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ON THE STRICTLY PARALLEL SUBBUNDLES OF VECTOR BUNDLES

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Abstract. In this paper are considered four propositions about strictly parallel subbundles of vector bundles. In propositions 1 and 2 are generalized some results of Walker [1] and Wong [2] in case of an arbitrary vector bundle, instead of tangent bundle. Proposition 3 generalizes one Wong's result [2] in global case. In proposition 4 are given necessary and sufficient conditions for existence of a connection in a vector bundle ξ such that given r linearly independent vector fields and s linearly independent 1-forms in ξ are parallel. That condition is given by (3).

Suppose that $\xi=(\varepsilon,\pi,M_n)$ is a vector bundle of rank m and that it is endowed with connection. A subbundle $\xi'=(\varepsilon',\pi',M_n)$ of ξ with rank r is said to be parallel, if for any two points $A,B\in M_n$ a vector in $(\pi')^{-1}(A)$ is displaced into a vector in $(\pi')^{-1}(B)$ by parallel transport along any curve from A to B. A subbundle $\xi'=(\varepsilon',\pi',M_n)$ of ξ with rank r is said to be strictly parallel, if it contains a basis of r parallel differentiable vector fields.

Proposition 1. A necessary and sufficient condition for a vector subbundle ξ' of rank r to be parallel, is that its basis of parallel vector fields $\{\lambda^i_{(\alpha)}\}$ $(\alpha\in\{1,\dots,r\},\ i\in\{1,\dots,m\})$ satisfies the recurent relations of the form

$$\nabla_k(\lambda_{(\alpha)})^i = A^\beta_{\alpha k}\lambda^i_{(\beta)} \tag{1}$$

where $A^\beta_{\alpha k}$ $(\alpha,\beta\in\{1,\dots,r\},\ k\in\{1,\dots,n\})$ are components of co-vector field for fixed α and β.

In a special case when $\xi=(\varepsilon,\pi,M_n)$ is the tangent bundle, the proof is given in [1]. The proof of proposition 1 is analogous to that proof.

The components

$$A^{\beta}_{\alpha k\ell} = \partial A^{\beta}_{\alpha k}/\partial x^{\ell} - \partial A^{\beta}_{\alpha\ell}/\partial x^{k} + A^{\delta}_{\alpha k}A^{\beta}_{\delta\ell} - A^{\delta}_{\alpha\ell}A^{\beta}_{\delta k} \qquad (2)$$

($\alpha,\beta,\delta \in \{1,\dots,r\}$, $k,\ell \in \{1,\dots,n\}$) for fixed α and β transform as a tensor skew-symmetric in k and ℓ. Furthermore, for fixed k and ℓ $A^{\beta}_{\alpha k\ell}$ has tensor character with respect to α and β under the change of the basis ([2]). So $A^{\beta}_{\alpha k\ell}$ is called double tensor.

In the case when $\xi=(\varepsilon,\pi,M_n)$ is the tangent bundle, Wong ([2], theorem 6.1) has proved the following proposition.

<u>Proposition 2</u>. If the vector subbundle $\xi'=(\varepsilon',\pi',M_n)$ is strictly parallel, then the double tensor associated with ξ' identically vanishes. Conversely, if the double tensor associated with a vector subbundle ξ' identically vanishes, then for each point $p \in M_n$ there exists a neighbourhood U of p such that the restriction of ξ' on U is a strictly parallel vector subbundle, i.e. ξ' is locally strictly parallel vector subbundle.

In general case the proof of theorem 2 is analogous to the Wong's proof.

Now we shall give an improvement of the second part of the proposition 2 in global case.

<u>Proposition 3</u>. Assume that M_n is simply-connected differentiable manifold. If the double tensor which is associated with a vector subbundle ξ' vanishes identically, then ξ' is strictly parallel vector subbundle.

<u>Proof</u>. We can suppose that M_n is connected manifold. Let $x \in M_n$, and we choose r linearly independent vectors $X_{(1)},\dots,X_{(r)} \in (\pi')^{-1}(x)$. For arbitrary point $y \in M_n$ there exists a path $z(t)$ which connects x and y, and the vectors $X_{(i)}$ $(i=1,\dots,r)$ can parallelly be transported to the point y. Using the lemma of factorization and the second part of the proposition 2, it can be verified that the transported vectors at y do not depend on the choice of the path $z(t)$ from x to y. It also can be verified that these r vector fields are parallel, and thus we obtain a basis of r linearly independent parallel vector fields for the vector subbundle ξ'. ||

Further we shall consider vector bundles without connection and our aim is to endow them with connections such that given vector fields and 1-forms be parallel.

Proposition 4. Assume that M_n is paracompact differentiable manifold and assume that $X_{(1)},\dots,X_{(r)}$ are linearly independent vector fields and $w^{(1)},\dots,w^{(s)}$ are linearly independent 1-forms on the vector bundle $\xi=(\varepsilon,\pi,M_n)$. Then there exists a connection on ξ such that $X_{(p)}$ and $w^{(q)}$ be parallel vector fields and 1-forms ($p\in\{1,\dots,r\}$, $q\in\{1,\dots,s\}$), iff for each $p\in\{1,\dots,r\}$ and for each $q\in\{1,\dots,s\}$

$$w^{(q)}(X_{(p)}) = C_p^q = \text{const.} \tag{3}$$

Proof. Let us suppose that $X_{(p)}$ has components $\lambda^i_{(p)}$, ($i\in\{1,\dots,m\}$, $p\in\{1,\dots,r\}$) and $w^{(q)}$ has components $\mu_j^{(q)}$, ($j\in\{1,\dots,m\}$, $q\in\{1,\dots,s\}$) with respect to the local coordinates. Then (3) is equivalent to

$$\lambda^i_{(p)}\mu_i^{(q)} = C_p^q = \text{const.} \tag{4}$$

If there exists a connection such that $X_{(p)}$ and $w^{(q)}$ ($p\in\{1,\dots,r\}$, $q\in\{1,\dots,s\}$) are parallel, then it is obvious that (4) is satisfied.

Conversely, let us suppose that the matrix $\left[\lambda^i_{(p)}\mu_i^{(q)}\right]$ has constant components and its rank is t. Then there exist invertible matrices $\left[c^p_\alpha\right]_{r\times r}$ and $\left[d^\beta_q\right]_{s\times s}$ with constant elements such that

$$\left[c^p_\alpha\lambda^i_{(p)}d^\beta_q\mu_i^{(q)}\right] = \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\lambda^i_{(p)}\mu_i^{(q)}$ do not depend on the coordinate system, then it follows that $\left[c^p_\alpha\right]$ and $\left[d^\beta_q\right]$ also do not depend on the coordinate system.

We can define new vector fields $\lambda'^i_{(\alpha)}$ and 1-forms $\mu'^{(\alpha)}_i$ by $\lambda'^i_{(\gamma)}=c^p_\gamma\lambda^i_{(p)}$ for $\gamma\in\{1,\dots,r\}$, and $\mu'^{(\alpha)}_j=d^\alpha_\beta\mu_j^{(\beta)}$ for $\alpha\in\{1,\dots,t\}$ and $\mu'^{(\beta)}_j=d^{\beta-(n-s)}_\gamma\mu_j^{(\gamma)}$ for $\beta\in\{m-s+t+1,\dots,m\}$. Since the vector fields and 1-forms are linearly independent, it can be verified that $r \le m-s+t$. The new vector fields and 1-forms satisfy

$$w'^{(\alpha)}(X'_{(\beta)}) = \delta^{\alpha}_{\beta} \tag{5}$$

($\beta \in \{1,\dots,r\}$, $\alpha \in \{1,\dots,t,m-s+t+1,\dots,m\}$).

Since M_n is a paracompact manifold, then there exists a locally finite open covering of coordinate neighbourhoods $\{U_i\}$, and let $\{f_i\}$ be the corresponding decomposition of the unit.

In arbitrary coordinate neighbourhood U_i can be chosen m-r vector fields $X'_{(\gamma)}$ ($\gamma \in \{r+1,\dots,m\}$) such that

$$w'^{(\alpha)}(X'_{(k)}) = \delta^{\alpha}_{k} \tag{6}$$

($k \in \{1,\dots,m\}$, $\alpha \in \{1,\dots,t,m-s+t+1,\dots,m\}$) and $X'_{(1)},\dots,X'_{(m)}$ to be linearly independent vector fields. Then $X'_{(1)},\dots,X'_{(m)}$ generate flat connection $\Gamma_{(i)}$ on U_i such that $X'_{(1)},\dots,X'_{(m)}$ are parallel vector fields on U_i. Since $X'_{(1)},\dots,X'_{(m)}$ are linearly independent, it follows from (6) that $w'^{(i)}$ ($i \in \{1,\dots,t,m-s+t+1,\dots,m\}$) are parallel 1-forms. Let us define a connection Γ on ξ by

$$\Gamma^{j}_{k\ell} = \sum_i f_i \Gamma^{j}_{k\ell(i)}$$

($j,k \in \{1,\dots,m\}$, $\ell \in \{1,\dots,n\}$). Then using that $\nabla_Y K = \sum_i f_i (\nabla_Y)_{(i)} K$ for arbitrary tensor field K, where ∇_Y and $(\nabla_Y)_{(i)}$ are covariant derivatives with respect to the connections Γ and $\Gamma_{(i)}$ respectively, we obtain that $X'_{(1)},\dots,X'_{(r)},w'^{(1)},\dots,w'^{(t)}$, $w'^{(m-s+t+1)},\dots,w'^{(m)}$ are parallel vector fields and 1-forms with respect to the connection Γ. So $X_{(1)},\dots,X_{(r)}$ and $w^{(1)}$, $\dots,w^{(s)}$ are parallel vector fields and 1-forms with respect to the connection Γ. ∎

R E F E R E N C E S

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ЗА СТРИКТНО ПАРАЛЕЛНИТЕ ПОДРАСЛОЈУВАЊА НА ВЕКТОРСКИТЕ РАСЛОЈУВАЊА

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Р е з и м е

Во овој труд се дадени четири тврдења. Во тврдењата 1 и 2 се обопштени некои резултати на Walker [1] и Wong [2] во случај на произволно векторско раслојување, наместо во тангентното раслојување. Тврдењето 3 генерализира еден резултат на Wong [2] во глобален случај. Во тврдењето 4 се дадени потребни и доволни услови за постоење на конексија во едно векторско раслојување ξ, така што дадени r линеарно независни векторски полиња и дадени s линеарно независни 1-форми во ξ бидат паралелни. Тој услов е даден со (3).