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ON HAHN-BANACH EXTENSION OF LINEAR N-FUNCTIONALS IN N-NORMED SPACES

NEERAJ SRIVASTAVA, S. BHATTACHARYA, AND S. N. LAL

Abstract. In this paper a new Hahn-Banach extension theorem for n-bounded linear n-functionals in complex n-normed linear spaces which generalize all the known results has been established.

1. Introduction

The results in 2-metric and real 2-normed linear spaces were extended by S. Gähler and A. Misiak to n-metric and real n-normed linear spaces ([1], [2], [3], [4], [7]), where n is any natural number. To develop the theory of n-normed linear spaces, extension theorems of Hahn-Banach type are crucial. It was White [8] who gave the required Hahn-Banach type extension theorem for the case $n = 2$ and in 1999 Malceski [6] gave the extension theorem for the general case. But in all these works only real n-normed linear spaces were considered. For a real development of the theory of n-normed linear spaces, the theory of complex n-normed linear spaces need to be developed. But the transition from real n-normed linear spaces to complex n-normed linear spaces is not trivial, as, for example, for $n = 2$, a 2-normed complex linear space $(E, ||.,.||)$ fails to be a 2-normed real linear space simply because for non-zero x in E, though $||x, ix|| = 0$ but when E is considered as a linear space over reals, $||x, ix||$ can not be zero as in this case x, ix are linearly independent elements of E. Another difficulty one may face to get Hahn-Banach type extension theorem for complex n-normed linear spaces from real n-normed linear spaces is that the Bohnenblust-Sobczyk technique fails for even $n = 2$ as described in [5]. In [5] a technique was developed to get complex version of White's result.

In the next section of this paper we establish the complex version of the Malceski's extension theorem for n-normed linear spaces.

Throughout this paper E is a linear space over **K** where **K** is the field of real or complex numbers.

We note the following :

Definition 1.1. *A mapping* $||., ..., .|| : E^n \mapsto \mathbf{R}$ *is called an* n*-norm on E, if for all* $x_i, y_1 \in E, i = 1, 2, \ldots, n$ *and* $\alpha \in \mathbf{K}$
(1.1) $||x_1, \ldots, x_n|| \geq 0$*, and* $||x_1, \ldots, x_n|| = 0$ *if and only if the set* $\{x_1, \ldots, x_n\}$ *is linearly dependent;*

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(1.2) $||x_1, \ldots, x_n|| = ||\pi(x_1), \ldots, \pi(x_n)||$ *for every bijection:*
$\pi : \{x_1, \ldots, x_n\} \mapsto \{x_1, \ldots, x_n\}$;
(1.3) $||\alpha\, x_1, \ldots, x_n|| = |\alpha| ||x_1, \ldots, x_n||$;
(1.4) $||x_1 + y_1, x_2, \ldots, x_n|| \leq ||x_1, x_2, \ldots, x_n|| + ||y_1, x_2, \ldots, x_n||$.
The pair $(E^n, ||., \ldots\ldots\ldots, .||)$ *is called an n-normed linear space over* $\mathbf{K}$.

Definition 1.2. *Let* $X_{i, i=1,2\ldots,n}$ *be* $n-$*linear subspaces of the linear space E. A mapping* $f : X_1 \times \cdots \times X_n \mapsto \mathbf{K}$ *is said to be a linear n-functional if for all* $x_i, y_i \in X_i$, $i = 1, 2 \ldots, n$,
(1.5) $f(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = \sum_{\substack{z_i \in \{x_i, y_i\} \\ i=1,2,\ldots,n}} f(z_1, z_2, \ldots, z_n)$
(1.6) $f(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n) = \alpha_1 \alpha_2 \ldots \alpha_n f(x_1, x_2, \ldots, x_n)$ *for all* $\alpha_i \in \mathbf{K}$, $i = 1, 2 \ldots, n$.

Definition 1.3. *The linear n-functional f is called n-bounded on* $X_1 \times \cdots \times X_n$ *if there exists a* $k > 0$ *such that for all* $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$,
(1.7) $|f(x_1, x_2, \ldots, x_n)| \leq k||x_1, x_2, \ldots, x_n||$.
For an n-bounded linear n-functional f with domain $X_1 \times \cdots \times X_n$, *the infimum of all* $k > 0$ *satisfying (1.7) is called the norm of f and is denoted by* $||f||$.

Proposition 1.1. *Let f be an n-bounded linear n-functional with domain* $\underset{i=1}{\overset{n}{X}} X_i$. *If* $\{x_1, x_2 \ldots, x_n\}$ *be such that a subset of it is linearly dependent where* $(x_1, \ldots, x_n) \in \underset{i=1}{\overset{n}{X}} X_i$, *then we have* $f(x_1, x_2, \ldots, x_n) = 0$.

Proof. Since $(x_1, x_2, \ldots, x_n) \in \underset{i=1}{\overset{n}{X}} X_i$ is such that $\{x_1, x_2, \ldots, x_n\}$ is linearly dependent as one of its subset is linearly dependent, we have $||x_1, x_2, \ldots, x_n|| = 0$. As f is an n-bounded linear n-functional over $\underset{i=1}{\overset{n}{X}} X_i$, there exists a $k > 0$ such that $|f(x_1, x_2, \ldots, x_n)| \leq k||x_1, x_2, \ldots, x_n|| = 0$, that is, $f(x_1, x_2, \ldots, x_n) = 0$. □

Proposition 1.2. *Let f be an* $n-$*bounded linear* $n-$*functional with domain* $\underset{i=1}{\overset{n}{X}} X_i$. *Then*
(1.8) $||f|| = \sup \left\{ \frac{|\mathrm{f}\,(\mathrm{x}_1, \ldots, x_n)}{||x_1, \ldots, x_n||} : (x_1, \ldots, \mathrm{x_n}) \in \underset{\mathrm{i}=1}{\overset{\mathrm{n}}{\mathrm{X}}} \mathrm{X_i} \text{ and } ||\mathrm{x}_1, \ldots, x_n|| \neq 0 \right\}$

Lemma 1.1. *Let* $(E, ||., \ldots, .||)$ *be an n-normed linear space. For all* $y, x_1, \ldots, x_{n-1} \in E$ *and* $\lambda_i \in \mathbf{K}, i = 1, 2 \ldots n-1$
(1.9) $||y, x_1, \ldots, x_{n-1}|| = ||y + \sum_{i=1}^{n-1} \lambda_i x_i, x_1, \ldots, x_{n-1}||$

Proof. As E is an $n-$normed linear space, we have

$$||y + \sum_{i=1}^{n-1} \lambda_i x_i, x_1, \ldots, x_{n-1}|| \leq ||y, x_1, \ldots, x_{n-1}|| + ||\sum_{i=1}^{n-1} \lambda_i x_i, x_1, \ldots, x_{n-1}||$$
$$= ||y, x_1, \ldots, x_{n-1}|| \quad \text{(by definition 1.1)}$$

Further more,

$$\begin{aligned}
||y, x_1, \ldots, x_{n-1}|| &= ||y + \sum_{i=1}^{n-1} \lambda_i x_i, + \left(-\sum_{i=1}^{n-1} \lambda_i x_i \right), x_1, \ldots, x_{n-1}|| \\
&\leq ||y + \sum_{i=1}^{n-1} \lambda_i x_i, x_1, \ldots, x_{n-1}|| + \left\| -\sum_{i=1}^{n-1} \lambda_i x_i, x_1, \ldots, x_{n-1} \right\| \\
&= ||y + \sum_{i=1}^{n-1} \lambda_i x_i, x_1, \ldots, x_{n-1}|| \quad \text{(again by definition 1.1)}
\end{aligned}$$

□

Combining the above two inequalities, the Lemma (1.1) follows.

Lemma 1.2. *Let $\{x_1, \ldots, x_{n-1}\}$ be a linearly independent subset of E, $\{x_1, \ldots, x_{n-1}\} \cup \{y_j : j \in I\}$ be a Hamel basis for E and if $N = span\{y_j : j \in I\}$. Then $||.||$ defined on N by $||x|| = ||x, x_1, \ldots, x_{n-1}||$ for $x \in N$ defines a norm on N.*

Proof. For every $x, yinN$, $\alpha \in \mathbf{K}$, we have

$$\begin{aligned}
||\alpha x|| &= ||\alpha x, x_1, \ldots, x_{n-1}|| \\
&= |\alpha| ||x, x_1, \ldots, x_{n-1}|| \\
&= |\alpha| ||x|| \\
||x + y|| &= ||x + y, x_1, x_2, \ldots, x_{n-1}|| \\
&\leq ||x, x_1, \ldots, x_{n-1}|| + ||y, x_1, \ldots, x_{n-1}|| \\
&= ||x|| + ||y||
\end{aligned}$$

and from the definition of $||.||$, $||x|| \geq 0$.

Now, let for x in N, $||x|| = 0$. Then $||x, x_1, \ldots, x_{n-1}|| = 0$ implies that $\{x, x_1, \ldots, x_{n-1}\}$ is linearly dependent and as $\{x_1, \ldots, x_{n-1}\}$ is linearly independent, x can be represented as a linear combination of $\{x_1, \ldots, x_{n-1}\}$ implies $x \in [x_1, \ldots, x_{n-1}]$. But as $x \in N$, we have a contradiction if $x \neq 0$. Therefore $x = 0$.

Conversely, if $x = 0$, then as $\{0, x_1, \ldots, x_{n-1}\}$ is linearly dependent and we have $||x|| = ||0, x_1, \ldots, x_{n-1}|| = 0$. Hence, $||.||$ defines a norm on N. □

Theorem 1.4. *Let M be a linear subspace of an $n-$normed linear space E over $\mathbf{K}$ and let $x_1, \ldots, x_{n-1} \in E$. If f is an $n-$bounded linear $n-$functional on $M \times [x_1] \times \cdots \times [x_{n-1}]$, then there exists an $n-$bounded linear $n-$functional F on $E \times [x_1] \times \cdots \times [x_{n-1}]$ satisfying*
(2.1) $||F|| = ||f||$
(2.2) $F(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) = f(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1})$

Proof. Let f be defined on $M \times [x_1] \times \cdots \times [x_{n-1}]$. If any $x_i, i = 1, 2, \ldots, n-1$ is the zero of E, then by Proposition 1.1, $f(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) = 0$ for every

$(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1})$ $M \times [x_1] \times \cdots \times [x_{n-1}]$. And also if $\{x_1, \ldots, x_{n-1}\}$ is linearly dependent then $\{x, x_1, \ldots, x_{n-1}\}$ is also linearly dependent for all $x \in M$ and so, $f(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}) = 0$ (by Proposition 1.1) for every $(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}) \in M \times [x_1] \times \cdots \times [x_{n-1}]$. Now by defining F on $E \times [x_1] \times \cdots \times [x_{n-1}]$ in both the cases by $F(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}) = 0$ for every $(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}) \in E \times [x_1] \times \cdots \times [x_{n-1}]$ the theorem follows.

Let now, $\{x_1, x_2, \ldots, x_{n-1}\}$ be linearly independent set. For an index set I, let $\{x_1, x_2, \ldots, x_{n-1}\} \cup \{y_j : j \in I\}$ be a Hamel basis for E. If N is the linear subspace of E generated by $\{y_j : j \in I\}$, then $E = [x_1, x_2, \ldots, x_{n-1}] \oplus N$. Define a mapping $\|\,.\,\|$ on N by $\|x\| = \|x, x_1, x_2, \ldots, x_{n-1}\|$ for every $x \in N$.

Then by Lemma 1.2, $(N, \|\,.\,\|)$ is a normed linear space over $\mathbf{K}$. We now consider two cases separately.

Case 1: Let for each $i, x_i \notin M$, $i = 1, 2, \ldots, n$. Then clearly $M \subset N$. Define $\tilde{f}$ on M by $\tilde{f}(x) = f(x, x_1, \ldots, x_{n-1})$ for all $x \in M$. Then $\tilde{f}$ is a linear functional on M and

$$|\tilde{f}(x)| = |f(x, x_1, x_2, \ldots, x_{n-1})| \leq \|f\| \|x, x_1, x_2, \ldots, x_{n-1}\| = \|f\| \|x\|$$

(as f is a $n-$bounded linear $n-$functional on $M \times [x_1] \times \cdots \times [x_{n-1}]$) that is, $\tilde{f}$ is bounded on M.

Now by Proposition 1.2,

$$\|f\| = \sup\left\{ \frac{|f(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}|}{\|x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}\|} : \begin{array}{c}(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}) \in M \times [x_1] \times \ldots \times [x_{n-1}] \\ \text{and} \quad \|x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}\| \neq 0\end{array} \right\}$$

$$= \sup\left\{ \frac{|\tilde{f}(x)|}{\|x\|} : x \in M, \|x\| \neq 0 \right\}$$

$$= \|\tilde{f}\|$$

Appealing to the Hahn-Banach theorem we get a bounded linear functional $\tilde{F}$ on N with

(2.3) $\|\tilde{F}\| = \|\tilde{f}\| = \|f|$ and

(2.4) $\tilde{F} = \tilde{f}$ on M.

Define F on $E \times [x_1] \times \cdots \times [x_{n-1}]$ by

$$F(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}) = \begin{cases} \left(\prod_{i=1}^{n-1} \lambda_i\right) \tilde{F}(x) & \text{if } x \in \mathrm{N} \\ \left(\prod_{i=1}^{n-1} \lambda_i\right) \tilde{F}(y) & \text{if } x = \sum_{i=1}^{n-1} \mu_i x_i + y \end{cases}$$

for all $\mu_i \in \mathbf{K}$ and $y \in \mathrm{N}$ (as $E = [x_1, \ldots, x_{n-1}] \oplus N\}$.

F is well defined on $E \times [x_1] \times \cdots \times [x_{n-1}]$ and is a linear $n-$functional on $E \times [x_1] \times \cdots \times [x_{n-1}]$. For $(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}) \in M \times [x_1] \times \cdots \times [x_{n-1}]$,

$$F(x, \lambda_1 x_1, \ldots, \lambda_{n-1}x_{n-1}) = \lambda_1 \ldots \lambda_{n-1} \tilde{F}(x)$$

$$= \lambda_1 \dots \lambda_{n-1} \tilde{f}(x)$$
$$= f(x, \lambda_1 x_1, \dots, \lambda_{n-1} x_{n-1})$$

and F satisfies (2.2).

Again, for $x \in E$, let $x = \sum_{i=1}^{n-1} \mu_i x_i + y$, where $\mu_i \in \mathbf{K}$ and $y \in$ N. For $\lambda_i \in \mathbf{K}$, $i = 1, 2, \dots n-1$, we have

$$\begin{aligned} |F(x, \lambda_1 x_1, \dots, \lambda_{n-1} x_{n-1})| &= |\lambda_1 \dots \lambda_{n-1} \tilde{F}(y)| \\ &\leq |\lambda_1, \dots, \lambda_{n-1}| ||f|| \, ||y|| \qquad \text{(using 2.3)} \\ &= |\lambda_1 \dots \lambda_{n-1}| \, ||f|| \, ||y, x_1, \dots, x_{n-1}|| \\ &= |\lambda_1 \dots \lambda_{n-1}| \, ||f|| || \sum_{i=1}^{n-1} \mu_i x_i + y, x_1, \dots, x_{n-1}|| \\ &\qquad \text{(using Lemma1.1)} \\ &= ||f|| \, ||x, \lambda_1 x_1, \dots, \lambda_{n-1} x_{n-1}|| \end{aligned}$$

and it follows that F is an $n-$bounded linear $n-$functional on $E \times [x_1] \times \cdots \times [x_{n-1}]$ with $||F|| \leq ||f||$ and hence $||F|| = ||f||$ and (2.1) holds.

Case 2: Let for some $k; i \leq k \leq n-1$, $x_i, \dots, x_{i_k} \in M$. For simplicity, we assume, $x_1, \dots, x_k \in M$. Let $\tilde{N} = [x_{k+1}, \dots, x_{n-1}] \oplus N$ and $P = \tilde{N} \cap M$. Define g on P by $g(x) = f(x, x_1, \dots, x_{n-1})$ for $x \in P$ and $||\,.\,||$ on P by $||x|| = ||x, x_1, \dots, x_{n-1}||$ for $x \in P$. Then as in case 1, g is a linear functional on P. Again $||\,.\,||$ is a norm on P as, for $x \in P$, we have $x = \sum_{i=k+1}^{n-1} \lambda_i x_i + y$, $y \in N$ and so,

$$\begin{aligned} ||x|| = 0 &\Rightarrow || \sum_{i=1}^{n-1} \lambda_i x_i + y, x_1, \dots, x_{n-1}|| = 0 \\ &\Rightarrow \left\{ || \sum_{i=k+1}^{n-1} \lambda_i x_i + y, x_1, \dots, x_{n-1} \right\} \text{ is linearly dependent .} \\ &\Rightarrow \alpha \left(|| \sum_{i=k+1}^{n-1} \lambda_i x_i + y \right) + \sum_{i=1}^{n-1} \alpha_i x_i = 0 \end{aligned}$$

for $\alpha, \alpha_1, \dots, \alpha_{n-1} \in K$, not all zero.

$$\begin{aligned} &\Rightarrow \alpha y + \sum_{i=1}^{k} \alpha_i x_i + y + \sum_{i=k+1}^{n-1} (\alpha \lambda_i + \alpha_i) x_i = 0 \\ &\Rightarrow \alpha = 0, \alpha_i = 0 \text{ for } i = 1, \dots, n-1 \end{aligned}$$

and $(\alpha \lambda_i + \alpha_i) = 0$ for $i = k+1, \dots, n-1$ (as $\{y, x_1, x_2, \dots, x_{n-1}\}$ is linearly independent)
$\Rightarrow \alpha = 0, \alpha_i = 0$ for $i = 1, 2 \dots, k$ and $\alpha_i = 0$ for $i = k+1, \dots, n-1$. This is

a contradiction if $x \neq 0$ and therefore $x = 0$. It now follows easily that $||\,.\,||$ is a norm on P.

We claim that $||g|| = ||f||$. Since $P \subseteq M$,

$$\begin{aligned}
||g|| &= \sup\left\{\frac{|g(x)|}{||x||}; x \in P, ||x|| \neq 0\right\} \\
&= \sup\left\{\frac{|f(x, x_1, \ldots, x_{n-1})|}{||x, x_1, \ldots, x_{n-1}||} : x \in P,\ ||x, x_1, \ldots, x_{n-1}|| \neq 0\right\} \\
&\leq \sup\left\{\frac{|f(x, x_1, \ldots, x_{n-1})|}{||x, x_1, \ldots, x_{n-1}||} : x \in M,\ ||x, x_1, \ldots, x_{n-1}|| \neq 0\right\} \\
&= ||\, f \,||.
\end{aligned}$$

That is $||g|| \leq ||f||$.

Again, for $x \in M$

$$\begin{aligned}
x &= \sum_{i=1}^{k} \lambda_i x_i + \sum_{i=k+1}^{n-1} \lambda_i x_i + y, \text{ where } y \in N \\
&= \sum_{i=1}^{k} \lambda_i x_i + z, \quad z \in \tilde{N}.
\end{aligned}$$

Therefore as $x \in M$ and $\sum\limits_{i=1}^{k} \lambda_i x_i \in M$ we have $z \in M$ also and so $z \in P$.

$$\begin{aligned}
\text{Now,}\quad ||x, x_1, \ldots, x_{n-1}|| &= ||\sum_{i=1}^{k} \lambda_i x_i + z, x_1, \ldots, x_{n-1}|| \\
&= ||z, x_1, \ldots, x_{n-1}|| \qquad \text{(by Lemma 1.1)} \\
&= ||z||
\end{aligned}$$

$$\begin{aligned}
\text{and,}\quad g(z) &= f(z, x_1, \ldots, x_{n-1}) \\
&= f\left(z + \sum_{i=1}^{k} \lambda_i x_i, x_1, \ldots, x_{n-1}\right) - f\left(\sum_{i=1}^{k} \lambda_i x_i, x_1, \ldots, x_{n-1}\right) \\
&= f(x, x_1, \ldots, x_{n-1}) \quad \text{(By Proposition 1.1)}
\end{aligned}$$

Hence, if $||x, x_1, \ldots, x_{n-1}|| \neq 0$ and $x \in M$, we have

$$\frac{|f\left(x, x_1, \ldots\ldots\ldots, x_{n-1}\right)|}{||x, x_1, \ldots\ldots\ldots, x_{n-1}||} = \frac{|g(z)|}{||z||} \leq ||g||,$$

and therefore,

$$\left\{ \frac{|f(x, x_1, \ldots, x_{n-1})|}{\|x, x_1, \ldots, x_{n-1}\|} : x \in M, \|x, x_1, \ldots, x_{n-1}\| \neq 0 \right\} \leq \|g\|$$

that is , $\|f\| \leq \|g\|$ and we finally get $\|g\| = \|f\|$.

Again, appealing to Hahn-Banach theorem, we get a G over E such that $\|G\| = \|g\| = \|f\|$ and $G(x) = g(x)$ for all $x \in P$.

Now, we define $F : E \times [x_1] \times \cdots \times [x_{n-1}] \to \mathbf{K}$ by $F(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) = \lambda_1 \ldots \lambda_{n-1} G(x)$, for $(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \in E \times [x_1] \times \ldots x[x_{n-1}]$.

Then clearly F is a linear $n-$functional on $E \times [x_1] \times \cdots \times [x_{n-1}]$.

Again, for $(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \in E \times [x_1] \times \cdots \times [x_{n-1}]$, with $\|x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}\| \neq 0$, we have

$$\frac{|F(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1})|}{\|x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}\|} = \frac{|\lambda_1 \ldots \lambda_{n-1}| \, |G(x)|}{|\lambda_1 \ldots \lambda_{n-1}| \, \|x, x_1, \ldots, x_{n-1}\|} = \frac{|G(x)|}{\|x\|}$$

and it follows that $\|F\| = \|G\| = \|f\|$.

Thus F is an $n-$bounded linear $n-$functional over $E \times [x_1] \times \cdots \times [x_{n-1}]$ with $\|F\| = \|f\|$.

Note that for $x \in M$, $x = \sum_{i=1}^{k} \mu_i x_i + z \;; \mu_i \in \mathbf{K}, \;\; i = 1, 2, \ldots, k, \;\; z \in \tilde{N}$.

Since $x_1, \ldots, x_k \in M$, we have $z \in M$ and so $z \in P$. For $(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) \in M \times [x_1] \times \cdots \times [x_{n-1}]$, we have,

$$F(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) = F\left(\sum_{i=1}^{k} \mu_i x_i + z, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1} \right)$$

$$= F\left(\sum_{i=1}^{k} \mu_i x_i, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1} \right) + F(z, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1})$$

(as F is a linear $n-$ functional)

$= 0 + \lambda_1 \ldots \lambda_{n-1} \; G(z)$ (By Proposition 1.1)

$= \lambda_1 \ldots \lambda_{n-1} \; g(z)$ as $z \in P$

$= \lambda_1 \ldots \lambda_{n-1} \; f(z, x_1, \ldots, x_{n-1})$

$$= f\left(\sum_{i=1}^{k} \mu_i x_i, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1} \right) + f(z, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1})$$

(By Proposition 1.1)

$$= f\left(\sum_{i=1}^{k} \mu_i x_i + z, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}\right)$$

$$= f\left(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}\right)$$

that is, $F(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1}) = f(x, \lambda_1 x_1, \ldots, \lambda_{n-1} x_{n-1})$
This completes the proof of the theorem in case 2 and the theorem is completely established. □

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Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi-221005, India
E-mail address: neeraj_bhuin@yahoo.co.in