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SKEW OPERATION ON (n,m)-GROUPS

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Abstract. In this paper a skew operation of (n,m)-groups is defined as a generalization of the notion of a skew operation of n-groups.

1. Preliminaries

Definition 1.1. [1] *Let $n \geq m+1$ and $(Q;A)$ be an (n,m)-groupoid $(A : Q^n \to Q^m)$. We say that $(Q;A)$ is an (n,m)-group iff the following statements hold:*

(|) *For every $i,j \in \{1,\ldots,n-m+1\}$, $i<j$, the following law holds*
$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-m}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-m})$$
[: $<i,j>$ -associative law][1]*; and*

(||) *For every $i \in \{1,\ldots,n-m+1\}$ and for every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds*
$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n.$$

Remark. For $m=1$ $(Q;A)$ is an n-group [3]. Cf. Chapter I in [11].

Definition 1.2. [6]: *Let $n \geq 2m$ and let $(Q;A)$ be an (n,m)-groupoid. Also, let $\mathbf{e}$ be a mapping of the set Q^{n-2m} into the set Q^m. Then, we say that $\mathbf{e}$ is a* **$\{1,n-m+1\}$-neutral operation** *of the (n,m)-groupoid $(Q;A)$ iff for every sequence a_1^{n-2m} over Q and for every $x_1^m \in Q^m$ the following equalities hold*
$$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m \text{ and } A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m.$$

Remark. For $m=1$ $\mathbf{e}$ is a $\{1,n\}$-neutral operation of the n-groupoid $(Q;A)$ [5]. Cf. Chapter II in [11].

Proposition 1.3. [6]: *Let $n \geq 2m$ and let $(Q;A)$ be an (n,m)-groupoid. Then there is at most one $\{1,n-m+1\}$-neutral operation of $(Q;A)$.*

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[1] $(Q;A)$ is an (n,m)-semigroup.

Proposition 1.4. [6]: *Every $(n,m)-$group $(n \geq 2m)$ has a $\{1, n-m+1\}-$neutral operation.*

See, also [10].

Remark. The paper [2] is mainly a survey on the known results for vector valued groupoids, semigroups and groups (up to 1988).

2. Auxiliary part

Definition 2.1. *Let $(Q;A)$ be an $(n,m)-$groupoid; $n > m$. Then:*

(α) $\overset{1}{A} \overset{def}{=} A$; *and*

(β) For every $s \in N$ and for every $x_1^{(s+1)(n-m)+m} \in Q$

$$\overset{s+1}{A}(x_1^{(s+1)(n-m)+m}) \overset{def}{=} A(\overset{s}{A}(x_1^{s(n-m)+m}), x_{s(n-m)+m+1}^{(s+1)(n-m)+m}).$$

Proposition 2.2. *Let $(Q;A)$ be an $(n,m)-$semigroup and $s \in N$. Then, for every $x_1^{(s+1)(n-m)+m} \in Q$ and for every $t \in \{1, \dots, s(n-m)+1\}$ the following equality holds*

$$\overset{s+1}{A}(x_1^{(s+1)(n-m)+m}) = \overset{s}{A}(x_1^{t-1}, A(x_t^{t+n-1}), x_{t+n}^{(s+1)(n-m)+m}).$$

Proposition 2.3. [1]: *Let $(Q;A)$ be an $(n,m)-$semigroup and $(i,j) \in N^2$. Then, for every $x_1^{(i+j)(n-m)+m} \in Q$ and for all $t \in \{1, \dots, i(n-m)+1\}$ the following equality holds*

$$\overset{i+j}{A}(x_1^{(i+j)(n-m)+m}) = \overset{i}{A}(x_1^{t-1}, \overset{j}{A}(x_t^{t+j(n-m)+m-1}), x_{t+j(n-m)+m}^{(i+j)(n-m)+m}).$$

Proposition 2.4. [1] *Let $(Q;A)$ be an $(n,m)-$group, $n \geq 2m$ and let $s \in N$. Then, $(Q;\overset{s}{A})$ is an $(s(n-m)+m, m)-$ group.*

3. Main part

Definition 3.1. *Let $(Q;A)$ be an $(n,m)-$group and $n \geq 2m+1$. Further on, let $^{-}$ be a mapping of the set Q into the set Q^m. Then, we shall say that mapping $^{-}$ is* **a skew operation** *of the $(n,m)-$group $(Q;A)$ iff for each $a \in Q$ there is (exactly one) $\overline{a} \in Q^m$ such that the following equality holds*

$$(0) \qquad A(\overset{n-m}{a}, \overline{a}) = \overset{m}{a}$$

Remark. For $m = 1$ skew operation is introduced in [3].

Proposition 3.2. *Let $(Q;A)$ be an $(n,m)-$group and $n \geq 2m+1$. Then for all $i \in \{1, \dots, n-m+1\}$ and for every $a \in Q$ the following equality holds*

$$A(\overset{i-1}{a}, \overline{a}, \overset{n-(i-1+m)}{a}) = \overset{m}{a}.$$

Sketch of the proof.

$$A(\overset{n-m}{a}, \overline{a}) \overset{(0)}{=} \overset{m}{a} \Rightarrow$$

$$A(\overset{i-1}{a}, A(\overset{n-m}{a}, \overline{a}), \overset{n-(i-1+m)}{a}) = A(\overset{i-1}{a}, \overset{m}{a}, \overset{n-(i-1+m)}{a}) \Rightarrow$$

$$A(\overset{i-1}{a}, A(\overset{n-m}{a}, \overline{a}), \overset{n-(i-1+m)}{a}) = A(\overset{n}{a}) \overset{1.1(|)}{\Longrightarrow}$$

$A(\overset{i-1}{a}, \overset{n-(i-1+m)}{a}, A(\overset{i-1}{a}, \bar{a}, \overset{n-(i-1+m)}{a})) = A(\overset{n}{a}) \Rightarrow$
$A(\overset{n-m}{a}, A(\overset{i-1}{a}, \bar{a}, \overset{n-(i-1+m)}{a})) = A(\overset{n-m}{a}, \overset{m}{a}) \overset{1.1(||)}{\Longrightarrow}$
$A(\overset{i-1}{a}, \bar{a}, \overset{n-(i-1+m)}{a}) = \overset{m}{a}.$

Proposition 3.3. *Let $(Q;A)$ be an (n,m)-group and $n \geq 2m+1$. Then for all $a, x_1^m \in Q$ the equality*
$$A(x_1^m, \overset{n-2m}{a}, \bar{a}) = x_1^m$$
holds.

Sketch of the proof.
$A(x_1^m, \overset{n-2m}{a}, \bar{a}) = y_1^m \Rightarrow$
$A(A(x_1^m, \overset{n-2m}{a}, \bar{a}), \overset{n-m}{a}) = A(y_1^m, \overset{n-m}{a}) \overset{1.1(|)}{\Longrightarrow}$
$A(x_1^m, \overset{n-2m}{a}, A(\bar{a}, \overset{n-m}{a})) = A(y_1^m, \overset{n-m}{a}) \overset{3.2, i=1}{\Longrightarrow}$
$A(x_1^m, \overset{n-2m}{a}, \overset{m}{a}) = A(y_1^m, \overset{n-m}{a}) \Rightarrow$
$A(x_1^m, \overset{n-m}{a}) = A(y_1^m, \overset{n-m}{a}) \overset{1.1(||)}{\Longrightarrow} x_1^m = y_1^m.$

Theorem 3.4. *Let $n \geq 2m+1$, $(Q;A)$ be an (n,m)-group, $\mathbf{e}$ its $\{1, n-m+1\}$-neutral operation and $^-$ its skew operation. Then for all $a \in Q$ the following equality holds*
$$\bar{a} = \mathbf{e}(\overset{n-2m}{a}).$$

Sketch of the proof.
$A(x_1^m, \overset{n-2m}{a}, \bar{a}) \overset{3.3}{=} x_1^m \ \wedge\ A(x_1^m, \overset{n-2m}{a}, \mathbf{e}(\overset{n-2m}{a})) \overset{1.4}{=} x_1^m \Rightarrow$
$A(x_1^m, \overset{n-2m}{a}, \bar{a}) = A(x_1^m, \overset{n-2m}{a}, \mathbf{e}(\overset{n-2m}{a})) \overset{1.1(||)}{\Longrightarrow} \bar{a} = \mathbf{e}(\overset{n-2m}{a}).$

Theorem 3.5. *Let $(Q;A)$ be an (n,m)-group, $\mathbf{e}$ its $\{1, n-m+1\}$-neutral operation, $^-$ its skew operation and $n > 3m$. Then for every sequence a_1^{n-m+1} over Q the following equality holds*
$$\mathsf{E}(\overset{m}{a_1}, \ldots, \overset{m}{a}_{n-m+1}) = \overset{n-2m-1}{A}(\bar{a}_{n-m-1}, \overset{n-3m}{a}_{n-m+1}, \ldots, \bar{a}_1, \overset{n-3m}{a_1})$$ [2],
where E is the $\{1, m(n-m)+1\}$-neutral operation of $(m(n-m)+m, m)$-group $(Q; \overset{m}{A})$.

Sketch of the proof.
$\overset{m}{A}(\overset{n-2m-1}{A}(\bar{a}_{n-m-1}, \overset{n-3m}{a}_{n-m+1}, \ldots, \bar{a}_1, \overset{n-3m}{a_1}), \overset{m}{a_1}, \ldots, \overset{m}{a}_{n-m-1}, x_1^m) \overset{3.4}{=}$
$\overset{m}{A}(\overset{n-2m-1}{A}(\mathbf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-3m}{a}_{n-m-1}, \ldots, \mathbf{e}(\overset{n-2m}{a_1}), \overset{n-3m}{a_1}), \overset{m}{a_1}, \ldots, \overset{m}{a}_{n-m-1}, x_1^m) \overset{2.3}{=}$
$\overset{n-m-1}{A}(\mathbf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-3m}{a}_{n-m-1}, \ldots, \mathbf{e}(\overset{n-2m}{a_1}), \overset{n-3m}{a_1}, \overset{m}{a_1}, \ldots, \overset{m}{a}_{n-m-1}, x_1^m) \overset{2.2}{=}$
$\overset{n-m-2}{A}(\mathbf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-3m}{a}_{n-m-1}, \ldots, A(\mathbf{e}(\overset{n-2m}{a_1}), \overset{n-3m}{a_1}, \overset{m}{a_1}, \overset{m}{a_2}), \overset{m}{a_3}, \ldots, \overset{m}{a}_{n-m-1}, x_1^m) \overset{1.2}{=}$
$\overset{n-m-2}{A}(\mathbf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-3m}{a}_{n-m-1}, \ldots, \overset{m}{a_2}, \overset{m}{a_3}, \ldots, \overset{m}{a}_{n-m-1}, x_1^m) =$

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[2] $n > 3m \Rightarrow \overset{n-3m}{a}_i \neq \emptyset$ $(i \in \{1, \ldots, n-m-1\})$.

$A(\mathsf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-3m}{a}_{n-m-1}, \overset{m}{a}_{n-m-1}, x_1^m) =$
$A(\mathsf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-2m}{a}_{n-m-1}, x_1^m) \overset{1.2}{=} x_1^m.$

Hence, by Prop. 2.4 and Prop. 1.4, we conclude that for every sequence a_1^{n-m-1} over Q and for all $x_1^m \in Q^m$ the following equality holds

$\overset{m}{A}(\overset{n-2m-1}{A}(\mathsf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-3m}{a}_{n-m-1}, \ldots, \mathsf{e}(\overset{n-2m}{a_1}), \overset{n-3m}{a_1}), \overset{m}{a}_1, \ldots, \overset{m}{a}_{n-m-1}, x_1^m) =$
$\overset{m}{A}(\mathsf{E}(\overset{m}{a}_1, \ldots, \overset{m}{a}_{n-m-1}), \overset{m}{a}_1, \ldots, \overset{m}{a}_{n-m-1}, x_1^m),$

where E is the $\{1, m(n-m)+1\}-$neutral operation of $(m(n-m)+m, m)-$group $(Q; \overset{m}{A})$. Finaly, whence, by Def. 1.1, we conclude that the proposition holds.

For $m = 1$ Th. 3.5 is reduced to:

Theorem 3.6. [13] *Let $(Q; A)$ be an $n-$group, e its $\{1, n\}-$neutral operation, $^{-}$ its skew operation and $n > 3$. Then for every sequence a_1^{n-2} over Q the following equality holds*

$$\mathsf{e}(a_1^{n-2}) = \overset{n-3}{A}(\overline{a}_{n-2}, \overset{n-3}{a}_{n-2}, \ldots, \overline{a}_1, \overset{n-3}{a_1}).$$

Remark. See, also VIII-2.9 and Appendix 2 in [11].

Remark 3.7. *In [9] topological $n-$groups for $n \geq 2$ are defined on $n-$groups as algebras $(Q; A, ^{-1})$ of the type $< n, n-1 >$ [[7], [8]; cf. Ch. III and Ch. IX in [11]]. In [12] topological $n-$groups for $n \geq 3$ are considered on $n-$groups as algebras $(Q; A, ^{-})$ of the type $< n, 1 >$ [[4]]. In [9] it is proved that for $n \geq 3$ these definitions are mutually equivalent. The key roole in the proof had Theorem 3.6. About topological $n-$groups see, also, Chapter VIII in [11].*

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