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THE ANTIAUTOMORPHISMS OF SIMPLE FINITE-DIMENSIONAL TERNARY ALGEBRAS

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Dedicated to Academician Ǵorǵi Čupona

Abstract. In this paper we consider antiautomorphisms of simple finite-dimensional ternary algebras.

1. Introduction

A classification of antiautomorphisms is motivated by a study of antipodes in Hopf $(2, 3)$-algebras ([Z]). It is shown in [Z] that Hopf $(2, 3)$-algebra can be embedded in the usual Hopf algebra. The structure of semisimple finite-dimensional Hopf algebras H, which have only one the irreducible non-one-dimensional representation, we found in [A]. This describtion is based on a classification of the antipodes in H. The results of the present paper can be used for analogous problems for $(2, 3)$-algebras.

In [N] and [N$_1$] there is found the structure of simple (1)-artinian $(2, n)$-rings in two cases, when the considered $(2, n)$-ring consists or not proper $(1, n)$-ideals.

In the first case, the ring R is isomorphic to $(2, n)$-ring of square matrices $\Delta^n_m(c, \varphi)$ over a skew field Δ, with a fixed automorphism φ, such that $\varphi^{n-1} = 1_\Delta$ and with central element c, $\varphi(c) = c$, with usual addition and n-ary multiplication

$$x_1 \cdots x_n = cx_1x_2^{\varphi^{n-2}} \cdots x_{n-1}^{\varphi}x_n,$$

for $x_1, \cdots , x_n \in \Delta_m$.

In the second case, the ring R has the form

$$Hom_k(V_1, V_2) \oplus Hom_k(V_2, V_3) \oplus \cdots \oplus Hom_k(V_{n-1}, V_1),$$

where k is a skew field not necessarily commutative, V_i, $1 \leqslant i \leqslant n - 1$, finite-dimensional (left) vector-spaces over k, $d_i = \dim_k V_i$. In the matrix language,

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$(2,n)$- ring R is representative as the ring of block-matrices of the form

$$\begin{pmatrix} & & & & d_1\{\overbrace{[*]}^{d_{n-1}} \\ d_2\{\overbrace{[*]}^{d_1} & & 0 & & \\ & d_3\{\overbrace{[*]}^{d_2} & & & \\ & & \ddots & & \\ & 0 & & d_{n-1}\{\overbrace{[*]}^{d_{n-2}} & \end{pmatrix}$$

In particular, for $n=3$, we consider $(2,3)$-rings R together with the multiplication

$$(\lambda x_1)x_2x_3 = x_1(\lambda x_2)x_3 = x_1x_2(\lambda x_3) = \lambda(x_1x_2x_3),$$

for each $\lambda \in k$. We also assume that k is an algebraically closed field.

In this paper, in the first case we consider antiautomorphisms of order 2 or involutions

$$\gamma : Mat(m,k) \to Mat(m,k),$$

defined by

$$\gamma(x) = \mu^{-1} z^t x^t z^{-1},$$

where $\mu = \pm 1$ and z is invertible matrix from $Mat(m,k)$ (T.2.1)

In the second case, we find antiautomorphisms in two cases, as:

$$\gamma \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} = \begin{pmatrix} 0 & {}^tA^{-1t}X^tB \\ {}^tB^{-1t}Y^tA & 0 \end{pmatrix}$$

where

$${}^tA = \lambda A, \quad {}^tB = \lambda B$$

and $\lambda = \pm 1$ (T.3.1) and

$$\gamma \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda^{-1}A^{-1t}Y^tA \\ \lambda^tA^{-1t}XA & 0 \end{pmatrix}$$

(T.3.2.).

2. Antiautomorphisms of (2,3)-algebras $R = \Delta^3_m(c,\varphi)$

In this section, we consider antiautomorphisms of $(2,3)$-algebra $R = \Delta^3_m(c,\varphi)$ ([N]). Note that as a vector space

$$\Delta^3_m(c,\varphi) = Mat(m,k),$$

where $k = \Delta$, with ternary multiplication

$$x_1x_2x_3 = cx_1x_2^{\varphi}x_3, \quad \varphi^2 = 1$$

and φ is an automorphism of the ring $Mat(m,k)$, c is a central element, such that $\varphi(c) = c$. Then, there exists a matrix $P \in Mat(m,k)$, such that $X^{\varphi} = PXP^{-1}$and $P^2 = \lambda E$.

Let characteristic of the field k is not equal to 2 and k is algebraically closed. Then, the polynomial $t^2 - \lambda$ has no multiple roots. Let ς and $-\varsigma$ be two different roots of the polynomial. Then, there exists an invertible matrix S, such that

$$SPS^{-1} = \begin{pmatrix} \varsigma E_s & 0 \\ 0 & -\varsigma E_{m-s} \end{pmatrix}.$$

In this case

$${}^tS^{-1t}P^tS = \begin{pmatrix} \varsigma E_s & 0 \\ 0 & -\varsigma E_{m-s} \end{pmatrix}.$$

Put

$$Q = {}^tS \cdot S.$$

Then, $QPQ^{-1} = {}^tP$ and ${}^tQ = Q$. Hence,

$$Q^{-1t}PQ = P.$$

Consider the map

$$\gamma : Mat(m,k) \to Mat(m,k),$$

defined by the rull

$$\gamma(X) = Q^{-1t}XQ.$$

We shall show that γ is an antiautomorphism of $(2,3)$-algebra R, i.e.

$$\gamma(cX_1PX_2P^{-1}X_3) = c\gamma(X_3)P\gamma(X_2)P^{-1}\gamma(X_1).$$

We have

$$\begin{aligned} \gamma(cX_1PX_2P^{-1}X_3) &= Q^{-1t}(cX_1PX_2P^{-1}X_3)Q \\ &= cQ^{-1t}X_3{}^tP^{-1t}X_2{}^tP^tX_1Q \\ &= cQ^{-1t}X_3QPQ^{-1t}X_2QP^{-1}Q^{-1t}X_1Q \end{aligned}$$

or

$${}^tP^{-1t}X_2{}^tP = QPQ^{-1t}X_2QP^{-1}Q^{-1} = {}^tP^tX_2{}^tP^{-1}.$$

But,

$$P^2 = \lambda E$$

and so

$$P^{-1} = \lambda^{-1}P, \quad \lambda^{-1t}P^tX_2{}^tP = {}^tP^tX_2\lambda^{-1t}P.$$

Therefore, γ is antiautomorphim. If γ_1 is arbitrary antiautomorphism, then $\gamma^{-1} \circ \gamma_1 = \alpha$ is an automorphism and therefore

$$\gamma_1 = \gamma \circ \alpha.$$

So, an arbitrary antiautomorphism γ_1 has the form

$$\gamma_1(X) = Q^{-1t}\alpha(X)Q.$$

Then, there exists an element $D \in Mat(m,k)$, such that

$$\alpha(X) = \alpha(E)D^{-1}XD$$

([N]). So,

$$\gamma_1(X) = Q^{-1t}D^tX^tD^{-1t}\alpha(E)Q = Z^tXZ^{-1}\gamma_1(E) \qquad (*)$$

where $Z = Q^{-1t}D$ and $\gamma_1(E) \in GL(m,k)$.

Suppose that $\gamma_1^2 = 1$, i.e.

$$X = \gamma_1(Z^tXZ^{-1}\gamma_1(E)) = Z^t\gamma_1(E)^tZ^{-1}X^tZZ^{-1}\gamma_1(E) \qquad (**)$$

for every X. The coefficients x_{ij} in matrix X on the right side are equal to

$$\sum_{r,s}(Z^t\gamma_1(E)^tZ^{-1})_{ir}x_{rs}({}^tZZ^{-1}\gamma_1(E))_{sj}.$$

Since x_{ij} is arbitrary element,

$$(Z^t\gamma_1(E)^tZ^{-1})_{ir}({}^tZZ^{-1}\gamma_1(E))_{sj} = \delta_{ir}\delta_{sj}. \qquad (***)$$

If

$$({}^tZZ^{-1}\gamma_1(E))_{sj} \neq 0$$

for some $s \neq j$, then

$$(Z^t\gamma_1(E)^tZ^{-1})_{ir} = \delta_{ir},$$

i.e.

$$Z^t\gamma_1(E)^tZ^{-1} = E$$

and ${}^t\gamma_1(E) = Z^{-1t}Z$, i.e. $\gamma_1(E) = Z^tZ^{-1}$.

Analogously, if

$$(Z^t\gamma_1(E)^tZ^{-1})_{ir} \neq 0,$$

for some $i \neq r$, then

$${}^tZZ^{-1}\gamma_1(E)) = E$$

and therefore

$$\gamma_1(E) = Z^tZ^{-1}.$$

In both cases, we obtain in $(*)$

$$\gamma_1(X) = Z^tXZ^{-1}Z^tZ^{-1} = Z^tX^tZ^{-1}.$$

Suppose that both of matrices

$$Z^t\gamma_1(E)^tZ^{-1}, \quad {}^tZZ^{-1}\gamma_1(E)$$

are diagonal. Then, we obtain in $(**)$

$$E = Z^t\gamma_1(E)^tZ^{-1t}ZZ^{-1}\gamma_1(E) = Z^t\gamma_1(E)Z^{-1}\gamma_1(E)$$

or

$$Z^t\gamma_1(E) = \gamma_1(E)^{-1}Z$$

and therefore we obtain in $(**)$

$$X = \gamma_1(E)^{-1}Z^tZ^{-1}X^tZZ^{-1}\gamma_1(E).$$

So

$$\gamma_1(E)^{-1}Z^tZ^{-1} = \mu E$$

and

$$\gamma_1(E) = \mu^{-1}Z^tZ^{-1}.$$

Using $(*)$, we obtain

$$\gamma_1(X) = Z^t X Z^{-1} \mu^{-1} Z^t Z^{-1} = \mu^{-1} Z^t X^t Z^{-1}.$$

From here

$$\begin{aligned} X = \gamma_1^2(X) &= \gamma_1(\mu^{-1} Z^t X^t Z^{-1}) \\ &= \mu^{-1} Z^t (\mu^{-1} Z^t X^t Z^{-1})^t Z^{-1} \\ &= \mu^{-2} Z Z^{-1} X^t Z^t Z^{-1} = \mu^{-2} X \end{aligned}$$

So $\mu^{-2} = 1$ and $\mu = \pm 1$.

Thus we have proved:

Theorem 2.1. *Let γ_1 be an involution. Then, there exists an invertible matrix Z, such that*

$$\gamma_1(X) = \mu^{-1} Z^t X^t Z^{-1}, \quad \mu = \pm 1$$

for each X. Conversely, any map γ_1 from $(*)$ is an involution.

3. Antiautomorphims of $(2,3)$-algebras $R = Hom_k(V_1, V_2) \oplus Hom_k(V_2, V_1)$

We describe antiautomorphisms of $(2,3)$-algebra

$$R = \left\{ \begin{pmatrix} 0 & [\overbrace{\ *\ }^{d_2}]\}d_1 \\ d_2\{[\underbrace{\ *\ }_{d_1}] & 0 \end{pmatrix} \right\}$$

($[N]_1$). For $X = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \in R$, the transpose has the form

$${}^tX = \begin{pmatrix} 0 & {}^tA \\ {}^tB & 0 \end{pmatrix} \in R.$$

So, the map

$$\gamma : X \to {}^tX$$

is an antiautomorphism of $(2,3)$-algebra R. If γ_1 is an arbitrary antiautomorphism, then $\gamma^{-1} \circ \gamma_1 = \alpha$ is an automorphism of $(2,3)$-algebra R. Hence,

$$\gamma_1 = \gamma \circ \alpha$$

i.e. every antiautomorphism of R is a composition of an automorphism α of R and the antiautomorphism γ. The automorphism α of R has form

$$\alpha(X) = PXP^{-1} \tag{A}$$

where

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(d_1 + d_2, k)$$

($[N_1]$). Thus

$$P \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} 0 & Y_1 \\ X_1 & 0 \end{pmatrix}. \tag{A$_1$}$$

Suppose first that the matrix P preserves the blocks of $\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix}$. Then, for each block X, there exists a unique block X_1, such that

$$P\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}P^{-1} = \begin{pmatrix} 0 & 0 \\ X_1 & 0 \end{pmatrix} \tag{B}$$

and for each Y, there exists a unique Y_1, such that

$$P\begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}P^{-1} = \begin{pmatrix} 0 & Y_1 \\ 0 & 0 \end{pmatrix}. \tag{C}$$

The formula (B) can be written as

$$P\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ X_1 & 0 \end{pmatrix}P. \tag{B$_1$}$$

Taking into account the form of the matrix P, in (B_1) we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} = \begin{pmatrix} BX & 0 \\ DX & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ X_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ X_1A & X_1B \end{pmatrix}.$$

Thus $BX = 0 = X_1B$ and $DX = X_1A$.

Since, X and X_1 are arbitrary, it follows that $B = 0$. Similary, (C) implies $C = 0$. So, we have proved:

Proposition 3.1. *If P from (A) defines an automorphism, which preserves blocks in $\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix}$, then $P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where A and B are invertible matrices of sizes d_1 and d_2, respectively.*

Put P in (A$_1$). Then,

$$\alpha\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} \cdot \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & AY \\ BX & 0 \end{pmatrix} \cdot \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} 0 & AYB^{-1} \\ BXA^{-1} & 0 \end{pmatrix}$$

is an automorphism of $(2,3)$-algebra. In this case, an arbitrary antiautomorphism $\gamma_1 = \gamma \circ \alpha$ has the form

$$\gamma_1\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} = \begin{pmatrix} 0 & {}^tA^{-1\,t}X^tB \\ {}^tB^{-1\,t}Y^tA & 0 \end{pmatrix}. \tag{A$_2$}$$

Let us describe the antiautomorphism γ_1, which is an involution, i.e. $\gamma_1^2 = 1$. By (A$_2$)

$$\gamma_1^2\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} = \begin{pmatrix} 0 & {}^tA^{-1}AYB^{-1\,t}B \\ {}^tB^{-1}BXA^{-1\,t}A & 0 \end{pmatrix} = \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix}$$

Then

$${}^tB^{-1}BXA^{-1\,t}A = X, \quad {}^tA^{-1}AYB^{-1\,t}B = Y. \tag{D}$$

The following Lemma is well known:

Lemma 3.1. *Let U, V be square matrices and $UY = YV$, for every Y. Then, $U = \lambda E$, $V = \lambda E$.*

Now, we have proved:

Theorem 3.1. *The antiautomorphism* (A_2) *is an involution if and only if*

$$ {}^tA = \lambda A, \quad {}^tB = \lambda B, \quad \lambda = \pm 1. $$

If P does not preserve blocks, then P is a composition of a permutation of the blocks and considered automorphism with the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ ([N$_1$]). The permutation of the blocks with $d_1 = d_2$ is defined by matrix $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$. So,

$$ P = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}. $$

The antiautomorphism

$$ \gamma_1 = \gamma \circ \alpha. $$

Then,

$$ \begin{aligned} \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} &\mapsto {}^t\left(\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & A^{-1} \\ B^{-1} & 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 & {}^tB^{-1} \\ {}^tA^{-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & {}^tX \\ {}^tY & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & {}^tA \\ {}^tB & 0 \end{pmatrix} \\ &= \begin{pmatrix} {}^tB^{-1t}Y & 0 \\ 0 & {}^tA^{-1t}X \end{pmatrix} \cdot \begin{pmatrix} 0 & {}^tA \\ {}^tB & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & {}^tB^{-1t}Y^tA \\ {}^tA^{-1t}X^tB & 0 \end{pmatrix} \end{aligned} $$

The map γ_1 is an involution if and only if $\gamma_1^2 = 1$, i.e.

$$ \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} = \begin{pmatrix} 0 & {}^tB^{-1t}({}^tB^{-1t}Y^tA)^tA \\ {}^tA^{-1t}({}^tA^{-1t}X^tB)^tB & 0 \end{pmatrix} = $$

$$ = \begin{pmatrix} 0 & {}^tB^{-1}AYB^{-1t}A \\ {}^tA^{-1}BXA^{-1t}B & 0 \end{pmatrix} $$

for every X, Y. By Lemma 3.1 ${}^tA^{-1}B = \lambda E$, i.e. $B = \lambda^tA$.

So, we have proved:

Theorem 3.2. *If $d_1 = d_2$ and γ_1 is not any involution from Theorem* 3.1. *then it has the form*

$$ \gamma_1 \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda^{-1}A^{-1t}Y^tA \\ \lambda^tA^{-1t}XA & 0 \end{pmatrix}. $$

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