\,\alpha'_{i-1}\overset{k-1}{a}\,\beta'_{m-i}\overset{i-1}{a}\right]_{m+1-i}\overset{k-1}{a}\,\alpha_{i-1}\overset{i-1}{a}\right]_{m+1-i} \overset{(III)}{\Rightarrow}$$

$$\left[\overset{m-i}{a}\,\alpha_{i-1}\overset{k-1}{a}\,\alpha_{i-1}\overset{i-1}{a}\right]_{m+1-i} = \left[\overset{m-i}{a}\,\alpha'_{i-1}\overset{k-1}{a}\,\alpha_{i-1}\overset{i-1}{a}\right]_{m+1-i} \overset{(I)}{\Rightarrow}$$

$$\left[\overset{m+k}{\alpha_{i-1}}\right]_{m+1-i} = \left[\overset{m-i}{a}\,\alpha'_{i-1}\,\overset{k-1}{a}\,\alpha_{i-1}\,\overset{i-1}{a}\right]_{m+1-i} \overset{(V)}{\Rightarrow}$$

$$\alpha_{i-1} = \left[\overset{m-i}{a}\,\alpha'_{i-1}\,\overset{k-1}{a}\left[\overset{m+1-i}{a}\,\alpha_{i-2}\,\overset{k-1}{a}\,x_i\,\overset{i-2}{a}\right]_{m+2-i}\overset{i-1}{a}\right]_{m+1-i} \overset{(IV)}{\Rightarrow}$$

$$\alpha_{i-1} = \left[\overset{m-i}{a}\,\alpha'_{i-1}\,\overset{k-1}{a}\,x_i\,\overset{i-1}{a}\right]_{m+1-i} \Rightarrow$$

$$\left[\overset{m+1-i}{a}\,\alpha_{i-2}\,\overset{k-1}{a}\,x_i\,\overset{i-2}{a}\right]_{m+2-i} = \left[\overset{m-i}{a}\left[\overset{m+1-i}{a}\,\alpha'_{i-2}\,\overset{k-1}{a}\,u_i\,\overset{i-2}{a}\right]_{m+2-i}\overset{k-1}{a}\,x_i\,\overset{i-1}{a}\right]_{m+1-i} \overset{(II)}{\Rightarrow}$$

$$\left[\overset{m+1-i}{a}\,\alpha_{i-2}\,\overset{k-1}{a}\,x_i\,\overset{i-2}{a}\right]_{m+2-i} = \left[\overset{m+1-i}{a}\,\alpha'_{i-2}\,\overset{k-1}{a}\left[\overset{m-i}{a}\,u_i\,\overset{k-1}{a}\,x_i\,\overset{i-1}{a}\right]_{m+1-i}\overset{i-2}{a}\right]_{m+2-i}.$$

Thus, we have

$$\left[\overset{m+1-i}{a}\,x_i\,\overset{k-1}{a}\left[\overset{m+1-i}{a}\,\alpha_{i-2}\,\overset{k-1}{a}\,x_i\,\overset{i-2}{a}\right]_{m+2-i}\overset{i-2}{a}\right]_{m+2-i} =$$

$$= \left[\overset{m+1-i}{a}\,x_i\,\overset{k-1}{a}\left[\overset{m+1-i}{a}\,\alpha'_{i-2}\,\overset{k-1}{a}\left[\overset{m-i}{a}\,u_i\,\overset{k-1}{a}\,x_i\,\overset{i-1}{a}\right]_{m+1-i}\overset{i-2}{a}\right]_{m+2-i}\overset{i-2}{a}\right]_{m+2-i} \overset{(IV)}{\Rightarrow}$$

$$\left[\overset{m+1-i}{a}\,x_i\,\overset{k-1}{a}\,x_i\,\overset{i-2}{a}\right]_{m+2-i} = \left[\overset{m+1-i}{a}\,x_i\,\overset{k-1}{a}\left[\overset{m-i}{a}\,u_i\,\overset{k-1}{a}\,x_i\,\overset{i-1}{a}\right]_{m+1-i}\overset{i-2}{a}\right]_{m+2-i} \overset{(I)}{\Rightarrow}$$

$$\left[\overset{m+k}{x_i}\right]_{m+2-i} = \left[\overset{m+1-i}{a}\,x_i\,\overset{k-1}{a}\left[\overset{m-i}{a}\,u_i\,\overset{k-1}{a}\,x_i\,\overset{i-1}{a}\right]_{m+1-i}\overset{i-2}{a}\right]_{m+2-i} \overset{(V)}{\Rightarrow}$$

$$x_i = \left[\overset{m+1-i}{a}\,x_i\,\overset{k-1}{a}\left[\overset{m-i}{a}\,u_i\,\overset{k-1}{a}\,x_i\,\overset{i-1}{a}\right]_{m+1-i}\overset{i-2}{a}\right]_{m+2-i}.$$

Since $x_i = \left[\overset{m+1-i}{a}\,x_i\,\overset{k-1}{a}\left[\overset{m-i}{a}\,u_i\,\overset{k-1}{a}\,x_i\,\overset{i-1}{a}\right]_{m+1-i}\overset{i-2}{a}\right]_{m+2-i}$ and $x_i, u_i \in A_{m+1-i}$, $2 \leq i \leq m$, we have

$$x_i = \left[\overset{m+1+k-i}{a}\,x_i\,\overset{i-2}{a}\right]_{m+2-i} =$$

$$= \left[\overset{m+1+k-i}{a}\left[\overset{m+1-i}{a}\,x_i\,\overset{k-1}{a}\left[\overset{m-i}{a}\,u_i\,\overset{k-1}{a}\,x_i\,\overset{i-1}{a}\right]_{m+1-i}\overset{i-2}{a}\right]_{m+2-i}\overset{i-2}{a}\right]_{m+2-i} \overset{(IV)}{=}$$

$$= \left[\overset{m+1-i}{a}\,a\,\overset{k-1}{a}\left[\overset{m-i}{a}\,u_i\,\overset{k-1}{a}\,x_i\,\overset{i-1}{a}\right]_{m+1-i}\overset{i-2}{a}\right]_{m+2-i} \overset{(II)}{=}$$

$$= \left[\overset{m-i}{a} \left[\overset{m+1-i}{a}\, a \overset{k-1}{a} u_i \overset{i-2}{a}\right]_{m+2-i} \overset{k-1}{a} x_i \overset{i-1}{a}\right]_{m+1-i} =$$

$$= \left[\overset{m-i}{a} \left[\overset{m+1+k-i}{a} u_i \overset{i-2}{a}\right]_{m+2-i} \overset{k-1}{a} x_i \overset{i-1}{a}\right]_{m+1-i} = \left[\overset{m-i}{a} u_i \overset{k-1}{a} x_i \overset{i-1}{a}\right]_{m+1-i} =$$

$$= \left[\overset{m-i}{a} u_i \overset{k-1}{a} \left[\overset{m-i}{a} x_i \overset{k+i-1}{a}\right]_{m+1-i} \overset{i-1}{a}\right]_{m+1-i} \overset{(IV)}{=} \left[\overset{m-i}{a} u_i \overset{k-1}{a}\, a \overset{i-1}{a}\right]_{m+1-i} =$$

$$= \left[\overset{m-i}{a} u_i \overset{k+i-1}{a}\right]_{m+1-i} = u_i.$$

Therefore,

$$\left(\left[\overset{m-1}{a} x_1 \overset{k}{a}\right]_m, x_2, \dots, x_m, \left[\overset{k}{a} x_{m+1} \overset{m-1}{a}\right]_1\right) = \left(\left[\overset{m-1}{a} u_1 \overset{k}{a}\right]_m, u_2, \dots, u_m, \left[\overset{k}{a} u_{m+1} \overset{m-1}{a}\right]_1\right),$$

i.e. φ is an injection.

(D3) Proof that φ is a surjection.

Let $x \in Q$. Then $\left[\overset{m-1}{a} x \overset{k}{a}\right]_m \in A_m$ and $\left[\overset{k}{a} x \overset{m-1}{a}\right]_1 \in A_0$. We will prove that

$$\left[\overset{i-1}{a} \left[\overset{k+i}{a} x \overset{m-i-1}{a}\right]_{i+1} \overset{m+k-i}{a}\right]_i \in A_i,\ 1 \le i \le m-1.$$

1. If $j \le i$ then

$$\left[\overset{j-1}{a} \left[\overset{i-1}{a} \left[\overset{k+i}{a} x \overset{m-i-1}{a}\right]_{i+1} \overset{m+k-i}{a}\right]_i \overset{m+k-j}{a}\right]_j \overset{(II)}{=} \left[\overset{i-1}{a} \left[\overset{k+i}{a} x \overset{m-i-1}{a}\right]_{i+1} \overset{k-1}{a} \left[\overset{j-1}{a}\, a \overset{k-1}{a} \overset{m-j}{a}\right]_j \overset{m-i}{a}\right]_i \overset{(V)}{=}$$

$$= \left[\overset{i-1}{a} \left[\overset{k+i}{a} x \overset{m-i-1}{a}\right]_{i+1} \overset{k-1}{a}\, a \overset{m-i}{a}\right]_i = \left[\overset{i-1}{a} \left[\overset{k+i}{a} x \overset{m-i-1}{a}\right]_{i+1} \overset{m+k-i}{a}\right]_i.$$

2. If $i < j$ then

$$\left[\overset{k+j-1}{a} \left[\overset{i-1}{a} \left[\overset{k+i}{a} x \overset{m-i-1}{a}\right]_{i+1} \overset{m+k-i}{a}\right]_i \overset{m-j}{a}\right]_j \overset{(II)}{=} \left[\overset{i-1}{a} \left[\overset{j-1}{a}\, a \overset{k-1}{a} \left[\overset{k+i}{a} x \overset{m-i-1}{a}\right]_{i+1} \overset{m-j}{a}\right]_j \overset{k-1}{a}\, a \overset{m-i}{a}\right]_i \overset{(II)}{=}$$

$$= \left[\overset{i-1}{a} \left[\overset{i}{a} \left[\overset{j-1}{a}\, a \overset{k-1}{a}\, a \overset{m-j}{a}\right]_j \overset{k-1}{a} x \overset{m-i-1}{a}\right]_{i+1} \overset{m+k-i}{a}\right]_i \overset{(V)}{=} \left[\overset{i-1}{a} \left[\overset{i}{a}\, a \overset{k-1}{a} x \overset{m-i-1}{a}\right]_{i+1} \overset{m+k-i}{a}\right]_i =$$

$$= \left[\overset{i-1}{a} \left[\overset{k+i}{a} x \overset{m-i-1}{a}\right]_{i+1} \overset{m+k-i}{a}\right]_i.$$

So, $\left[\overset{i-1}{a} \left[\overset{k+i}{a} x \overset{m-i-1}{a}\right]_{i+1} \overset{m+k-i}{a}\right]_i \in A_i,\ 1 \le i \le m-1.$

Let

$$\varphi\left(\left[\overset{m-1}{a} x \overset{k}{a}\right]_m, \left[\overset{m-2}{a} \left[\overset{m+k-1}{a} x\right]_m \overset{k+1}{a}\right]_{m-1}, \dots, \left[\left[\overset{k+1}{a} x \overset{m-2}{a}\right]_2 \overset{m+k-1}{a}\right]_1, \left[\overset{k}{a} x \overset{m-1}{a}\right]_1\right) = \alpha_m.$$

We have

$$\alpha_1 = \left[\overset{m-1}{a}\, x\, \overset{k-1}{a} \left[\overset{m-2}{a} \left[\overset{m+k-1}{a}\, x\right]_m \overset{k+1}{a}\right]_{m-1}\right]_m \overset{(II)}{=}$$

$$= \left[\overset{m-1}{a}\, x\, \overset{k-1}{a} \left[\overset{m-1}{a}\, a\, \overset{k-1}{a} \left[\overset{m-2}{a}\, x\, \overset{k-1}{a}\, aa\right]_{m-1}\right]_m\right]_m \overset{(IV)}{=} \left[\overset{m-1}{a}\, x\, \overset{k-1}{a} \left[\overset{m-2}{a}\, x\, \overset{k-1}{a}\, aa\right]_{m-1}\right]_m \overset{(II)}{=}$$

$$= \left[\overset{m-2}{a} \left[\overset{m-1}{a}\, x\, \overset{k-1}{a}\, x\right]_m \overset{k-1}{a}\, aa\right]_{m-1} \overset{(I)}{=} \left[\overset{m-2}{a} \left[\overset{m+k}{x}\right]_m \overset{k-1}{a}\, aa\right]_{m-1} \overset{(V)}{=} \left[\overset{m-2}{a}\, x\, \overset{k+1}{a}\right]_{m-1};$$

$$\alpha_2 = \left[\overset{m-2}{a}\, \alpha_1\, \overset{k-1}{a} \left[\overset{m-3}{a} \left[\overset{m+k-2}{a}\, xa\right]_{m-1} \overset{k+2}{a}\right]_{m-2} a\right]_{m-1} \overset{(II)}{=}$$

$$= \left[\overset{m-2}{a}\, \alpha_1\, \overset{k-1}{a} \left[\overset{m-2}{a}\, a\, \overset{k-1}{a} \left[\overset{m-3}{a}\, x\, \overset{k-1}{a}\, a\overset{2}{a}\right]_{m-2} a\right]_{m-1} a\right]_{m-1} \overset{(IV)}{=}$$

$$= \left[\overset{m-2}{a}\, \alpha_1\, \overset{k-1}{a} \left[\overset{m-3}{a}\, x\, \overset{k-1}{a}\, a\overset{2}{a}\right]_{m-2} a\right]_{m-1} \overset{(II)}{=} \left[\overset{m-3}{a} \left[\overset{m-2}{a}\, \alpha_1\, \overset{k-1}{a}\, xa\right]_{m-1} \overset{k-1}{a}\, a\overset{2}{a}\right]_{m-2} =$$

$$= \left[\overset{m-3}{a} \left[\overset{m-2}{a} \left[\overset{m-2}{a}\, x\, \overset{k+1}{a}\right]_{m-1} \overset{k-1}{a}\, xa\right]_{m-1} \overset{k-1}{a}\, a\overset{2}{a}\right]_{m-2} \overset{(III)}{=}$$

$$= \left[\overset{m-3}{a} \left[\overset{m-2}{a}\, x\, \overset{k-1}{a}\, xa\right]_{m-1} \overset{k-1}{a}\, a\overset{2}{a}\right]_{m-2} \overset{(I)}{=} \left[\overset{m-3}{a} \left[\overset{m+k}{x}\right]_{m-1} \overset{k-1}{a}\, a\overset{2}{a}\right]_{m-2} \overset{(V)}{=} \left[\overset{m-3}{a}\, x\, \overset{k+2}{a}\right]_{m-}$$

$$\vdots$$

$$\alpha_i = \left[\overset{m-i-1}{a}\, x\, \overset{k+i}{a}\right]_{m-i}, 1 \le i \le m-1.$$

Then,

$$\alpha_m = \left[\alpha_{m-1}\, \overset{k-1}{a}\, x\, \overset{m-1}{a}\right]_1 = \left[\left[x\, \overset{m+k-1}{a}\right]_1 \overset{k-1}{a}\, x\, \overset{m-1}{a}\right]_1 \overset{(III)}{=} \left[x\, \overset{k-1}{a}\, x\, \overset{m-1}{a}\right]_1 \overset{(I)}{=}$$

$$= \left[\overset{m+k}{x}\right]_1 \overset{(V)}{=} x$$

and therefore φ is a surjection.

(D4) Proof that φ is $(m+k, m)-$homomorphism.

Let

$$\gamma_j = \left(\left[\overset{m-1}{a}\, x_{j,1}\, \overset{k}{a}\right]_m, x_{j,2}, \dots, x_{j,m}, \left[\overset{k}{a}\, x_{j,m+1}\, \overset{m-1}{a}\right]_1\right) \in A_m \times \dots \times A_1 \times A_0, j \in \mathbb{N}_{m+k}.$$

Then

$$\varphi\left([\gamma_1^{m+k}]_i\right) =$$

$$= \varphi\left(\left[\overset{m-1}{a}\, x_{i,1}\, \overset{k}{a}\right]_m, x_{i,2}, \dots, x_{i,m+1-i}, x_{i+k,m+2-i}, \dots, \left[\overset{k}{a}\, x_{i+k,m+1}\, \overset{m-1}{a}\right]_1\right) =$$

$$= \left[\overset{i-1}{a}\ \alpha_{m-i}\ \overset{k-1}{a}\ \beta_{i-1}\ \overset{m-i}{a}\right]_i = \alpha_m.$$

$$[\varphi(\gamma_1)...\varphi(\gamma_{m+k})]_i \overset{(I)}{=} \left[\overset{i-1}{a}\ \varphi(\gamma_i)\ \overset{k-1}{a}\ \varphi(\gamma_{i+k})\ \overset{m-i}{a}\right]_i = \left[\overset{i-1}{a}\ \alpha'_m\ \overset{k-1}{a}\ \alpha''_m\ \overset{m-i}{a}\right]_i =$$

$$= \left[\overset{i-1}{a}\left[\overset{i-1}{a}\ \alpha'_{m-i}\ \overset{k-1}{a}\ \beta'_{i-1}\ \overset{m-i}{a}\right]_i \overset{k-1}{a}\left[\overset{i-1}{a}\ \alpha''_{m-i}\ \overset{k-1}{a}\ \beta''_{i-1}\ \overset{m-i}{a}\right]_i \overset{m-i}{a}\right]_i \overset{(III)}{=}$$

$$= \left[\overset{i-1}{a}\ \alpha'_{m-i}\ \overset{k-1}{a}\left[\overset{i-1}{a}\ \alpha''_{m-i}\ \overset{k-1}{a}\ \beta''_{i-1}\ \overset{m-i}{a}\right]_i \overset{m-i}{a}\right]_i \overset{(IV)}{=} \left[\overset{i-1}{a}\ \alpha'_{m-i}\ \overset{k-1}{a}\ \beta''_{i-1}\ \overset{m-i}{a}\right]_i =$$

$$= \left[\overset{i-1}{a}\ \alpha_{m-i}\ \overset{k-1}{a}\ \beta_{i-1}\ \overset{m-i}{a}\right]_i = \alpha_m.$$

Thus, φ is $(m+k,m)$−homomorphism.
Hence, $(A_m \times ... \times A_1 \times A_0; [\ ]) \cong \mathbf{Q}$. □

3. A CHARACTERIZATION OF $(m+k,m)$−BANDS

In the sequel we will give a characterization of $(m+k,m)$−bands using the usual rectangular bands, where a rectangular band is a semigroup $(Q;*)$ that satisfies the following two identities $x*y*z = x*z$ and $x*x = x$, for each x, y, $z \in Q$.

Proposition 3.1. $\mathbf{Q} = (Q;[\ ])$ *is an* $(m+k,m)$−*band if and only if there are rectangular bands* $(Q;*_i)$, $i \in \mathbb{N}_m$, *such that*

$(i)\,(x *_i y) *_j z = x *_i (y *_j z)\,, j \le i;$

$(ii)\,(x *_j y) *_i z = x *_i z, j \le i;$

$(iii) x *_j (y *_i z) = x *_j z, j \le i;$

and $\left[x_1^{m+k}\right]_i = x_i *_i x_{i+k}$, $x_1^{m+k} \in Q^{m+k}$, $i \in \mathbb{N}_m$.

Proof. Suppose $\mathbf{Q} = (Q;[\ ])$ is an $(m+k,m)$−band. According to Proposition 2.2, (I), (II), (III), (IV) and (V) are satisfied in $\mathbf{Q}$. For a fixed in Q, let $*_i$ $i \in \mathbb{N}_m$, be an operation defined on Q, by $x *_i y = \left[\overset{i-1}{a}\ x\ \overset{k-1}{a}\ y\ \overset{m-i}{a}\right]_i$. Then using (I), (II), (III), (IV) and (V) we can obtain that $(Q;*_i)$, $i \in \mathbb{N}_m$ are rectangular bands, satisfying (i), (ii) and (iii) and $\left[x_1^{m+k}\right]_i = x_i *_i x_{i+k}$, $x_1^{m+k} \in Q^{m+k}$, $i \in \mathbb{N}_m$.

Conversely, let $(Q;*_i)$, $i \in \mathbb{N}_m$, be rectangular bands, satisfying (i), (ii) and (iii) and $\left[x_1^{m+k}\right]_i = x_i *_i x_{i+k}$, $x_1^{m+k} \in Q^{m+k}$, $i \in \mathbb{N}_m$.

Clearly, $Q = (Q;[\ ])$ is an $(m+k,m)$−groupoid.

In order to prove that $Q = (Q;[\ ])$ is an $(m+k,m)$−semigroup, we need to go through the following three cases: (1) $k = m$; (2) $k > m$, i.e. $k = m+s, s \ge 1$, and (3) $k < m$.

(1) $\left[\left[x_1^{2m}\right] x_{2m+1}^{3m}\right]_i = \left[x_1^{2m}\right]_i *_i x_{2m+i} = (x_i *_i x_{i+m}) *_i x_{i+2m} = x_i *_i x_{i+2m}$.

We will prove that $\left[x_1^j \left[x_{j+1}^{j+2m}\right] x_{j+2m+1}^{3m}\right]_i = x_i *_i x_{i+2m}$.

a) Let $i \leq j$. Then $i+m \leq j+m$. We obtain, $i+m=j+t$, for $1 \leq t \leq m$. It is also true that $i+m=j+t \leq m+t$. So $i \leq t$. Then

$$\left[x_1^j \left[x_{j+1}^{j+2m}\right] x_{j+2m+1}^{3m}\right]_i = x_i *_i \left[x_{j+1}^{j+2m}\right]_t = x_i *_i (x_{j+t} *_t x_{j+t+m}) \overset{(iii)}{=}$$
$$= x_i *_i x_{j+t+m} = x_i *_i x_{i+2m}.$$

b) Let $j < i$. $j < i \leq m$ implies $j < m$. Let $j+t=m$, then $i = j+\lambda$, where $1 \leq \lambda \leq t$ and $i+m > j+m$.

$$\left[x_1^j \left[x_{j+1}^{j+2m}\right] x_{j+2m+1}^{3m}\right]_i = \left[x_{j+1}^{j+2m}\right]_\lambda *_{j+\lambda} x_{j+2m+\lambda} =$$
$$= (x_{j+\lambda} *_\lambda x_{j+\lambda+m}) *_{j+\lambda} x_{j+2m+\lambda} \overset{(ii)}{=} x_{j+\lambda} *_{j+\lambda} x_{j+2m+\lambda} = x_i *_i x_{i+2m}.$$

Hence, $\left[\left[x_1^{2m}\right] x_{2m+1}^{3m}\right]_i = \left[x_1^j \left[x_{j+1}^{j+2m}\right] x_{j+2m+1}^{3m}\right]_i$, for any $i \in \mathbb{N}_m$, $0 \leq j \leq m$. So, $(Q; [\,])$ is an $(2m, m)$ semigroup.

(2) $\left[\left[x_1^{2m+s}\right] x_{2m+s+1}^{3m+2s}\right]_i = \left[x_1^{2m+s}\right]_i *_i x_{2m+s+s+i} = (x_i *_i x_{i+m+s}) *_i x_{i+2m+2s} = x_i *_i x_{i+2m+2s}$. We will prove that $\left[x_1^j \left[x_{j+1}^{j+2m+s}\right] x_{j+2m+s+1}^{3m+2s}\right]_i = x_i *_i x_{i+2m+2s}$.

a) Let $i < j \leq m+s$.

a1) Let $j \leq s$. Then $i+s+m > s+m \geq j+m$. $\left[x_1^j \left[x_{j+1}^{j+2m+s}\right] x_{j+2m+s+1}^{3m+2s}\right]_i = x_i *_i x_{j+2m+s+s-j+i} = x_i *_i x_{i+2m+2s}$.

a2) Let $s < j \leq m+s$. If $s+t=j$ then $s+t \leq m+s$ i.e. $t \leq m$. So, $1 \leq t \leq m$.

a2.1) $i \leq t$. Then $i+m+s \leq t+m+s = j+m$ and $j \leq m+s \leq i+m+s \leq j+m$

$$\left[x_1^j \left[x_{j+1}^{j+2m+s}\right] x_{j+2m+s+1}^{3m+2s}\right]_i =$$
$$= \left[x^{s+t} \left[x_{j+1}^{j+2m+s}\right]_1 \cdots \left[x_{j+1}^{j+2m+s}\right]_{m-t} \left[x_{j+1}^{j+2m+s}\right]_{m-t+1} \cdots \left[x_{j+1}^{j+2m+s}\right]_{m-t+t} x_{j+2m+s+1}^{3m+2s}\right]_i =$$
$$= x_i *_i \left[x_{j+1}^{j+2m+s}\right]_{m-t+i} = x_i *_i (x_{j+m-t+i} *_{m-t+i} x_{j+m-t+i+m+s})$$

Since $i \leq m-t+i$, using (iii) we have

$$x_i *_i (x_{j+m-t+i} *_{m-t+i} x_{j+m-t+i+m+s}) \overset{(iii)}{=} x_i *_i x_{j+m-t+i+m+s} =$$
$$= x_i *_i x_{s+t+m-t+i+m+s} = x_i *_i x_{i+2m+2s}.$$

a2.2) $i > t$. Then $i+m+s > t+m+s = j+m$.

$$\left[x_1^j \left[x_{j+1}^{j+2m+s}\right] x_{j+2m+s+1}^{3m+2s}\right]_i =$$
$$= \left[x_1^{s+t} \left[x_{j+1}^{j+2m+s}\right]_1 \cdots \left[x_{j+1}^{j+2m+s}\right]_{m-t} \left[x_{j+1}^{j+2m+s}\right]_{m-t+1} \cdots \left[x_{j+1}^{j+2m+s}\right]_{m-t+t} x_{j+2m+s+1}^{3m+2s}\right]_i =$$
$$= x_i *_i x_{j+2m+s+i-t} = x_i *_i x_{s+t+2m+s+i-t} = x_i *_i x_{i+2m+2s}.$$

b) Let $i=j$. $\left[x_1^i \left[x_{i+1}^{i+2m+s}\right] x_{i+2m+s+1}^{3m+2s}\right]_i = x_i *_i x_{i+2m+s+s} = x_i *_i x_{i+2m+2s}$.

c) $i > j$. Then $i \leq m$ implies that $m > j$. Let $j+t=m$. So, $i = j+\lambda$, $1 \leq \lambda \leq t$

(clearly, $i+m+s>j+m$). We have

$$\left[x_1^j\left[x_{j+1}^{j+2m+s}\right]x_{j+2m+s+1}^{3m+2s}\right]_i=\left[x_{j+1}^{j+2m+s}\right]_\lambda *_i x_{j+2m+s+\lambda+s}=$$
$$=(x_{j+\lambda}*_\lambda x_{j+\lambda+m+s})*_i x_{j+2m+s+\lambda+s}$$

Since $\lambda\le i$, using (ii) we have

$$(x_{j+\lambda}*_\lambda x_{j+\lambda+m+s})*_i x_{j+2m+s+\lambda+s}\overset{(ii)}{=}x_{j+\lambda}*_i x_{j+2m+s+\lambda+s}=x_i *_i x_{i+2m+2s}.$$

Then $\left[\left[x_1^{2m+s}\right]x_{2m+s+1}^{3m+2s}\right]_i=\left[x_1^j\left[x_{j+1}^{j+2m+s}\right]x_{j+2m+s+1}^{3m+2s}\right]_i$, for any $i\in\mathbb{N}_m$, $0\le j\le m$. So $(Q;[\,])$ is a $(2m+s,m)$-semigroup.

(3) Since $k<m$, let $k+t=m$, $t\ge 1$.
First, we will prove that $\left[\left[x_1^{m+k}\right]x_{m+k+1}^{m+2k}\right]_i=x_i *_i x_{i+2k}$.

a) Let $i\le t$. Then $i+k\le t+k=m$. We have

$$\left[\left[x_1^{m+k}\right]x_{m+k+1}^{m+2k}\right]_i=\left[x_1^{m+k}\right]_i *_i \left[x_1^{m+k}\right]_{i+k}=(x_i *_i x_{i+k})*_i(x_{i+k}*_{i+k}x_{i+2k})=$$
$$=x_i *_i(x_{i+k}*_{i+k}x_{i+2k})$$

Since $i\le i+k$, (iii) implies that:
$x_i *_i(x_{i+k}*_{i+k}x_{i+2k})\overset{(iii)}{=}x_i *_i x_{i+2k}$.
b) Let $t<i\le m$. Then $i=t+\lambda$, $1\le\lambda\le k$ and $i+k=t+\lambda+k=m+\lambda$.

$$\left[\left[x_1^{m+k}\right]x_{m+k+1}^{m+2k}\right]_i=\left[x_1^{m+k}\right]_i *_i x_{m+k+\lambda}=(x_i *_i x_{i+k})*_i x_{m+k+\lambda}=$$
$$=x_i *_i x_{m+k+\lambda}=x_i *_i x_{t+k+k+\lambda}=x_i *_i x_{i+2k}.$$

Further on we will prove that $\left[x_1^j\left[x_{j+1}^{j+m+k}\right]x_{j+m+k+1}^{m+2k}\right]_i=x_i *_i x_{i+2k}$.
c) Let $i\le j$. Then $i\le j<j+t$ implies $i+k<j+t+k=j+m$. Moreover, $i+k>k\ge j$ i.e. $j<i+k<j+m$. Let $i+k=j+\lambda$ (then $i+k=j+\lambda\le k+\lambda$ i.e. $i\le\lambda$).
We obtain

$$\left[x_1^j\left[x_{j+1}^{j+m+k}\right]x_{j+m+k+1}^{m+2k}\right]_i=$$
$$=\left[x_1^i x_{i+1}^j\left[x_{j+1}^{j+m+k}\right]_1\dots\left[x_{j+1}^{j+m+k}\right]_\lambda\left[x_{j+1}^{j+m+k}\right]_{\lambda+1}\dots\left[x_{j+1}^{j+m+k}\right]_m x_{j+m+k+1}^{m+2k}\right]_i=$$
$$=x_i *_i\left[x_{j+1}^{j+m+k}\right]_\lambda=x_i *_i(x_{j+\lambda}*_\lambda x_{j+\lambda+k}).$$

Since $i\le\lambda$, (iii) implies that:
$x_i *_i(x_{j+\lambda}*_\lambda x_{j+\lambda+k})\overset{(iii)}{=}x_i *_i x_{j+\lambda+k}=x_i *_i x_{i+k+k}=x_i *_i x_{i+2k}$.
d) Let $j<i$.
d1) Let $j+1\le i\le j+t$ i.e. $i=j+\lambda$, $1\le\lambda\le t$.
Then $i+k=j+\lambda+k\le j+t+k=j+m$. Also, $i+k=j+\lambda+k\le k+\lambda+k$, therefore $i\le\lambda+k$.

$$\left[x_1^j\left[x_{j+1}^{j+m+k}\right]x_{j+m+k+1}^{m+2k}\right]_i=\left[x_{j+1}^{j+m+k}\right]_\lambda *_i\left[x_{j+1}^{j+m+k}\right]_{\lambda+k}=$$

$= (x_{j+\lambda} *_\lambda x_{j+\lambda+k}) *_i (x_{j+\lambda+k} *_{\lambda+k} x_{j+\lambda+k+k})$.

Since $\lambda \leq i$, using (ii) we have

$(x_{j+\lambda} *_\lambda x_{j+\lambda+k}) *_i (x_{j+\lambda+k} *_{\lambda+k} x_{j+\lambda+k+k}) \overset{(ii)}{=} x_{j+\lambda} *_i (x_{j+\lambda+k} *_{\lambda+k} x_{j+\lambda+k+k})$.

Since $i \leq \lambda + k$, by (iii) we have

$x_{j+\lambda} *_i (x_{j+\lambda+k} *_{\lambda+k} x_{j+\lambda+k+k}) \overset{(iii)}{=} x_{j+\lambda} *_i x_{j+\lambda+k+k} = x_i *_i x_{i+2k}$.

d2) Let $j + t < i$ i.e. $i = j + t + \lambda$, $1 \leq \lambda \leq k - j$. Then $j + t + k < i + k$ i.e. $j + m < i + k$.

$\left[x_1^j \left[x_{j+1}^{j+m+k}\right] x_{j+m+k+1}^{m+2k}\right]_i = \left[x_{j+1}^{j+m+k}\right]_{t+\lambda} *_i x_{j+m+k+\lambda} =$

$= (x_{j+t+\lambda} *_{t+\lambda} x_{j+t+\lambda+k}) *_i x_{j+k+t+k+\lambda}$.

Since $t + \lambda \leq i$, using (ii) we have

$$(x_{j+t+\lambda} *_{t+\lambda} x_{j+t+\lambda+k}) *_i x_{j+k+t+k+\lambda} \overset{(ii)}{=} x_{j+t+\lambda} *_i x_{j+k+t+k+\lambda} = x_i *_i x_{i+2k}.$$

Then $\left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right]_i = \left[x_1^j \left[x_{j+1}^{j+m+k}\right] x_{j+m+k+1}^{m+2k}\right]_i$, for any $i \in \mathbb{N}_m$ $0 \leq j \leq m$. So, $(Q; [\,])$ is an $(m+k, m)-$semigroup, when $k < m$.

Since $(Q; *_i)$, $i \in \mathbb{N}_m$, are rectangular bands, satisfying (i), (ii) and (iii) and $\left[x_1^{m+k}\right]_i = x_i *_i x_{i+k}$, $x_1^{m+k} \in Q^{m+k}$, $i \in \mathbb{N}_m$, it follows that in $(Q; [\,])$ (I), (II), (III), (IV) and (V) are satisfied.

Hence, according to Proposition 2.2, $\mathbf{Q}$ is an $(m+k, m)-$band. □

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