Математички Билтен
14 (XL)
1990 (73-77)
Скопје, Југославија

# ON THE SOLUTION OF ARBITRARY SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

Kostadin Trenčevski

Abstract. In this paper it is obtained the exact solution for arbitrary system of linear differential equations.

The aim of this paper is to prove the following theorem.

Theorem. Let $f_{ij}(s)\in C^{\infty}$ $(i,j\in\{1,\dots,n\})$, and let us define the functions $P_{ik}^{[\ell]}$ $(i,k\in\{1,\dots,n\},\ \ell\in\{0,1,2,\dots\})$ by

$$\begin{cases} P_{ik}^{[0]} = \delta_{ik} \\ P_{ik}^{[1]} = f_{ik} \\ P_{ik}^{[\ell+1]} = \frac{d}{ds}P_{ik}^{[\ell]} + \sum_{r=1}^{n} P_{ir}^{[1]}P_{rk}^{[\ell]}\ (\ell\in\{1,2,3,\dots\}). \end{cases} \tag{1}$$

If the series $\sum_{\ell=0}^{\infty}\frac{1}{\ell!}P_{ik}^{[\ell]}(t-s)^{\ell}$ $(i,k\in\{1,\dots,n\})$ are convergent, and if it is admissible to differentiate them by parts, then the general solution of the following system

$$\frac{dy_i}{ds} + \sum_{j=1}^{n} f_{ij}y_j = g_i,\quad i=1,\dots,n \tag{2}$$

of linear differential equations is given by

$$y_i = \sum_{k=1}^{n}\int_0^s\Big(\sum_{\ell=0}^{\infty}\frac{1}{\ell!}P_{ik}^{[\ell]}(t-s)^{\ell}\Big)g_k(t)\,dt + \sum_{k=1}^{n} C_k\sum_{\ell=0}^{\infty}P_{ik}^{[\ell]}\frac{(-s)^{\ell}}{\ell!}, \tag{3}$$
$$i=1,\dots,n$$

where $C_k$ $(k\in\{1,\dots,n\})$ are constants.

Proof.

$$\frac{dy_i}{ds} + \sum_{j=1}^{n} f_{ij}y_j = \sum_{k=1}^{n}\int_0^s\Big(\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\frac{dP_{ik}^{[\ell]}}{ds}(t-s)^{\ell}\Big)g_k(t)\,dt +$$

$$+\sum_{k=1}^{n}\int_{0}^{s}\Big(\sum_{\ell=1}^{\infty}\frac{1}{(\ell-1)!}P_{ik}^{[\ell]}(t-s)^{\ell-1}\Big)g_k(t)\,dt+\sum_{k=1}^{n}P_{ik}^{[0]}g_k(s)+$$

$$+\sum_{k=1}^{n}C_k\sum_{\ell=0}^{\infty}(-1)^{\ell}\left[\frac{dP_{ik}^{[\ell]}}{ds}\frac{s^{\ell}}{\ell!}+P_{ik}^{[\ell]}\frac{s^{\ell-1}}{(\ell-1)!}\right]+$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{0}^{s}\Big(\sum_{\ell=0}^{\infty}\frac{1}{\ell!}P_{ij}^{[1]}P_{jk}^{[\ell]}(t-s)^{\ell}\Big)g_k(t)\,dt+$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{n}C_k\sum_{\ell=0}^{\infty}P_{ij}^{[1]}P_{jk}^{[\ell]}\frac{(-s)^{\ell}}{\ell!}=$$

$$=\sum_{k=1}^{n}\int_{0}^{s}\Big(\sum_{\ell=0}^{\infty}\Big(\frac{dP_{ik}^{[\ell]}}{ds}+\sum_{j=1}^{n}P_{ij}^{[1]}P_{jk}^{[\ell]}\Big)\frac{(t-s)^{\ell}}{\ell!}\Big)g_k(t)\,dt+$$

$$+\sum_{k=1}^{n}C_k\sum_{\ell=0}^{\infty}\Big(\frac{dP_{ik}^{[\ell]}}{ds}+\sum_{j=1}^{n}P_{ij}^{[1]}P_{jk}^{[\ell]}\Big)\frac{(-s)^{\ell}}{\ell!}+g_i(s)-$$

$$-\sum_{k=1}^{n}\int_{0}^{s}\Big(\sum_{\ell=0}^{\infty}\frac{1}{\ell!}P_{ik}^{[\ell+1]}(t-s)^{\ell}\Big)g_k(t)\,dt-$$

$$-\sum_{k=1}^{n}C_k\sum_{\ell=0}^{\infty}P_{ik}^{[\ell+1]}\frac{(-s)^{\ell}}{\ell!}=$$

$$=g_i(s)+\sum_{k=1}^{n}\int_{0}^{s}\Big(\sum_{\ell=0}^{\infty}P_{ik}^{[\ell+1]}(t-s)^{\ell}\frac{1}{\ell!}\Big)g_k(t)\,dt-$$

$$-\sum_{k=1}^{n}\int_{0}^{s}\Big(\sum_{\ell=0}^{\infty}P_{ik}^{[\ell+1]}(t-s)^{\ell}\frac{1}{\ell!}\Big)g_k(t)\,dt+$$

$$+\sum_{k=1}^{n}C_k\sum_{\ell=0}^{\infty}P_{ik}^{[\ell+1]}\frac{(-s)^{\ell}}{\ell!}-$$

$$-\sum_{k=1}^{n}C_k\sum_{\ell=0}^{\infty}P_{ik}^{[\ell+1]}\frac{(-s)^{\ell}}{\ell!}=$$

$$=g_i(s).\ \|$$

If $f_{ij}\notin C^{\infty}$, but they are continuous, then we can find a series of polynomials $T_{ij}^{(k)}$ such that $\max_{0\le s\le A}|f_{ij}(s)-T_{ij}^{(k)}(s)|<\varepsilon$ for $k>N(\varepsilon)$ and $i,j\in\{1,\dots,n\}$. Since $T_{ij}^{(k)}\in C^{\infty}$, the solution of the following system

$$\frac{dy_i^{(k)}}{ds} + \sum_{j=1}^{n} T_{ij}^{(k)} y_j^{(k)} = g_i, \quad i=1,\dots,n$$

can be found from the above theorem, and then one can prove that

$$y_i(s) = \lim_{k\to\infty} y_i^{(k)}(s)$$

if $s\in[0,A]$ and $i\in\{1,\dots,n\}$.

Now we shall consider three simple examples, using the above theorem.

Example 1. Let us consider the following linear differential equation

$$y' + ay = g(s)$$

where $a$ = const. Then it is obvious that $P_{11}^{[\ell]} = a^\ell$ ($\ell\in\{0,1,2,\dots\}$), and we obtain

$$y = \int_0^s \sum_{\ell=0}^{\infty} \frac{(t-s)^\ell}{\ell!} a^\ell g(t)dt + C \sum_{\ell=0}^{\infty} \frac{a^\ell(-s)^\ell}{\ell!} =$$

$$= \int_0^s e^{at-as} g(t)dt + Ce^{-as} =$$

$$= e^{-as}\left[C+\int_0^s e^{at} g(t)dt\right].$$

Example 2. Let us consider the following linear differential equation

$$y^{(n)}+a_{n-1}y^{(n-1)}+\dots+a_1y'+a_0y = 0$$

with constant coefficients. It yields to the following system

$$\begin{cases} y_1'+a_{n-1}y_1+a_{n-2}y_2+\dots+a_1y_{n-1}+a_0y = 0 \\ y_2'-y_1 = 0 \\ y_3'-y_2 = 0 \\ \dots\dots\dots \\ y_{n-1}'-y_{n-2} = 0 \\ y'-y_{n-1} = 0. \end{cases}$$

Since $a_0, a_1, \dots, a_{n-1}$ are constants, we obtain from (1) that

$$P^{[0]} = I_{n\times n}$$

$$P^{[1]} = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \\ -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 & 0 \end{bmatrix}$$

and $P^{[\ell]} = (P^{[1]})^{\ell}$ $(\ell\in\{2,3,\dots\})$. Now we obtain from (3) the general solution for y

$$y(s) = \sum_{k=1}^{n} C_k \sum_{\ell=0}^{\infty} \left[(P^{[1]})^{\ell}\right]_{nk} \cdot \frac{(-s)^{\ell}}{\ell!} = \sum_{k=1}^{n} C_k (\exp[(-s)P^{[1]}])_{nk}.$$

Now let us suppose that $b_1,\dots,b_n$ are the roots of the polinomial $t^n+a_{n-1}t^{n-1}+\dots+a_0$, and suppose that $b_i\neq b_j$ if $i\neq j$. Then there exists a matrix Q such that

$$-P^{[1]} = Q\begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & b_n \end{bmatrix}Q^{-1}$$

because $b_1,\dots,b_n$ are eigenvalues for the matrix $-P^{[1]}$. Using this equality, we obtain the general solution in the following form

$$y(s) = \sum_{r,i,k=1}^{n} Q_{ni}\left(\exp s\begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{bmatrix}\right)_{ir} Q^{-1}_{rk}C_k =$$

$$= \sum_{r,i,k=1}^{n} Q_{ni}\delta_{ir}e^{sb_i}Q^{-1}_{rk}C_k =$$

$$= \sum_{i=1}^{n}\left(\sum_{k=1}^{n} Q_{ni}Q^{-1}_{ik}C_k\right)e^{s\alpha_i} =$$

$$= \sum_{i=1}^{n} C'_i e^{s\alpha_i}.$$

Example 3. Let us consider the following system of differential equations

$$\begin{cases} y_1' + v(s)y_1(s) + u(s)y_2(s) = g_1(s) \\ y_2' + u(s)y_1(s) + v(s)y_2(s) = g_2(s) \end{cases}$$

where $v(s) = \dfrac{s+a}{(s+a)^2+b^2}$, $u(s) = \dfrac{b}{(s+a)^2+b^2}$, a=const, and b=const.
From (1) we obtain

$$P^{[0]} = I_{2\times2}, \quad P^{[1]} = \begin{bmatrix} v & u \\ -u & v \end{bmatrix},$$

$$P^{[2]} = \begin{bmatrix} v' & u' \\ -u' & v' \end{bmatrix} + \begin{bmatrix} v & u \\ -u & v \end{bmatrix}\begin{bmatrix} v & u \\ -u & v \end{bmatrix} = \begin{bmatrix} v'-u^2+v^2 & u'+2uv \\ -u'-2uv & v'-u^2+v^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and hence $P^{[\ell]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ if $\ell \in \{2,3,4,\ldots\}$. Now we obtain from (3) that

$$y_1(s) = \int_0^s g_1(t)dt - v(s)\int_0^s (s-t)g_1(t)dt - u(s)\int_0^s (s-t)g_2(t)dt + $$
$$+ C_1(1-sv(s)) + C_2(-su(s)),$$

$$y_2(s) = \int_0^s g_2(t)dt + u(s)\int_0^s (s-t)g_1(t)dt - v(s)\int_0^s (s-t)g_2(t)dt + $$
$$+ C_1(su(s)) + C_2(1-sv(s)).$$

## R E F E R E N C E S

[1] Trenčevski, K.: A formula for the k-th covariant derivative Serdica, Vol. 15, 197-202, 1989

## ЗА РЕШЕНИЕТО НА ПРОИЗВОЛЕН СИСТЕМ ОД ЛИНЕАРНИ ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ

Костадин Тренчевски

### Р е з и м е

Во овој труд се докажува следнава теорема: Нека $f_{ij}(s) \in C^\infty$ и нека се дефинирани функции $P_{ik}^{[\ell]}$ $(i,k \in \{1,\ldots,n\},\ \ell \in \{0,1,2,\ldots\})$ со (1). Ако редовите $\sum_{\ell=0}^{\infty} \frac{1}{\ell!} P_{ik}^{[\ell]} (t-s)^\ell$ $(i,k \in \{1,\ldots,n\})$ се конвергентни и ако е дозволено нивно диференцирање член по член, тогаш (3) претставува општо решение на системот (2).

Како илустрација за користење на оваа теорема, на крајот се дадени три примери.