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A NOTE ON $(1,2)$-GPR-CLOSED SETS

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Abstract. The aim of this paper is to introduce the notion of gpr-closed sets in bitopological spaces and to study its properties. Their corresponding continuous and irresolute functions are also defined and studied.

1. Introduction

The study of bitopological space was initiated by Kelly [2] in the year 1963. Recently Balachandran,Sundaram and many others [1, 8, 7] defined different weak froms of semi-open, pre-open, regular open and α-open sets in bitopological spaces. In this paper, we make use of the (1,2)-regular-open set defined in [4] to define our (1,2)-generalized pre-regular closed sets. Comparisons are made with the other generalized closed sets defined in [4]. We also study the properties of their corresponding open sets and (1,2)-gpr-continuous and (1,2)-gpr-irresolute functions are also defined and studied.

Throughout this paper (X, τ_1, τ_1), (Y, σ_1, σ_2) and (Z, τ_1, τ_2) (or simply X, Y and Z) always represent bitopological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of (X, τ_1, τ_2), τ_i-$cl(A)$ and τ_i-$int(A)$ represent the closure and interior of A with respect to the topology τ_i for $i = 1, 2$. Now the recollection of the following definitions will stand in good measure in the further study of this paper.

Definition 1. *([3]) Let (X, τ_1, τ_2) be a bitopological space and $A \subset X$.*

(i) A is called $\tau_1\tau_2$-open if $A \in \tau_1 \cup \tau_2$.

(ii) A is called $\tau_1\tau_2$-closed if $A^c \in \tau_1 \cup \tau_2$.

(iii) τ_1 τ_2-closure of A is denoted as $\tau_1\tau_2$-$cl(A)$ and is defined as the intersection of all $\tau_1\tau_2$-closed set containing A.

(iv) $\tau_1\tau_2$-interior of A is denoted as $\tau_1\tau_2$-$int(A)$ and is defined as the union of all $\tau_1\tau_2$-open set contained in A.

Definition 2. *A subset A of X is called (1,2)-α-open* [3] *(resp. (1,2)-semi-open* [4], *(1,2)-pre-open* [4], *(1,2)-semi-pre-open* [4], *(1,2)-regular-open* [4]*) if $A \subset \tau_1$-$int(\tau_1\tau_2$-$cl(\tau_1$-$int(A)))$ (resp. $A \subset \tau_1\tau_2$-$cl(\tau_1$-$int(A))$, $A \subset \tau_1$-$int(\tau_1\tau_2$-$cl(A))$, $A \subset \tau_1\tau_2$-$cl(\tau_1$-$int(\tau_1\tau_2$-$cl(A)))$, $A = \tau_1$-$int(\tau_1\tau_2$-$cl(A))$).*

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The family of all (1,2)-α-open, (1,2)-semi-open, (1,2)-pre-open, (1,2)-semi-pre-open and (1,2)-regular open are denoted by (1,2)-$\alpha O(X)$, (1,2)-$SO(X)$, (1,2)-$PO(X)$ (1,2)-$SPO(X)$ and (1,2)-$RO(X)$, respectively.

Definition 3. *([4]) A subset A of X is called (1,2)-α-closed (resp. (1,2)-semi-closed, (1,2)-pre-closed, (1,2)-semi-pre-closed, (1,2)-regular-closed) if τ_1-$cl(\tau_1\tau_2$-$int(\tau_1$-$cl(A)) \subset A$ (resp. $\tau_1\tau_2$-$int(\tau_1$-$cl(A)) \subset A$, τ_1-$cl(\tau_1\tau_2$-$int(A)) \subset A$, τ_1-$int(\tau_1\tau_2$-$cl(\tau_1$-$int(A))) \subset A$, τ_1-$cl(\tau_1\tau_2$-$int(A)) = A$).*

The (1,2)-α-closure (resp. (1,2)-semi-closure, (1,2)-pre-closure and (1,2)-semi-pre-closure) of a subset A of X is denoted by (1,2)-α-$cl(A)$ (resp. (1,2)-$scl(A)$, (1,2)-$pcl(A)$, (1,2)-$spcl(A)$) and is defined as the intersection of all (1,2)-α-closed (resp. (1,2)-semi-closed, (1,2)-pre-closed, (1,2)-semi-pre-closed) sets containing A . The (1,2)-α-interior of A and the (1,2)-pre-interior of A is denoted as (1,2)-α-$int(A)$ and (1,2)-$pint(A)$, respectively and is defined as the union of all (1,2)-α-open and (1,2)-pre-open sets respectively containing A.

The family of all (1,2)α-closed sets, (1,2)semi-closed sets,1,2)pre-closed sets and (1,2)semi-pre-closed sets are denoted as (1,2)$\alpha CL(X)$,(1,2)$SCL(X)$, (1,2)$PCL(X)$and (1,2)$SPCL(X)$respectively.

Definition 4. *A subset A of X is called*

(i) (1,2)-αg-closed [4] if (1,2)-$\alpha cl(A) \subset U$ where $A \subset U$ and $U \in$(1,2)-$\alpha O(X)$.

(ii) (1,2)-sg-closed [4] if (1,2)-$scl(A) \subset U$ whenever $A \subset U$ and $U \in$(1,2)-$SO(X)$.

(iii) (1,2)-gs-closed [4] if (1,2)-$scl(A) \subset U$ wherever $A \subset U$ and $U \in$(1,2)-$\alpha O(X)$.

(iv) (1,2)-pg-closed [5] if $pcl(A) \subset U$ whenever $A \subset U$ and $U \in$(1,2)-$PO(X)$.

(v) (1,2)-gp-closed [5] if (1,2)-$pcl(A) \subset U$ whenever $A \subset U$ and $U \in$(1,2)-$\alpha O(X)$.

(vi) (1,2)-gsp-closed [5] if (1,2)-$spcl(A) \subset U$ whenever $A \subset U$ and $U \in$(1,2)-$\alpha O(X)$.

The family of all (1,2)-αg-closed (resp.(1,2)-sg-closed, (1,2)-gs-closed, (1,2)-pg-closed, (1,2)-gp-closed, (1,2)-gsp-closed) sets is denoted as (1,2)-$\alpha CL(X)$ (resp. (1,2)-$SGCL(X)$, (1,2)-$GSCL(X)$, (1,2)-$PGCL(X)$, (1,2)-$GPCL(X)$, (1,2)-$GSPCL(X)$)

Definition 5. *A space X is said to be (1,2)-α-hyperconnected [6] if every (1,2)-α-open set is (1,2)-α-dense in X, that is (1,2)-$\alpha cl(A) = X$.*

Definition 6. *A map $f : X \to Y$ is said to be (1,2)-αg-continuous [4] if the inverse image of (1,2)-α-closed set in Y is (1,2)-αg-closed.*

2. (1,2)-GENERALIZED PRE-REGULAR-CLOSED SETS

Definition 7. *A subset A of (X, τ_1, τ_2) is called (1,2)-regular-generalized-closed (briefly (1,2)-rg-closed) if (1,2)-$\alpha cl(A) \subset U$ whenever $A \subset U$ and $U \in$(1,2)-$RO(X)$.*

Definition 8. *A subset A of (X, τ_1, τ_2) is called (1,2)-generalized pre-regular-closed (briefly (1,2)-gpr-closed) if (1,2)-$pcl(A) \subset U$, whenever $A \subset U$ and $U \in$(1,2)-$RO(X)$.*

The family of all (1,2)-rg-closed and (1,2)-gpr-closed sets of X is denoted by (1,2)-$RGCL(X)$ and (1,2)-$GPRCL(X)$, respectively.

Theorem 9. *Every (1,2)-regular-generalized-closed set is (1,2)-gpr-closed.*

Proof. Let $A \subset X$ be a (1,2)-rg-closed set. Obviously (1,2)-$pcl(A) \subset$(1,2)-$\alpha cl(A)$. Hence, A is (1,2)-gpr-closed. □

Remark 10. The converse of the above theorem need not always be true as shown by the following example.

Example 11. Let $X = \{a, b, c, d\}$ and $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Then we have $\tau_2 = \{\emptyset, X, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$, (1,2)-$\alpha O(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and (1,2)-$RO(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$.

Remark 12. ¿From Example 11 we can say that

(i) (1,2)-gpr-closed and (1,2)-sg-closed sets are independent of each other.
(a) $\{a, d\} \in$(1,2)-$GPRCL(X)$ but $\{a, d\} \notin$(1,2)-$SGCL(X)$,
(b) $\{b\} \in$(1,2)-$SGCL(X)$ but $\{b\} \notin$(1,2)-$GPRCL(X)$.
(ii) (1,2)-gs-closed and (1,2)-gpr-closed sets are independent of each other.
(a) $\{c\} \in$(1,2)-$GPRCL(X)$ but $\{c\} \notin$(1,2)-$GSCL(X)$,
(b) $\{b\} \in$(1,2)-$GSCL(X)$ but $\{b\} \notin$(1,2)-$GPRCL(X)$.
(iii) (1,2)-gsp-closed and (1,2)-gpr-closed sets are independent of each other.
(a) $\{b\} \in$(1,2)-$GSPCL(X)$ but $\{b\} \notin$(1,2)-$GPRCL(X)$,
(b) $\{a, b, c\} \in$(1,2)-$GPRCL(X)$ but $\{a, b, c\} \notin$(1,2)-$GSPCL(X)$.

Theorem 13. *Every (1,2)-gp-closed set is (1,2)-gpr-closed.*

Proof. Let A be a (1,2)-gp-closed in a space X. Then (1,2)-$pcl(A) \subset U$ whenever $A \subset U$ and $U \in$(1,2)-$\alpha O(X)$. Since (1,2)-$RO(X) \subset$(1,2)-$\alpha O(X)$, the proof is completed. □

Remark 14. Converse of Theorem 13 need not always be true. In Example 11, $\{a, b, c\} \in$(1,2)-$GPRCL(X)$ but $\{a, b, c\} \in$(1,2)-$GPCL(X)$.

Theorem 15. *Let A be a (1,2)-regular-open subset of a space X. Then*
(1) If A is (1,2)-regular-open and (1,2)-gpr-closed, then A is (1,2)-pre-closed.
(2) If A is (1,2)-regular-open and (1,2)-gr-closed, then A is (1,2)-α-closed.

Proof. (1): Let A be (1,2)-regular-open and (1,2)-gpr-closed. Then (1,2)-$pcl(A) \subset A$ implies that A is (1,2)-pre-closed.

(2): Since A is (1,2)-regular-open and (1,2)-gr-closed, it is (1,2)-α-open and hence (1,2)-$\alpha cl(A) \subset A$. Thus, A is (1,2)-α-closed. □

Remark 16. The intersection of (1,2)-gpr-closed sets need not be a (1,2)-gpr-closed.

Example 17. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}\}$. We obtain (1,2)-$RO(X) = \{\emptyset, X, \{a\}, \{b\}\}$ and (1,2)-$GPRCL(X) = \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}, \{a, b\}\}$. Then $\{a, b\}$, $\{a, c\} \in$(1,2)-$GPRCL(X)$ but $\{a\} \notin$(1,2)-$GPRCL(X)$.

Remark 18. The union of (1,2)-gpr-closed sets also need not be a (1,2)-gpr-closed as shown in the following example.

Example 19. Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$ and $\tau_2 = \{\phi, X, \{a, b\}\}$. Then $\{a\}$ and $\{b\}$ are (1,2)-gpr-closed but $\{a, b\}$ is not a (1,2)-gpr-closed set

Theorem 20. *If A be a (1,2)-gpr-closed set in X, then (1,2)-pcl$(A) - A$ does not contain any nonempty (1,2)-regular closed set.*

Proof. Let F be a (1,2)-regular-closed set such that $F \subset$(1,2)-*pcl*$(A) - A$. Then $F \subset$(1,2)-*pcl*(A) but F is not a subset of A. On the other hand, $A \subset X - F$ and $X - F$ is (1,2)-regular-open. Also A is (1,2)-gpr-closed implies (1,2)-*pcl*$(A) \subset X - F$ and then $F \subset X -$(1,2)-*pcl*(A). This is a contradiction. □

Remark 21. The converse of Theorem 20 need not always be true. In Example 11, $\{c\}$ is not (1,2)-gpr-closed, while (1,2)-*spcl*$\{c\}) - \{c\} = \{d\}$, which does not contain any non-empty (1,2)-regular-closed set.

Theorem 22. *Let A be a (1,2)-gpr-closed set in X. Then A is (1,2)-pre-closed if and only if (1,2)-pcl$(A) - A$ is (1,2)-regular-closed.*

Proof. **Necessity:** Let A be (1,2)-pre-closed. Then (1,2)-*pcl*$(A) = A$. Hence, (1,2)-*pcl*$(A) - A = \emptyset$ is a regular closed set.

Sufficiently: Suppose (1,2)-*pcl*$(A) - A$ is (1,2)-regular-closed. As A is (1,2)-gpr-closed by Theorem 20, (1,2)-*pcl*$(A) - A = \emptyset$. Hence, A is (1,2)-pre-closed. □

Definition 23. *Let X be a bitopological space and $A \subset X$ and $x \in X$. Then x is said to be a (1,2)-pre-limit point of A if every (1,2)-pre-open set containing x contains a point of A different from x.*

Definition 24. *The set of all (1,2)-pre-limit point of A is said to be (1,2)-pre-derived set and is denoted by (1,2)-$D_p(A)$.*

Let us recall that x is a (1,2)-α-limit point of A if every (1,2)-α-open set containing x contains a point of A different from x and the set of all (1,2)-α-limit points of A is set to be (1,2)-α-derived set and is denoted by (1,2)-$D(A)$.

Theorem 25. *Let A and B be (1,2)-gpr-closed in a space X. If*
(i) (1,2)-$D(A) \subset$(1,2)-$D_p(A)$ and
(ii) (1,2)-$D(B) \subset$(1,2)-$D_p(B)$, then $A \cup B$ is (1,2)-gpr-closed.

Proof. For any set $E \subset X$, (1,2)-$D_p(E) \subset$(1,2)-$D(E)$ which implies (1,2)-$D_p(A) \subset$(1,2)-$D(A)$. By assumption (1,2)-$D(A) =$(1,2)-$D_p(A)$ and so (1,2)-$D(B) =$(1,2)-$D_p(B)$. Let $A \cup B \subset U$ and U be (1,2)-regular-open. Since A and B are (1,2)-gpr-closed, then (1,2)-$pcl(A) \subset U$ and (1,2)-$pcl(B) \subset U$. Now (1,2)-$\alpha cl(A \cup B) \subset$(1,2)-$\alpha cl(A) \cup$(1,2)-$\alpha cl(B)$ =(1,2)-$pcl(A) \cup$(1,2)-$pcl(B) \subset U$. But (1,2)-$pcl(A \cup B) \subset$(1,2)-α-$cl(A \cup B) \subset U$. Hence, $A \cup B$ is (1,2)-gpr-closed. □

Theorem 26. *If A is (1,2)-gpr-closed and $A \subset B \subset$(1,2)-$pcl(A)$, then B is (1,2)-gpr-closed.*

Proof. Let $B \subset U$ and U be (1,2)-regular-open. Then $A \subset U$ and hence (1,2)-$pcl(A) \subset U$. By assumption, $B \subset$(1,2)-$pcl(A)$. Then (1,2)-$pcl(B) \subset$(1,2)-$pcl(A) \subset U$. Hence, B is (1,2)-gpr-closed. □

Theorem 27. *For a space X, the following condition are equivalent:*
(i) X is (1,2)-α-hyperconnected.
(ii) Every subset of X is (1,2)-rg-closed and X is connected.

Proof. $(i) \Rightarrow (ii)$: Since X is (1,2)-α-hyperconnected, the only (1,2)-regular-open subset of X are $\emptyset$ and X. Hence every subset of X is (1,2)-rg-closed. Also in [8] it is shown that every (1,2)-hyperconnected space is connected.

$(ii) \Rightarrow (i)$: Let A be a nonempty (1,2)-α-open subset of X. By assumption A is (1,2)-rg-closed. Then A is (1,2)-α-closed. This is a contradiction . □

Definition 28. *(1,2)-gpr closure of a subset A of a space X is denoted by (1,2)-$gprcl(A)$ and is defined as intersection of all (1,2)-gpr-closed sets containing A.*

Proposition 29. *Let A and B be subsets of X. The following hold:*
(i) (1,2)-$gprcl(\emptyset) = \emptyset$ and (1,2)-$gprcl(X) = X$.
(ii) If $A \subset B$, then (1,2)-$gprcl(A) \subset$(1,2)-$gprcl(B)$.
(iii) $A \subset$(1,2)-$gprcl(A)$.
(iv) (1,2)-$gprcl(A \cup B) \supset$(1,2)-$gprcl(A) \cup$(1,2)-$gprcl(B)$.

Remark 30. If $A \subset X$ is (1,2)-gpr-closed, then (1,2)-$gprcl(A) = A$. But the converse need not always be true can be shown by Example 11 in which (1,2)-$gprcl\{a\} = \{a,b,d\} \cap \{a,b,c\} = \{a,b\}$, but $\{a,b\}$ is not a (1,2)-gpr-closed set.

Proposition 31. *Let A and B be subset of Y. Then (1,2)-$gprcl(A \cap B) \subset$(1,2)-$gprcl(A) \cap$(1,2)-$gprcl(B)$.*

Proposition 32. *Let A be a subset of X. Then $x \in$(1,2)-$gprcl(A)$ iff $V \cap A \neq \emptyset$ for every (1,2)-gpr-open set V containing x.*

Proof. **Necessity:** Suppose that there exists a (1,2)-gpr-open set V containing x such that $V \cap A = \emptyset$. Then $A \subset X - V$ and $X - V$ is (1,2)-gpr-closed. So (1,2)-$gprcl(A) \subset X - V$ implies $x \notin$(1,2)-$gprcl(A)$. This is a contradiction.

Sufficeency: Suppose $x \notin$(1,2)-$gprcl(A)$. Then there exists a (1,2)-gpr-closed subset F containing A such that $x \notin F$. Then $x \in X - F$ and $X - F$ is (1,2)-gpr-open and $(X - F) \cap A = \emptyset$. This is a contradiction. □

3. (1,2)-GENERALIZED PRE-REGULAR-OPEN SETS

Definition 33. *A set A of X is called (1,2)-generalized pre-regular open (briefly (1,2)-gpr-open) if its complement is (1,2)-generalized pre-regular closed.*

Remark 34. For a subset A of a space X, (1,2)-$pcl(X - A) = X-$(1,2)-$pint(A)$.

Theorem 35. *Let A be a subset of a space X. Then A is a (1,2)-gpr-open iff $F \subset$(1,2)-$pint(A)$ whenever F is a (1,2)-regular closed set and $F \subset A$.*

Proof. **Necessity:** Let A be a (1,2)-gpr-open set. Let F be (1,2)-regular-closed and $F \subset A$. Then $X - A \subset X - F$ and $X - F$ is (1,2)-regular-open and $X - A$ is (1,2)-gpr-closed. Since (1,2)-$pcl(X - A) \subset X - F$, then $X-$(1,2)-$pint(A) \subset X - F$ and so $F \subset$(1,2)-$pint(A)$.

Sufficiency: Suppose F is (1,2)-regular-closed set and $F \subset A$ with $F \subset pint(A)$. Let $X - A \subset U$ where U is (1,2)-regular-open. Then $X - U \subset A$ and hence by assumption, $X - U \subset$(1,2)-$pint(A)$. We have $X - pint(A) \subset U$. Since (1,2)-$pcl(X - A) \subset U$, $X - A$ is (1,2)-gpr-closed and hence A is (1,2)-gpr-open. □

Theorem 36. *If (1,2)-$pint(A) \subset B \subset A$ and A is (1,2)-gpr-open, then B is (1,2)-gpr-open.*

Proof. Since (1,2)-$pint(A) \subset B \subset A$, then $X - A \subset X - B \subset X-$(1,2)-$pint(A)$, that is $X - A \subset X - B \subset$(1,2)-$pcl(X - A)$. Then by Theorem 26, $X - B$ is (1,2)-gpr-closed. Hence B is (1,2)-gpr-open. □

Theorem 37. *If $A \subset X$ is (1,2)-gpr-closed, then (1,2)-$pcl(A) - A$ is (1,2)-gpr-open.*

Proof. Let A be a (1,2)-gpr-closed. Let F be a (1,2)-regular-closed set such that $F \subset$ (1,2)-$pcl(A) - A$. Then by Theorem 20, (1,2)-$pcl(A) - A = \emptyset$ implies $F = \emptyset$. Hence $F \subset$(1,2)-$pint((1,2)\text{-}pcl(A) - A)$. This shows that (1,2)-$pcl(A) - A$ is (1,2)-gpr-open. □

Remark 38. The converse of Theorem 37 need not always be true as shown by Example 11 in which $A = \{a, c\}$, (1,2)-$pcl(A) - A = \{d\}$ which is (1,2)-gpr-open but A is not (1,2)-gpr-closed.

Lemma 39. *Let X be a space and $x \in X$. Then $X - \{x\}$ is either (1,2)-regular-open or (1,2)-gpr-closed.*

Proof. If $X - \{x\}$ is not (1,2)-regular-open, then the only open set containing $X - \{x\}$ is X and hence $X - \{x\}$ is (1,2)-gpr-closed. □

Theorem 40. *If (1,2)-$PO(X) =$(1,2)-$PCL(X)$, then (1,2)-$GPRCL(X) = P(X)$.*

Proof. Suppose that $A \subset X$ and $A \subset F$ and F is (1,2)-regular-open in X. Since F is (1,2)-pre-open,then by assumption F is also (1,2)-pre-closed. Hence (1,2)-$pcl(A) \subset F$ and so A is (1,2)-gpr-closed. Hence (1,2)-$GPRCL(X) = P(X)$. □

Lemma 41. *For a subset A of a space X, (1,2)-$pcl(A) = A \cup \tau_1\text{-}cl(\tau_1\tau_2\text{-}int(A))$.*

Proof. Since $A \subset$(1,2)-$pcl(A)$, $\tau_1\text{-}cl(\tau_1\tau_2\text{-}int(A)) \subset \tau_1\text{-}cl(\tau_1\tau_2\text{-}int((1,2)\text{-}pcl(A))$. Since (1,2)-$pcl(A)$ is (1,2)-pre-closed, $\tau_1\text{-}cl(\tau_1\tau_2\text{-}int((1,2)\text{-}pcl(A)) \subset$(1,2)-$pcl(A)$. Thus, $A \cup \tau_1\text{-}cl(\tau_1\tau_2\text{-}int(A)) \subset$(1,2)-$pcl(A)$.

Since $\tau_1\tau_2$-$int(A) \subset A$, then τ_1-$cl(\tau_1\tau_2$-$int(A)) \subset \tau_1$-$cl(A)$ and so $A \cup \tau_1$-$cl(\tau_1\tau_2$-$int(A)) \subset A \cup \tau_1$-$cl(A)$. Now $\tau_1\tau_2$-$int(A \cup \tau_1$-$cl(\tau_1\tau_2$-$int(A)) \subset \tau_1\tau_2$-$int(A \cup \tau_1$-$cl(A)) \subset \tau_1\tau_2$-$intA \cup \tau_1\tau_2$-$int(\tau_1$-$cl(A))$. Since $\tau_1\tau_2$-$int(A \cup \tau_1$-$cl(\tau_1\tau_2$-$int(A)) \subset \tau_1\tau_2$-$intA \cup \tau_1$-$cl(A)$, then τ_1-$cl(\tau_1\tau_2$-$int(A \cup \tau_1$-$cl(\tau_1\tau_2$-$int(A))) \subset \tau_1$-$cl(\tau_1\tau_2$-$int(A) \cup \tau_1$-$cl(A) \subset \tau_1$-$cl(\tau_1\tau_2$-$int(A) \cup A$. Hence $A\cup$ τ_1-$cl(\tau_1\tau_2$-$int(A))$ is a (1,2)-pre-closed set. Therefore (1,2)-$pcl(A \cup \tau_1$-$cl(\tau_1\tau_2$-$int(A))) = A \cup \tau_1$-$cl(\tau_1\tau_2$-$int(A))$. Since $A \subset A \cup \tau_1$-$cl(\tau_1\tau_2$-$int(A))$, (1,2)-$pcl(A)$ $\subset$(1,2)-$pcl(A \cup \tau_1$-$cl(\tau_1\tau_2$-$int(A)) = A \cup \tau_1$-$cl(\tau_1\tau_2$-$int(A)$.

Thus, we obtain (1,2)-$pcl(A) = A \cup \tau_1$-$cl(\tau_1\tau_2$-$int(A))$. □

Theorem 42. *If A is (1,2)-semi-open and (1,2)-gpr-closed, then it is (1,2)-rg-closed.*

Proof. Suppose $A \subset F$ where F is (1,2)-regular-open. Since A is (1,2)-gpr-closed, (1,2)-$pcl(A) \subset F$. Since A is (1,2)-semi-open, $A \subset \tau_1\tau_2$-$cl(\tau_1$-$int(A)) \subset \tau_1$-$cl((\tau_1\tau_2$-$int(A))$ implies by Lemma 41, (1,2)-$pcl(A) = A$ and also $A \subset \tau_1$-$cl(A)$ implies τ_1-$cl(\tau_1\tau_2$-$int(A)) \subset \tau_1$-$cl(\tau_1\tau_2$-$int(\tau_1$-$cl(A))$. As $A \subset \tau_1$-$cl(\tau_1\tau_2$-$int(A))$, we have $A \subset \tau_1$-$cl(\tau_1\tau_2$-$int(\tau_1$-$cl(A)$. Hence A is (1,2)-α-closed (by Lemma 3.1 [8]). Then we have (1,2)-$\alpha cl(A)$ =(1,2)-$pcl(A) = A$. Thus, A is (1,2)-rg-closed. □

Theorem 43. *Let X be a bitopological space and A, $B \subset X$. If B is (1,2)-gpr-open and $A \supset$(1,2)-pint(B), then $A \cap B$ is (1,2)-gpr-open.*

Proof. Since B is (1,2)-gpr-open and $A \supset$(1,2)-$pint(B)$, we have (1,2)-$pint(B) \subset A \cap B \subset B$. Hence by Theorem 36 $A \cap B$ is (1,2)-gpr-open. □

Theorem 44. *Let (1,2)-PO(X) be closed under finite intersection. If A and B are (1,2)-gpr-open, then $A \cap B$ is (1,2)-gpr-open.*

Proof. Let $X - (A \cap B) = (X - A) \cup (X - B) \subset F$ where F is regular-open. This implies $X - A \subset F$ and $X - B \subset F$. Since A and B are (1,2)-gpr-open, (1,2)-$pcl(X - A) \subset F$ and (1,2)-$pcl(X - B) \subset F$. So (1,2)-$pcl(X - (A \cap B)) \subset F$. Therefore $A \cap B$ is (1,2)-gpr-open. □

Definition 45. *For any subset $A \subset X$, (1,2)-gprint(A) is defined as the union of all (1,2)-gpr-open sets contained in A.*

Proposition 46. *For a subset A of a bitopological space X, $X-$(1,2)-gprint(A) =(1,2)-gprcl$(X - A)$.*

Proof. Let $x \in X-$(1,2)-$gprint(A)$. Then $x \notin$(1,2)-$gprint(A)$, i.e. every (1,2)-gpr-open set B containing x is is not a subset of A. Then B intersects $X - A$ and so $x \in$(1,2)-$gprcl(X - A)$. Hence $X-$(1,2)-$gprint(A) \subset$(1,2)-$gprcl(X - A)$.

Conversely, let $x \in$(1,2)-$gprcl(X - A)$. Then every (1,2)-gpr-open set B containing x intersects $X - A$ which implies that the (1,2)-gpr-open set B containing x is not a subset of A, i.e. $x \notin$(1,2)-$gprint(A)$ implies $x \in X-$(1,2)-$gprint(A)$ and so (1,2)-$gprcl(X - A) \subset X-$(1,2)-$gprint(A)$. □

4. (1,2)-GPR-CONTINUOUS AND (1,2)-GPR-IRRESOLUTE FUNCTIONS

Definition 47. *A function $f : X \to Y$ is called (1,2)-gpr-continuous ((1,2)-rg-continious)if $f^{-1}(V)$ is (1,2)-gpr-closed ((1,2)-rg-closed)in X for every (1,2)-α-closed set in Y.*

Definition 48. *(i)A function $f : X \to Y$ is called (1,2)-gpr-irresolute if $f^{-1}(V)$ is (1,2)-gpr-closed in X for every (1,2)-gpr-closed set in Y.*

(ii) A function $f : X \to Y$ is called (1,2)-strongly-pre closed if the image of each (1,2)-pre-closed set is in X is a (1,2)-pre-closed set in Y.

Theorem 49. *Every (1,2)-gpr-irresolute function is (1,2)-gpr-continuous.*

The following example shows that the converse is not reversible.

Example 50. Let $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and $\tau_2 = \{\emptyset, X, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. We have (1,2)-$GPRCL(X) = \{\emptyset, X, \{c\}, \{d\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let $Y = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_2 = \{\emptyset, X, \{c\}, \{a, c\}\}$. We have (1,2)-$GPRCL(X) = \{\emptyset, X, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Then the function $f : X \to Y$, defined as $f(a) = a$, $f(b) = c$, $f(c) = b$ and $f(d) = d$, is (1,2)-gpr-continuous but not (1,2)-gpr-irresolute.

Remark 51. Every (1,2)-αg-closed set is (1,2)-gpr-closed.

Theorem 52. *(i) If $f : X \to Y$ is (1,2)-αg-continuous, then f is (1,2)-gpr-continuous.*

(ii) If $f : X \to Y$ is (1,2)-rg-continuous, then f is (1,2)-gpr-continuous.

Remark 53. The converse of Theorem 52 need not always be true.

Remark 54. Let $X = \{a, b, c, d\}\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and $\tau_2 = \{\emptyset, X, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Then we obtain (1,2)-$RGCL(X) = \{\emptyset, X, \{d\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$, (1,2)-$\alpha GCL(X) = \{\emptyset, X\{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$, (1,2)-$GPRCL(X) = \{\emptyset, X, \{c\}, \{d\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Let $Y = \{p, q, r\}$, $\tau_1 = \{\emptyset, Y, \{p\}, \{q\}, \{p, q\}, \{p, r\}\}$ and $\tau_2 = \{\emptyset, Y, \{a\}\}$. Then $\tau_1\tau_2$-$CL(Y) = \{\emptyset, Y, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}\}$ and (1,2)-$\alpha cl(Y) = \{\emptyset, Y, \{q\}, \{r\}, \{q, r\}, \{p, r\}\}$. The function $f : X \to Y$, defined as $f(a) = f(b) = p$, $f(c) = q$, $f(d) = r$, is (1,2)-gpr-continuous but it is neither (1,2)-rg-continuous nor (1,2)-αg-continuous.

Proposition 55. *Let $f : X \to Y$ and $g : Y \to Z$ be any two functions.*

(i) $g \circ f$ is (1,2)-gpr-continuous if g is (1,2)-gpr-continuous and f is (1,2)-gpr-continuous.

(ii) $g \circ f$ is (1,2)-gpr-irresolute if g is (1,2)-gpr-irresoulte and f is (1,2)-gpr-irresolute.

(iii) $g \circ f$ is (1,2)-gpr-continuous, if g is (1,2)-gpr-continuous and f is (1,2)-gpr-irresolute.

Theorem 56. *If $f : X \to Y$ is (1,2)-gpr-continuous, then $f((1,2)\text{-}gpcl(A)) \subset (1,2)\text{-}\alpha cl(f(A))$ for every subset A of X.*

Proof. Let A be a subset of X. Since (1,2)-$\alpha cl(f(A))$ is (1,2)-α-closed in Y, then $f^{-1}((1,2)\text{-}\alpha cl(f(A))$ is (1,2)-gpr-closed. Since $A \subset f^{-1}(f(A)) \subset f^{-1}((1,2)\text{-}\alpha cl(f(A)))$, then (1,2)-$gprcl(A) \subset f^{-1}((1,2)\text{-}\alpha cl(f(A))$ and hence $f((1,2)\text{-}gprcl(A)) \subset (1,2)\text{-}\alpha cl(f(A))$. □

Definition 57. *A function $f : X \to Y$ is said to be (1,2)-regular irresolute if for any (1,2)-regular-open set V of $f^{-1}(V)$ is (1,2)-regular-open.*

Theorem 58. *Let $f : X \to Y$ be (1,2)-regular irresolute and (1,2)-strongly-pre-closed, then for every (1,2)-gpr-closed set A of X, $f(A)$ is (1,2)-gpr-closed in Y.*

Proof. Let A be (1,2)-gpr-closed in X. Let $f(A) \subset U$ where U is (1,2)-regular-open. Then $A \subset f^{-1}(U)$. Since f is (1,2)-regular-irresolute, $f^{-1}(U)$ is (1,2)-regular-open in X. So, (1,2)-$pcl(A) \subset f^{-1}(U)$. Then $f((1,2)\text{-}pcl(A)) \subset U$. Since f is (1,2)-pre-closed, $(1,2)\text{-}pcl(f(A)) \subset (1,2)\text{-}pcl(f(1,2)\text{-}pcl(A)) = f((1,2)\text{-}pcl(A)) \subset U$. Thus, $f(A)$ is (1,2)-gpr-closed in Y. □

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