EVALUATION OF THE FINITE HYPERGEOMETRIC SERIES $F\left(-n, \frac{1}{2}, j+1; 4\right)$

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1. *Introduction and summary.* An interesting binomial sum

$$(1.1)\qquad F\left(-n, \frac{1}{2}, j+1; 4z\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{k}\binom{j+k}{k}^{-1} z^k$$

arises from the definition of the hypergeometric series. The series does not seem to have been evaluated in closed form in general. A few special cases have appeared in the literature. Because of the wellknown formula of G uss for the special case $F(a, b, c; 1)$, we know that

$$F\left(-n, \frac{1}{2}, j+1; 1\right) = 2^{-2n}\binom{2n+2j}{n+j}\binom{2j}{j}^{-1}.$$

Sandham [3] posed as an easy problem to show that

$$(1.2)\qquad F\left(-2n, \frac{1}{2}, n+1; 4\right) = 1,$$

and Spiegel [4] proved this Euler's integral. Looking at the simplicity of (1.2) suggested to me that progress might be made with (1.1) in the case $z = 1/4$. The results given below represent a condensation to the study of combinatorial identities the results seem to be of interest, though the general pattern of the coefficients below has not been determined.

It is convenient to consider (1.1) according to the parity of n, and we consider below

$$(1.3)\qquad S_j^n = F\left(-2n, \frac{1}{2}, n+j+1; 4\right)$$

and

(1.4) $$R_j^n = E\left(-2n-1, \frac{1}{2}, n+j+1; 4\right).$$

We obtain the recurrence relations

(1.5) $$2\,\frac{2n+2j+1}{n+j+1}\,S_{j+1}^n = 3S_j^n + R_j^n$$

and

(1.6) $$2\,\frac{2n+2j+1}{n+j+1}\,R_{j+1}^n = 3R_j^n + S_{j-1}^{n+1}.$$

which, together with some initial values, are sufficient to generate the following Table of special values.

TABLE OF VALUES

j	S_j^n	R_j^n
-1	$3 \quad (n \geqslant 1)$	$-\frac{5n+2}{n} \quad (n \geqslant 1)$
0	1	-1
1	$\frac{n+1}{2n+1}$	0
2	$\frac{2(n+1)(n+2)}{2(2n+1)(2n+3)}$	$\frac{n+2}{2(2n+3)}$
3	$\frac{(n+2)(n+3)(11n+10)}{4(2n+1)(2n+3)(2n+5)}$	$\frac{5(n+2)(n+3)}{4(2n+3)(2n+5)}$
4	$\frac{(n+2)(n+3)(n+4)(43n+35)}{8(2n+1)(2n+3)(2n+5)(2n+7)}$	$\frac{21(n+2)(n+3)(n+4)}{8(2n+3)(2n+5)(2n+7)}$

The formulas in the Table are valid for $n \geqslant 0$ except as indicated.

For the general case we find that

(1.7) $$S_j^n = 2^{1-j}\, g_j(n) \prod_{i=1}^{j} (2n+2i-1)^{-1} \qquad j \geqslant 1,$$

and

(1.8) $$R_j^n = 2^{1-j}\, f_j(n) \prod_{i=2}^{j} (2n+2i-1)^{-1}, \qquad j \geqslant 1.$$

As a mater of fact these follow by repeated use of (1.5) and (1.6). Here $g_j(n)$ and $f_j(n)$ are some polynomials in n of degree j (for g) and $j-1$ (for f) whose form we have not determined exactly.

Finally, we obtain the finite series

$$F\left(-n, \frac{1}{2}, j+1; 4z\right) = \tag{1.9}$$
$$= 2^{-2j}\binom{2j}{j}^{-1}\frac{1}{t}\sum_{k=0}^{t=1}\left(\cos\frac{2\pi k}{t}\right)^{2j}\left\{1-4z\left(\sin\frac{2\pi k}{t}\right)^2\right\}^n$$

which is valid for integers $t > 2n + 2j$, and is of the type considered by Good [2], Carlitz [1] and others, for Legendre polynomials and other classical polynomials.

2. *Proofs of the formulas.* We apply the well-known integral formula

$$\binom{2k}{k}\binom{j+k}{k}^{-1} = 2^{2j+2k+1}\binom{2j}{j}^{-1}\frac{1}{\pi}\int_0^{\pi/2}\sin^{2k}x\cos^{2j}x\,dx \tag{2.1}$$

to (1.1) and this yields

$$F\left(-n, \frac{1}{2}, j+1; 4z\right) = 2^{2j+1}\binom{2j}{j}^{-1}\int_0^{\pi/2}\cos^{2j}x\,(1-4z\sin^2 x)^n\,dx = \tag{2.2}$$
$$= 2^{2j}\binom{2j}{j}^{-1}\int_0^1(\cos 2\pi x)^{2j}\,(1-4z\sin^2 2\pi x)^n\,dx.$$

which is symmetrical in sin and cos because of (2.1). On may also say that (2.2) follow from Euler's integral transformation.

It is from this integral that we find (1.9) at onse because of the fundamental lemma: If $f(x)$ is any polynomial in x of degree n, then

$$\frac{1}{t}\sum_{k=0}^{t=1} f\left(\cos\frac{2\pi k}{t}\right) = \int_0^1 f(\cos 2\pi x)\,dx, \text{ for } t > n. \tag{2.3}$$

Taking account of the trigonometric identity $\cos x(1 - 4\sin^2 x) = \cos 3x$ we obtain from (2.2) and (1.3)—(1.4)

$$S_j^n = 2^{2n+2j+1}\binom{2n+2j}{n+j}^{-1}\frac{1}{\pi}\int_0^{\pi/2}\cos^{2j}x\cos^{2n}3x\,dx, \tag{2.4}$$

and

$$(2.5) \qquad R_j^{n*} = 2^{2n+2j+1} \binom{2n+2j}{n+j}^{-1} \frac{1}{\pi} \int_0^{\pi/2} \cos^{2j-1} x \cos^{2n+1} 3x \, dx.$$

Combining these we obtain

$$S_j^n - R_j^n = 2^{2n+2j+1} \binom{2n+2j}{n+j}^{-1} \frac{1}{\pi} \int_0^{\pi/2} \cos^{2j-1} x \cos^{2n} 3x (\cos x - \cos 3x) dx$$

which simplifies (using $\cos x - \cos 3x = 4\cos x - 4\cos^3 x$) to

$$S_j^n - R_j^n = 4S_j^n - 2\frac{2n+2j+1}{n+j+1} S_{j+1}^n,$$

which is precisely (1.5).

To obtain (1.6) we must only simplify the difference

$$2\frac{2n+2j+1}{n+j+1} R_{j+1}^n - 3R_j^n$$

in a similar way.

From (2.4) we have Sandham's formula $S_0^n = 1$ at once. To show that $R_1^n = 0$ it is sufficient to note that

$$\int_0^{\pi/2} \cos x \cos^{2n+1} 3x \, dx = 0$$

since the integrand can be written as a sum of cosines of multiples of $2x$.

The value of R_2^n now follows from (1.6) using S_0^n and R_1^n. The value of R_0^n is found as follows. When $j = 0$ the integrand in (2.5) may be written (using identities) in the form $(-1 + 2\cos 2x)\cos^{2n} 3x$, so that

$$R_0^n = -2^{2n+1} \binom{2n}{n}^{-1} \frac{1}{\pi} \int_0^{\pi/2} \cos^{2n} 3x \, dx +$$

$$+ 2^{2n+2} \binom{2n}{n}^{-1} \frac{1}{\pi} \int_0^{\pi/2} \cos^{2n} 3x \cos 2x \, dx$$

$= -1$, since the second integral is identically zero (being that the integrand is a sum of consines of multiples of $2x$).

The value of S_1^n now follows from (1.5) using S_0^n and R_0^n. The remainder of the Table is now easily obtained by using (1.5) and (1.6).

It is true that the recurrence relations (1.5)−(1.6) could be combined into a single relation in various ways, but it is advantageous to have them in the form given.

If we next compare (1.7) with (2.4) and (1.8) with (3.5) we find

$$(2.6)\quad f_j(n) = j!\ 2^{2n+2j}\binom{n+j}{n}\binom{2n}{n}^{-1}\frac{1}{2n+1}\int_0^{\pi/2}\cos^{2j-1}x\cos^{2n+1}3x\,dx$$

and

$$(2.7)\qquad g_j(n) = j!2^{2n+2j}\binom{n+j}{n}\binom{2n}{n}^{-1}\int_0^{\pi/2}\cos^{2j}x\cos^{2n}3x\,dx.$$

These relations reveal in a way why it is relatively easy to express the original series in terms of n but not so easy in terms of j. For we may convert an integral of the form

$$\int_0^{\pi/2}\cos^p x\cos^q 3x\,dx$$

into a series of terms by using $\cos 3x = \cos x(1-4\sin^2 x)$, and then use the binomial theorem and integrate term by term. This gives a series the number of whose terms depends on q. But it is more difficult to replace $\cos x$ by something in terms of $\cos 3x$. This would give a series the number of whose terms depends on p. Expansions of this sort would allow us to sum some other similar series.

It is possible to use the Euler transformation

$$F(a,\ b,\ c;\ z) = (1-z)^{-a}\left(a,\ c-b,\ c;\ \frac{z}{z-1}\right)$$

to convert our series into another form, and the result is

$$F\left(-n,\ \frac{1}{2},\ j+1;\ 4z\right) = \binom{2j}{j}^{-1}\sum_{k=0}^{n}\binom{n}{k}\binom{2j+2k}{j+k}2^{-2k}(4z)^k(1-4z)^{n-k}$$

and this might shed more light on the sum, but we have not found a simple result.

3. *Condensation of results.* It is interesting to note that the values tabulated above for the several special cases may be expressed in a condensed form by use of the bracket function. Recalling that $[x]$ denotes the greatest integer $\leqslant x$ it is not difficult to see that part of the table we gave above may be summariced in the following formulas:

$$(3.1)\quad F\left(-n,\ \frac{1}{2},\ \left[\frac{n}{2}\right];\ 4\right) = 3-\left(8+\frac{4}{n}\right)\frac{(-1)^n+1}{2},\quad n\geqslant 2,$$

which combines S^n_{-1} and R^n_{-1};

(3.2) $$F\left(-n,\ \frac{1}{2},\ \left[\frac{n+1}{2}\right];\ 4\right) = 2(-1)^n + 1, \quad n \geqslant 1,$$

which combines S^n_{-1} and R^n_0;

(3.3) $$F\left(-n,\ \frac{1}{2},\ \left[\frac{n+2}{2}\right];\ 4\right) = (-1)^n, \quad n \geqslant 0,$$

which combines S^n_0 and R^n_0;

(3.4) $$F\left(-n,\ \frac{1}{2},\ \left[\frac{n+3}{2}\right];\ 4\right) = \frac{(-1)^n + 1}{2}, \quad n \geqslant 0,$$

which combines S^n_0 and R^n_1;

(3.3) $$F\left(-n,\ \frac{1}{2},\ \left[\frac{n+4}{2}\right];\ 4\right) = \frac{n+2}{2(n+1)}\ \frac{(-1)^n + 1}{2}, \quad n \geqslant 0,$$

which combines S^n_1 and R^n_1.

What may be the general rule we have not discovered.

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(ABSTRACT)

Recurence relations, depending on the parity of n, are given for the binomial sum

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\binom{j+k}{k}^{-1}$$

from which the sum may be calculated for any integer j. A short table for $-1 \leqslant j \leqslant 4$ is given. Various other results appear, and the bracket function is used to simplify the expressions. For example, it shown tha

$$F\left(-n, \frac{1}{2}, \left[\frac{n}{2}\right]; 4\right) = 3 - \left(8 + \frac{4}{n}\right)\frac{(-1)^n + 1}{2},$$

for $n \geqslant 2$. A closed form the general case in not found, but some information about the general sum is found. The sum is found to be a quotient of polynomias, the denominator polynomials having a very simple form.