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# THE RELATIVE ROTHBERGER PROPERTY AND PIXLEY ROY SPACES

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**Abstract.** We characterize subsets of the real line which have strong measure zero in all finite powers in terms of a selection property of the correspsponding Pixley-Roy space.

## INTRODUCTION

For a space $X$ with topology $\tau$, $\mathsf{PR}(X) = \{F \subset X : F \text{ finite}\}$. For $F \in \mathsf{PR}(X)$, and for $V$ an open subset of $X$ with $F \subset V$, $[F, V] = \{G \in \mathsf{PR}(X) : F \subseteq G \subset V\}$. Then $\mathcal{B} = \{[F, V] : F \in \mathsf{PR}(X) \text{ and } F \subset V \subseteq X \text{ open}\}$ is a basis for a topology, denoted $\mathsf{PR}(\tau)$, on $\mathsf{PR}(X)$. We shall call $\mathcal{B}$ the *standard basis* for this topology. The space $(\mathsf{PR}(X), \mathsf{PR}(\tau))$ is the *Pixley-Roy space of* $(X, \tau)$.

The symbol $\mathsf{C}_p(X)$ denotes the space of continuous real-valued functions with domain $X$, topologized by the topology of pointwise convergence. For Tychonoff spaces there is an extensive "duality" theory between the space $X$ and the corresponding hyperspace $\mathsf{C}_p(X)$. Generally speaking, covering properties of finite powers of $X$ correspond to properties of the closure operator of $\mathsf{C}_p(X)$. Examples of duality results related to selection principles can be found in [9] and [10].

The "duality" theory between the space $X$ and the corresponding hyperspace $\mathsf{PR}(X)$ is less well developed. Some work has been done on the relation between selection principles holding in a space $X$, and selection principles holding in the Pixley-Roy space $\mathsf{PR}(X)$ - for example in [7] and [12]. Experience suggests that for appropriate $X$ there should be a rich duality theory between $X$ and $\mathsf{PR}(X)$. In this paper we will give some more evidence of this by exploring the classical strong measure zero property of Borel in the context of Pixley-Roy spaces. The result given here belongs to the area of relative selection principles as were considered in [1], [2], [5] and [6].

## STRONG MEASURE ZERO AND ROTHBERGER'S PROPERTY

A metric space $(X, d)$ has strong measure zero if there is for each sequence $(\epsilon_n : n \in \mathbb{N})$ of positive real numbers also a partition $X = \cup_{n\in\mathbb{N}} X_n$ where for each $n$ the diameter of $X_n$ is less than $\epsilon_n$. Borel conjectured that only countable and finite sets of real numbers have strong measure zero. Sierpiński early on proved that the Continuum Hypothesis can be used to construct uncountable sets of real

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numbers of strong measure zero. A few years later K. Gödel proved that the Continuum Hypothesis is consistent relative to the consistency of classical mathematics: Consequently Sierpiński's result implies that it is relatively consistent that Borel's Conjecture is false. In 1976 Laver published a proof that it is also relatively consistent that the only sets of real numbers which have strong measure zero are the countable and finite sets of real numbers. Thus, even in the case of the real line Borel's Conjecture is not decidable by the traditional Zermelo Fraenkel Axioms of Mathematics.

In his study of Borel's Conjecture F. Rothberger considered a special case of the following selection principle:

> $\mathsf{S}_1(\mathcal{A},\mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each $n$ we have $B_n \in A_n$ and $\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Let $\mathcal{O}_X$ be the collection of open covers of $X$. Rothberger proved that if a metrizable space $X$ has the property $\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X)$ then it has strong measure zero. But Rothberger also proved that the converse is not true. More specifically, the Continuum Hypothesis implies that there are sets of real numbers which have strong measure zero, but do not have the Rothberger selection property.

The secret to describing strong measure zeroness for sets of real numbers in terms of the selection property $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ was to *relativize* the selection property. Let $Y \subset X$ be a subspace of $X$. We let $\mathcal{O}_X$ denote the open covers of $X$, and $\mathcal{O}_{XY}$, the covers of $Y$ by sets open in $X$. An open cover $\mathcal{U}$ of a space is said to be an $\omega$–cover if the space itself is not an element of $\mathcal{U}$, but for each finite subset $F$ of $X$ tehre is a $U \in \mathcal{U}$ such that $F \subseteq U$. The symbol $\Omega_X$ denotes the collection of $\omega$–covers of the space $X$, and $\Omega_{XY}$ denotes the collection of $\omega$-covers of $Y$ by sets open in $X$. The following was proved in [11]:

**Theorem 1.** *Let $X$ be a metrizable $\sigma$–compact space. The following are equivalent for a subset $Y$ of $X$:*

1. *In each metric $d$ for $X$, $Y$ has strong measure zero.*
2. $X \models \mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_{XY})$.

And for finite powers: For metric $d$ on $X$, let $d^n$ denote the product metric on $X^n$.

**Theorem 2.** *Let $X$ be a metrizable $\sigma$–compact space. The following are equivalent for a subset $Y$ of $X$:*

1. *For each metric $d$ for $X$, $Y^n$ has strong measure zero with respect to $d^n$.*
2. $X \models \mathsf{S}_1(\Omega_X, \Omega_{XY})$.

**Games, and Ramsey Theory.** The following game, $\mathsf{G}_1(\mathcal{A},\mathcal{B})$, is naturally associated with $\mathsf{S}_1(\mathcal{A},\mathcal{B})$: In the $n$-th inning ONE chooses an $O_n \in \mathcal{A}$, and TWO responds with a $T_n \in O_n$. A play $O_1, T_1, \cdots, O_n, T_n, \cdots$ is won by TWO if $\{T_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$; else, ONE wins. It is evident that if ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$, then $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ holds. The converse is not always true. When it is true, it presents a powerful tool in analysing the properties of $\mathcal{A}$ and $\mathcal{B}$.

Also Ramsey theory is related to the selection principle $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$. For a set $S$ and for $n \in \mathbb{N}$ the symbol $[S]^n$ denotes $\{F \subseteq S : |F| = n\}$. The symbol

$$\mathcal{A} \rightarrow (\mathcal{B})^n_k$$

denotes that for each $A \in \mathcal{A}$ and for each function $f : [A]^n \rightarrow \{1, \cdots, k\}$ there is a $B \subset A$ and an $i \in \{1, \cdots, k\}$ such that $B \in \mathcal{B}$, and $f[F] = i$ for each $F \in [B]^n$.

**Dense families of open sets.** The following notions were introduced and studied in [7] and [12]. They are important in developing a "duality" theory for a space and its Pixley-Roy hyperspace. For a space $X$ let $\mathcal{D}_X$ be the collection of families $\mathcal{U}$ of open sets with $\cup\mathcal{U}$ dense in $X$. Let $\mathcal{D}^\Omega_X$ be the collection of elements $\mathcal{U}$ of $\mathcal{D}_X$ such that no element of $\mathcal{U}$ is a dense subset of $X$ and for each finite set $\mathcal{F}$ of nonempty open subsets of the space there is an $A \in \mathcal{U}$ such that for each $F \in \mathcal{F}$, $F \cap A \neq \emptyset$.

These notions can evidently be relativized as follows: Let $Y$ be a subset of $X$. Then $\mathcal{D}_{XY}$ denotes the families $\mathcal{U}$ of open subsets of $X$ for which $(\cup\mathcal{U}) \cap Y$ is a dense subset of $Y$. And $\mathcal{D}^\Omega_{XY}$ denotes the collection of $\mathcal{U} \in \mathcal{D}_{XY}$ such that no element of $\mathcal{U}$ is dense relative to $Y$, but for each finite family $\mathcal{F}$ of nonempty open subsets of $Y$ there is a $U \in \mathcal{U}$ such that for each $F \in \mathcal{F}$ we have $F \cap U \neq \emptyset$.

In [12] we examined among other things the selection principle $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ for the cases when $\mathcal{A}$ and $\mathcal{B}$ are members of $\{\mathcal{D}_X, \mathcal{D}^\Omega_X\}$.

The proofs of Theorems 3 and 4 are very similar to those of Theorems 14 and 23 of [12] and will be omitted here:

**Theorem 3.** *For a topological space $X$ and a subspace $Y$ of $X$, the following are equivalent:*

1. $X \models \mathsf{S}_1(\mathcal{D}_X, \mathcal{D}_{XY})$.
2. *ONE has no winning strategy in the game* $\mathsf{G}_1(\mathcal{D}_X, \mathcal{D}_{XY})$.

**Theorem 4.** *Let $X$ be a space for which each element of $\mathcal{D}^\Omega_X$ has a countable subset in $\mathcal{D}^\Omega_X$. Then for subspace $Y$ of $X$ the following are equivalent:*

1. $X \models \mathsf{S}_1(\mathcal{D}_X, \mathcal{D}_{XY})$.
2. *For each $k$,* $\mathcal{D}^\Omega_X \rightarrow (\mathcal{D}_{XY})^2_k$.

**Pixley-Roy spaces, Rothberger's property and strong measure zero.** There is a strong connection between the theory $\mathcal{O}_X$ and the theory of $\mathcal{D}_{\mathsf{PR}(X)}$. For the Rothberger selection principle the following is known:

**Theorem 5.** *For a metrizable Lindelöf space $X$ the following are equivalent:*

1. $X \models \mathsf{S}_1(\Omega_X, \Omega_X)$.
2. *For each $n \in \mathbb{N}$,* $(X^n \models \mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X))$
3. $\mathsf{PR}(X) \models \mathsf{S}_1(\mathcal{O}_{\mathsf{PR}(X)}, \mathcal{D}_{\mathsf{PR}(X)})$
4. $\mathsf{PR}(X) \models \mathsf{S}_1(\mathcal{D}_{\mathsf{PR}(X)}, \mathcal{D}_{\mathsf{PR}(X)})$
5. $\mathsf{PR}(X) \models \mathsf{S}_1(\mathcal{D}_{\Omega\,\mathsf{PR}(X)}, \mathcal{D}_{\mathsf{PR}(X)})$
6. $\mathsf{PR}(X) \models \mathsf{S}_1(\mathcal{D}_{\Omega\,\mathsf{PR}(X)}, \mathcal{D}_{\Omega\,\mathsf{PR}(X)})$

In this theorem, 1 $\Leftrightarrow$ 2 is due to Masami Sakai [8], 2 $\Leftrightarrow$ 3 is due to Peg Daniels [7], and the remaining equivalences were proved in [12]. More examples of such "duality" between $X$ and $\mathsf{PR}(X)$ can be found in [7] and [12]. Now we connect the Pixley-Roy topology with the notion of strong measure zero metric spaces. Our main result is

**Theorem 6.** *Let $X$ be a $\sigma$-compact metrizable space. The following are equivalent for subspace $Y$ of $X$:*

1. *For each $n$, $Y^n$ has strong measure zero in all metrizations of $X^n$.*
2. $X \models \mathsf{S}_1(\Omega_X, \Omega_{XY})$
3. $\mathsf{PR}(X) \models \mathsf{S}_1(\mathcal{D}_{\mathsf{PR}(X)}, \mathcal{D}_{\mathsf{PR}(X)\mathsf{PR}(Y)})$

In the course of the proof of Theorem 6 we will use the following lemma:

**Lemma 7.** *Let $(X, d)$ be a metric space and let $Y \subset X$ be such that $X \models \mathsf{S}_1(\Omega_X, \Omega_{XY})$. Then for any $\mathsf{F}_\sigma$-subset $F$ of $X$ we have $F \models \mathsf{S}_1(\Omega_F, \Omega_{F(F\cap Y)})$.*

**Proof** : For let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open (in $F$) $\omega$-covers of $F$. Since $F$ is an $\mathsf{F}_\sigma$-set, write $F = \cup_{n\in\mathbb{N}} F_n$ where for each $n$ we have $F_n \subseteq F_{n+1}$ and $F_n$ is closed in $X$. For each $n$, for each $U \in \mathcal{U}_n$, choos a $V_U \subset X$ open in $X$ and with $V_U \cap F = U$. Then define for each $n$ $\mathcal{V}_n = \{V_U \cap (X \setminus F_n) : U \in \mathcal{U}_n\}$. Each $\mathcal{V}_n$ is an $\omega$-cover of $X$ by sets open in $X$. Applying $\mathsf{S}_1(\Omega_X, \Omega_Y)$ to this sequence we choose for each $n$ a $W_n \in \mathcal{V}_n$ such that $\{W_n : n \in \mathbb{N}\}$ is an $\omega$-cover of $Y$. For each $n$ choose a $U_n \in \mathcal{U}_n$ such that $W_n = V_{U_n} \cup (X \setminus F_n)$. Then for each $n$ a $U_n \in \mathcal{U}_n$, and $\{U_n : n \in \mathbb{N}\}$ is an $\omega$-cover of $Y \cap F$ by sets open in $F$. $\diamondsuit$

**Proof** : **(of Theorem 6)**: 1 $\Leftrightarrow$ 2: This part uses the $\sigma$-compactness of $X$, and was proved in [11]. For 2 $\Leftrightarrow$ 3 we only need that the metric space $X$ is Lindelöf. 2 $\Rightarrow$ 3: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of elements of $\mathcal{D}_{\mathsf{PR}(X)}$. Since $X$ is $\sigma$-compact, $\mathsf{PR}(X)$ has the countable chain condition and we may thus assume each $\mathcal{U}_n$ is countable. Moreover, by refining $\mathcal{U}_n$ if necessary, we may assume each element of $\mathcal{U}_n$ is an element of the standard basis $\mathcal{B}$.

Now let $(B_n : n < \infty)$ bijectively enumerate the set of finite unions of elements from some countable basis for $X$. If each $[F, U] \in \mathcal{U}_n$ is replaced by countably many $[F, B_m]$ such that for each $m$ we have $F \subseteq B_m \subseteq \overline{B}_m \subseteq U$, then the refinement of $\mathcal{U}_n$ thus obtained is also an element of $\mathcal{D}_{\mathsf{PR}(X)}$. Thus, we may assume that each element of $\mathcal{U}_n$ is of the form $[F, B_m]$, and that for each $[S, B_m] \in \mathcal{U}_n$, the set $\{B_j : S \subseteq B_j \subseteq \overline{B}_j \subseteq B_m\}$ is an $\omega$-cover of $B_m$.

For each $n$ enumerate $\mathcal{U}_n$ bijectively as $\{[F^n_m, B^n_m] : m \in \mathbb{N}\}$. Also, write $\mathbb{N} = \cup_{n\in\mathbb{N}} Y_n$ so that each $Y_n$ is infinite and for $m < n$ we have $Y_m \cap Y_n = \emptyset$.

Fix $n$ and for each $m \in Y_n$ write $\mathcal{V}_m = \{B^m_k : \overline{B}^m_k \subseteq B_n\}$. Then $(\mathcal{V}_m : m \in Y_n)$ is a sequence of $\omega$-covers of $B_n$. Since $B_n$ is an open set in metrizable space $X$, it is an $\mathsf{F}_\sigma$-set. Apply, by Lemma 7, $\mathsf{S}_1(\Omega_{B_n}, \Omega_{B_n(B_n\cap Y)})$ to find for each $n \in Y_m$ a $V_m \in \mathcal{V}_m$ such that $\{V_m : m \in Y_n\}$ is an $\omega$-cover of $B_n \cap Y$. For each $m \in Y_n$ choose a $k_m$ such that $V_m = B^m_{k_m}$.

To see that $\{[F^m_{k_m}, B^m_{k_m}] : m \in \mathbb{N}\}$ is in $\mathcal{D}_{\mathsf{PR}(X)\,\mathsf{PR}(Y)}$, consider a basic open set $[F, U]$ with $F \subset Y$. Choose an $n$ such that $F \subseteq B_n \subseteq U$. Then choose an $m \in Y_n$ with $F \subseteq B^m_{k_m}$. Then $[F, U] \cap [S^m_{k_m}, B^m_{k_m}] \neq \emptyset$. This completes the proof of 1 $\Rightarrow$ 2.

$3 \Rightarrow 2$: Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of $\omega$-covers of $X$. For each $n$ put $\mathcal{U}_n = \{[F,U] : F \in \mathsf{PR}(X) \text{ and } F \subset U \in \mathcal{V}_n\}$. Then each $\mathcal{U}_n$ is a member of $\mathcal{D}_{\mathsf{PR}(X)}$. Apply $\mathsf{S}_1(\mathcal{D}_{\mathsf{PR}(X)}, \mathcal{D}_{\mathsf{PR}(X)\,\mathsf{PR}(Y)})$ to choose for each $n$ an $[F_n, U_n] \in \mathcal{U}_n$ so that $\{[F_n, U_n] : n \in \mathbb{N}\} \in \mathcal{D}_{\mathsf{PR}(X)\,\mathsf{PR}(Y)}$. Consider any finite subset $F$ of $Y$, and an open set $U$ with $F \subset U$. Then choose an $n$ such that $[F,U] \cap [F_n, U_n] \neq \emptyset$. It follows that $F \subseteq U_n$. Consequently, $\{U_n : n \in \mathbb{N}\}$ is a member of $\Omega_{XY}$. $\diamondsuit$

Now using the general theory for $\mathcal{D}$ as outlined before, we obtain the following corollaries.

**Corollary 8.** *For metrizable Lindelöf space $X$ and subspace $Y$ the following are equivalent:*

1. $X \models \mathsf{S}_1(\Omega_X, \Omega_{XY})$.
2. $\mathsf{PR}(X) \models \mathsf{S}_1(\mathcal{D}_{\mathsf{PR}(X)}, \mathcal{D}_{\mathsf{PR}(X)\mathsf{PR}(Y)})$.
3. *ONE has no winning strategy in* $\mathsf{G}_1(\mathcal{D}_{\mathsf{PR}(X)}, \mathcal{D}_{\mathsf{PR}(X),\mathsf{PR}(Y)})$.
4. *For each $n$,* $\mathcal{D}^{\Omega}_{\mathsf{PR}(X)} \rightarrow (\mathcal{D}_{\mathsf{PR}(X),\mathsf{PR}(Y)})^2_n$.

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