Математички Билтен
31 (LVII)
2007 (33–52)
Скопје, Македонија

ISSN 0351–336X

# ALGEBRAIC STRUCTURE OF THE KERNEL OF THE n-SEMINORM

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## Abstract

Abstract. In [4], the notion of n-norm is introduced as generalization of the notion of 2-norm, introduced in [1]. In [3], the notion of n-seminorm is introduced as generalization of the notion of 2-seminorm. Equivalent definition of 2-norm is given in [2]. In the first part of this paper, we give equivalent definition of n-seminorm. In the second part of the paper, we give a characterization of the kernel of the n-seminorm as a classe of subsets of $X^n$ and we construct two examples of 2-subspace of 2-vector space.

## 1. Introduction

In [4], A. Misiak introduced the notion of $n$-norm and $n$-normed space as generalization of the notion of 2-norm and 2-normed space.

**Definition 1.** Let $X$ be a vector space over the field of real numbers and let $\dim X \geq n$. The function $\|\cdot,\cdots,\cdot\|: X^n \to \mathbf{R}$ which satisfies the conditions:

(N1) $\|x_1,\ldots,x_n\| \geq 0$ and $\|x_1,\ldots,x_n\| = 0$ if and only if the set $\{x_1,\ldots,x_n\}$ is linearly dependent;

(N2) $\|x_1,\ldots,x_n\| = \|x_{\pi(1)},\ldots,x_{\pi(n)}\|$ for any $x_1,\ldots,x_n \in X$ and for every permutation $\pi: \{1,\ldots,n\} \to \{1,\ldots,n\}$;

(N3) $\|x_1,\ldots,x_{i-1},\alpha x_i,x_{i+1},\ldots,x_n\| = |\alpha|\,\|x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n\|$, for any $x_1,\ldots,x_n \in X$ and for every scalar $\alpha \in \mathbf{R}$ and for all

$i \in \{1, 2, \ldots, n\}$;

(N4) $\|x_1 + x_1', x_2, \ldots, x_n\| \leq \|x_1, x_2, \ldots, x_n\| + \|x_1', x_2, \ldots, x_n\|$, for any $x_1, x_1', x_2, \ldots, x_n \in X$,

is called $n$-norm of the vector space $X$, and the ordered pair $(X, \|\cdot, \ldots, \cdot\|)$ is called n-normed real space.

**Note.** Clearly, a 2-normed space is a special case of an n-normed space. So, it is natural to expect that some of the properties of 2-normed space will hold for $n$-normed space. The technique of proving them is much more complicated. In the subsequent part, we will give some of this analogy.

**Lemma 1.** For each $i$, for arbitrary real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n$ and for any vectors $x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n$, the equality

$$\|x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n\| = \|x_1, \ldots, x_{i-1}, x_i + \sum_{j=1, j\neq i}^{n} \alpha_j x_j, x_{i+1}, \ldots, x_n\|.$$

holds. (1)

**Proof.** Using the fact that the vectors $x_1, \ldots, x_{i-1}, -\sum_{j=1, j\neq i}^{n} \alpha_j x_j, x_{i+1}, \ldots, x_n$ are linearly dependent, from (N1) of the definition of $n$-norm, we get that $\|x_1, \ldots, x_{i-1}, -\sum_{j=1, j\neq i}^{n} \alpha_j x_j, x_{i+1}, \ldots, x_n\| = 0$. Using this and (N2) and (N4), we have

$$\|x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n\| = \|x_1, \ldots, x_{i-1}, x_i + \sum_{j=1, j\neq i}^{n} \alpha_j x_j - \sum_{j=1, j\neq i}^{n} \alpha_j x_j, x_{i+1}, \ldots, x_n\| \leq \|x_1, \ldots, x_{i-1}, x_i + \sum_{j=1, j\neq i}^{n} \alpha_j x_j, x_{i+1}, \ldots, x_n\| + \|x_1, \ldots, x_{i-1}, -\sum_{j=1, j\neq i}^{n} \alpha_j x_j, x_{i+1}, \ldots, x_n\|$$
$$= \|x_1, \ldots, x_{i-1}, x_i + \sum_{j=1, j\neq i}^{n} \alpha_j x_j, x_{i+1}, \ldots, x_n\|.$$

So,

$$\|x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n\| \leq \|x_1, \ldots, x_{i-1}, x_i + \sum_{j=1, j\neq i}^{n} \alpha_j x_j, x_{i+1}, \ldots, x_n\|. \tag{2}$$

Using the fact that the vectors $x_1, \ldots, x_{i-1}, \sum_{j=1, j\neq i}^{n} \alpha_j x_j, x_{i+1}, \ldots, x_n$ are linearly dependent from (N1) we get that

$\left\| \sum_{j=1,j\neq i}^{n} \alpha_j x_j, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \right\| = 0$. Again, from this and (N2) and (N4), we have

$$\left\| x_1, \dots, x_{i-1}, x_i + \sum_{j=1,j\neq i}^{n} \alpha_j x_j, x_{i+1}, \dots, x_n \right\| =$$
$$= \left\| x_i + \sum_{j=1,j\neq i}^{n} \alpha_j x_j, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \right\|$$
$$\leq \| x_i, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \|$$
$$+ \left\| \sum_{j=1,j\neq i}^{n} \alpha_j x_j, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \right\|$$
$$= \| x_i, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \| = \| x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n \| .$$

So,

$$\left\| x_1, \dots, x_{i-1}, x_i + \sum_{j=1,j\neq i}^{n} \alpha_j x_j, x_{i+1}, \dots, x_n \right\| \leq \| x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n \| . \tag{3}$$

From the inequalities (2) and (3) we get the equality (1). □

The natural generalization of lemma 1 is the following lemma 2.

**Lemma 2.** If $X$ is a real n-normed space, $a_{ip}, i \in \{1, 2, \dots, k\}$, $p \in \{1, 2, \dots, n\}$ are given real numbers and $x_i \in X, i \in \{1, 2, \dots, n\}$, then

$$\begin{aligned} &\| a_{11}x_1 + \dots + a_{1n}x_n, a_{21}x_1 + \dots + a_{2n}x_n, \dots, a_{k1}x_1 + \dots + a_{kn}x_n, x_{k+1}, \dots, x_n \| = \\ &= \| a_{11}x_1 + \dots + a_{1k}x_k, a_{21}x_1 + \dots + a_{2k}x_k, \dots, a_{k1}x_1 + \dots + a_{kn}x_k, x_{k+1}, \dots, x_n \| . \end{aligned} \tag{4}$$

**Proof.** Using Lemma 1, we get

$$\| a_{11}x_1 + \dots + a_{1n}x_n, a_{21}x_1 + \dots + a_{2n}x_n, \dots, a_{k1}x_1 + \dots + a_{kn}x_n, x_{k+1}, \dots, x_n \| =$$
$$= \| a_{11}x_1 + \dots + a_{1n}x_n, a_{21}x_1 + \dots + a_{2n}x_n, \dots, a_{k1}x_1 + \dots + a_{kn}x_n - a_{kk+1}x_{k+1} - \dots - a_{kn}x_n, x_{k+1}, \dots, x_n \|$$
$$= \| a_{11}x_1 + \dots + a_{1n}x_n, a_{21}x_1 + \dots + a_{2n}x_n, \dots, a_{k-11}x_1 + \dots + a_{k-1n}x_n, a_{k1}x_1 + \dots + a_{kk}x_k, x_{k+1}, \dots, x_n \| .$$

Repeating the above technique for $p = k-1, k-2, \dots, 3, 2, 1$, we get the equality (4). □

**Lemma 3.** If $A = [a_{ij}]_{n\times n}$ upper-triangular matrix, i.e. $a_{ij} = 0$ for $i > j$ and $(x_1, \dots, x_n) \in X^n$, then

$$\begin{aligned} \| A(x_1, \dots, x_n)^T \| &= | a_{11}a_{22}a_{33} \dots a_{n-1n-1}a_{nn} | \, \| x_1, \dots, x_n \| \\ &= | \det A | \, \| x_1, \dots, x_n \| . \end{aligned} \tag{5}$$

**Proof.** Since the matrix $A$ is upper-triangular, we have

$$\begin{aligned}&\|A(x_1,x_2,\dots,x_n)^T\| =\\ =&\|a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n, a_{22}x_2+a_{23}x_3+\cdots\\ &+a_{2n}x_n,\dots,a_{n-1n-1}x_{n-1}+a_{n-1n}x_n, a_{nn}x_n\| = (*)\end{aligned}$$

Using Lemma 1, Lemma 2 and (N3), we get

$$\begin{aligned}(*) =&|a_{nn}|\,\|a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n, a_{22}x_2+a_{23}x_3+\cdots\\ &+a_{2n}x_n,\dots,a_{n-1n-1}x_{n-1}+a_{n-1n}x_n, x_n\|\\ =&|a_{nn}|\,\|a_{11}x_1+a_{12}x_2+\cdots+a_{1n-1}x_{n-1}, a_{22}x_2+a_{23}x_3+\cdots\\ &+a_{2n-1}x_{n-1},\dots,a_{n-1n-1}x_{n-1}, x_n\| = (**)\end{aligned}$$

From (N3) and Lemma 2, we get

$$\begin{aligned}(**) =&|a_{nn}a_{n-1n-1}|\,\|a_{11}x_1+a_{12}x_2+\cdots+a_{1n-2}x_{n-2}, a_{22}x_2+a_{23}x_3+\cdots\\ &+a_{2n-2}x_{n-2},\dots,a_{n-2n-2}x_{n-2}, x_{n-1}, x_n\|.\end{aligned}$$

Repeating the above technique consecutively for $k = n-2,\dots,3,2,1$, we get the equality (5). □

In the proves of the following properties, we will often use the next theorem.

**Theorem 1.** For each determinant of order $k$,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k}\\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k}\\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{kk}\end{vmatrix}, \text{ such that}$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-2} & a_{1k-1}\\ a_{21} & a_{22} & \cdots & a_{2k-2} & a_{2k-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-21} & a_{k-22} & \cdots & a_{k-2k-2} & a_{k-2k-1}\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-2} & a_{k-1k-1}\end{vmatrix} \neq 0, \text{ the following equality}$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k}\\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k}\\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{kk}\end{vmatrix} = \frac{1}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-2} & a_{1k-1}\\ a_{21} & a_{22} & \cdots & a_{2k-2} & a_{2k-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-21} & a_{k-22} & \cdots & a_{k-2k-2} & a_{k-2k-1}\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-2} & a_{k-1k-1}\end{vmatrix}}$$

$$\cdot\left\|\begin{array}{cc}
\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-2} & a_{1k-1} \\ a_{21} & a_{22} & \cdots & a_{2k-2} & a_{2k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-21} & a_{k-22} & \cdots & a_{k-2k-2} & a_{k-2k-1} \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-2} & a_{k-1k-1} \end{vmatrix} &
\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-2} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k-2} & a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-21} & a_{k-22} & \cdots & a_{k-2k-2} & a_{k-2k} \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-2} & a_{k-1k} \end{vmatrix} \\
\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-2} & a_{1k-1} \\ a_{21} & a_{22} & \cdots & a_{2k-2} & a_{2k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-21} & a_{k-22} & \cdots & a_{k-2k-2} & a_{k-2k-1} \\ a_{k1} & a_{k2} & \cdots & a_{kk-2} & a_{kk-1} \end{vmatrix} &
\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-2} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k-2} & a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-21} & a_{k-22} & \cdots & a_{k-2k-2} & a_{k-2k} \\ a_{k1} & a_{k2} & \cdots & a_{kk-2} & a_{kk} \end{vmatrix}
\end{array}\right\|$$

holds. (See [5]).

## 2. Main results

### 2.1. Equivalent definitions of n-norm and n-seminorm

The next theorem is a generalization of Lemma 3.

**Theorem 2.** If $A \in M_n(\mathbf{R})$ is arbitrary matrix and $(x_1, x_2, \ldots, x_n) \in X^n$, then $\|A(x_1, x_2, \ldots, x_n)^T\| = |\det A|\, \|x_1, x_2, \ldots, x_n\|$.

**Proof.** We will consider two cases.

**Case 1.** Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $\det A \neq 0$. Then there exists an even permutation $\pi\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} (\pi(k) = i_k, k = 1, 2, \ldots, n)$ and matrix

$$B = \begin{bmatrix} a_{1i_1} & a_{1i_2} & a_{1i_3} & \cdots & a_{1i_{n-1}} & a_{1i_n} \\ a_{2i_1} & a_{2i_2} & a_{2i_3} & \cdots & a_{2i_{n-1}} & a_{2i_n} \\ a_{3i_1} & a_{3i_2} & a_{3i_3} & \cdots & a_{3i_{n-1}} & a_{3i_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1i_1} & a_{n-1i_2} & a_{n-1i_3} & \cdots & a_{n-1i_{n-1}} & a_{n-1i_n} \\ a_{ni_1} & a_{ni_2} & a_{ni_3} & \cdots & a_{ni_{n-1}} & a_{ni_n} \end{bmatrix}$$

such that $\det A = \det B$, and for any $k$, $k \in \{1, 2, \ldots, n\}$, it holds

$$\begin{vmatrix} a_{1i_1} & a_{1i_2} & \cdots & a_{1i_k} \\ a_{2i_1} & a_{2i_2} & \cdots & a_{2i_k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{ki_1} & a_{ki_2} & \cdots & a_{ki_k} \end{vmatrix} \neq 0\,.$$

So, without loss of generality, we may assume that the matrix $A$ is such that

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix} \neq 0, \quad \text{for} \quad k \in \{1, 2, \ldots, n\}.$$

Using the definition of n-norm and Lemma 1, we have

$$
\begin{aligned}
&\|A(x_1,x_2,\ldots,x_n)^T\| = \\
&= \left\| \sum_{i=1}^{n} a_{1i}x_i, \sum_{i=1}^{n} a_{2i}x_i, \sum_{i=1}^{n} a_{2i}x_i, \ldots, \sum_{i=1}^{n} a_{n-1i}x_i, \sum_{i=1}^{n} a_{ni}x_i \right\| \\
&= \left\| \sum_{i=1}^{n} a_{1i}x_i, \sum_{i=1}^{n} a_{2i}x_i - \frac{a_{21}}{a_{11}} \sum_{i=1}^{n} a_{1i}x_i, \sum_{i=1}^{n} a_{2i}x_i - \frac{a_{31}}{a_{11}} \sum_{i=1}^{n} a_{1i}x_i, \ldots, \right. \\
&\quad \left. \sum_{i=1}^{n} a_{n-1i}x_i - \frac{a_{n-11}}{a_{11}} \sum_{i=1}^{n} a_{1i}x_i, \sum_{i=1}^{n} a_{ni}x_i - \frac{a_{n1}}{a_{11}} \sum_{i=1}^{n} a_{1i}x_i \right\| \\
&= \left\| \sum_{i=1}^{n} a_{1i}x_i, \sum_{i=1}^{n} \left(a_{2i} - \frac{a_{21}}{a_{11}}a_{1i}\right)x_i, \sum_{i=1}^{n} \left(a_{3i} - \frac{a_{31}}{a_{11}}a_{1i}\right)x_i, \ldots, \right. \\
&\quad \left. \sum_{i=1}^{n} \left(a_{n-1i} - \frac{a_{n-11}}{a_{11}}a_{1i}\right)x_i, \sum_{i=1}^{n} \left(a_{ni} - \frac{a_{n1}}{a_{11}}a_{1i}\right)x_i \right\| \\
&= \left\| \sum_{i=1}^{n} a_{1i}x_i, \frac{1}{a_{11}} \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{21} & a_{2i} \end{vmatrix} x_i, \frac{1}{a_{11}} \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{31} & a_{3i} \end{vmatrix} x_i, \ldots, \right. \\
&\quad \left. \frac{1}{a_{11}} \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{n-11} & a_{n-1i} \end{vmatrix} x_i, \frac{1}{a_{11}} \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{n1} & a_{ni} \end{vmatrix} x_i \right\| \\
&= \frac{1}{|a_{11}|^{n-1}} \left\| \sum_{i=1}^{n} a_{1i}x_i, \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{21} & a_{2i} \end{vmatrix} x_i, \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{31} & a_{3i} \end{vmatrix} x_i, \ldots, \right. \\
&\quad \left. \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{n-11} & a_{n-1i} \end{vmatrix} x_i, \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{n1} & a_{ni} \end{vmatrix} x_i \right\|.
\end{aligned}
$$

In the next step, for each $j \geq 3$, on the $j$- th place, we apply the transformation

$$
\begin{aligned}
&\sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{j1} & a_{ji} \end{vmatrix} x_i - \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{j1} & a_{j2} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{21} & a_{2i} \end{vmatrix} x_i \\
&= \sum_{i=2}^{n} \left( \begin{vmatrix} a_{11} & a_{1i} \\ a_{j1} & a_{ji} \end{vmatrix} - \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{j1} & a_{j2} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \begin{vmatrix} a_{11} & a_{1i} \\ a_{21} & a_{2i} \end{vmatrix} \right) x_i
\end{aligned}
$$

$$= \sum_{i=2}^{n} \frac{\begin{vmatrix} a_{11} & a_{1i} \\ a_{j1} & a_{ji} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{j1} & a_{j2} \end{vmatrix} \begin{vmatrix} a_{11} & a_{1i} \\ a_{21} & a_{2i} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} x_i$$

$$= \frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \sum_{i=2}^{n} \left( \begin{vmatrix} a_{11} & a_{1i} \\ a_{j1} & a_{ji} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{j1} & a_{j2} \end{vmatrix} \begin{vmatrix} a_{11} & a_{1i} \\ a_{21} & a_{2i} \end{vmatrix} \right) x_i$$

$$= \frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \sum_{i=2}^{n} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{1i} \\ a_{21} & a_{2i} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{j1} & a_{j2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{1i} \\ a_{j1} & a_{ji} \end{vmatrix} \end{vmatrix} x_i$$

$$= \frac{a_{11}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \sum_{i=2}^{n} \frac{1}{a_{11}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{1i} \\ a_{21} & a_{2i} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{j1} & a_{j2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{1i} \\ a_{j1} & a_{ji} \end{vmatrix} \end{vmatrix} x_i$$

$$= \frac{a_{11}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{12} & a_{1i} \\ a_{21} & a_{22} & a_{2i} \\ a_{j1} & a_{j2} & a_{ji} \end{vmatrix} x_i .$$

Using the property (N3) of $n$-norm, we get

$$\|A(x_1, x_2 \ldots, x_n)^T\| = \frac{1}{|a_{11}|} \frac{1}{\left| \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right|^{n-2}}$$

$$\left\| \sum_{i=1}^{n} a_{1i} x_i, \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{21} & a_{2i} \end{vmatrix} x_i, \sum_{i=3}^{n} \begin{vmatrix} a_{11} & a_{12} & a_{1i} \\ a_{21} & a_{22} & a_{2i} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} x_i, \ldots, \sum_{i=3}^{n} \begin{vmatrix} a_{11} & a_{12} & a_{1i} \\ a_{21} & a_{22} & a_{2i} \\ a_{n1} & a_{n2} & a_{ni} \end{vmatrix} x_i \right\| .$$

Repeating the technique $(k-1)$ times, we get the equality

$$\|A(x_1, x_2 \ldots, x_n)^T\| =$$
$$\frac{1}{|a_{11}|} \frac{1}{\left| \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right|} \cdots \frac{1}{\left| \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-2} & a_{1k-1} \\ a_{21} & a_{22} & \cdots & a_{2k-2} & a_{2k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-21} & a_{k-22} & \cdots & a_{k-2k-2} & a_{k-2k-1} \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-2} & a_{k-1k-1} \end{vmatrix} \right|^{n-(k-1)}}$$
$$\cdot \ \|b_{k-11}, b_{k-12}, \ldots, b_{k-1n}\|$$

where

$$b_{k-11}=\sum_{i=1}^{n}a_{1i}x_i,\, b_{k-12}=\sum_{i=2}^{n}\begin{vmatrix} a_{11} & a_{1i}\\ a_{21} & a_{2i}\end{vmatrix}x_i,\ldots,$$

$$b_{k-1k-1}=\sum_{i=k-1}^{n}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-2} & a_{1i}\\ a_{21} & a_{22} & \cdots & a_{2k-2} & a_{2i}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-21} & a_{k-22} & \cdots & a_{k-2k-2} & a_{k-2i}\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-2} & a_{k-1i}\end{vmatrix}$$

$$b_{k-1k}=\sum_{i=k}^{n}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1i}\\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2i}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1i}\\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{ki}\end{vmatrix}x_i,$$

$$b_{k-1k+1}=\sum_{i=k}^{n}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1i}\\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2i}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1i}\\ a_{k+11} & a_{k+12} & \cdots & a_{k+1k-1} & a_{k+1i}\end{vmatrix}x_i,\ldots,$$

$$b_{k-1n}=\sum_{i=k}^{n}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1i}\\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2i}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1i}\\ a_{n1} & a_{n2} & \cdots & a_{nk-1} & a_{ni}\end{vmatrix}x_i\,.$$

In the $k$-th step, we apply transformation on the vectors $b_{k-1k+1}, b_{k-1k+2},\ldots, b_{k-1n}$, i.e. on the vector $b_{k-1j}, k+1\leq j\leq n$ we apply the transformation

$$b_{k-1j}=\sum_{i=k}^{n}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1i}\\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2i}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1i}\\ a_{j1} & a_{j2} & \cdots & a_{jk-1} & a_{ji}\end{vmatrix}x_i-$$

$$-\sum_{i=k}^{n}\frac{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1i}\\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2i}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1i}\\ a_{j1} & a_{j2} & \cdots & a_{jk-1} & a_{ji}\end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k}\\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k}\\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{kk}\end{vmatrix}}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1i}\\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2i}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1i}\\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{ki}\end{vmatrix}x_i$$

$$= \sum_{i=k}^{n} \left( \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1i} \\ a_{j1} & a_{j2} & \cdots & a_{jk-1} & a_{ji} \end{vmatrix} - \frac{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k} \\ a_{j1} & a_{j2} & \cdots & a_{jk-1} & a_{jk} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k} \\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{kk} \end{vmatrix}} \right.$$

$$\left. \cdot \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1i} \\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{ki} \end{vmatrix} \right) x_i = \frac{1}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k} \\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{kk} \end{vmatrix}}$$

$$\cdot \sum_{i=k}^{n} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k} \\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{kk} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1i} \\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{ki} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k} \\ a_{j1} & a_{j2} & \cdots & a_{jk-1} & a_{jk} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1i} \\ a_{j1} & a_{j2} & \cdots & a_{jk-1} & a_{ji} \end{vmatrix} \end{vmatrix} x_i$$

$$= \frac{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-2} & a_{1k-1} \\ a_{21} & a_{22} & \cdots & a_{2k-2} & a_{2k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-21} & a_{k-22} & \cdots & a_{k-2k-2} & a_{k-2k-1} \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-2} & a_{k-1k-1} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k} \\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{kk} \end{vmatrix}} \sum_{i=k+1}^{n} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2k} & a_{2i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} & a_{ki} \\ a_{j1} & a_{j2} & \cdots & a_{jk} & a_{ji} \end{vmatrix} x_i .$$

(The last equality is true because of Theorem 1)
Using the property (N3) of $n$-norm, we have

$$\|A(x_1, x_2, \ldots, x_n)^T\| = \frac{1}{|a_{11}|} \frac{1}{\left| \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right|} \cdots$$

$$\cdot \frac{1}{\left|\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-2} & a_{1k-1} \\ a_{21} & a_{22} & \cdots & a_{2k-2} & a_{2k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-21} & a_{k-22} & \cdots & a_{k-2k-2} & a_{k-2k-1} \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-2} & a_{k-1k-1} \end{vmatrix}\right|}$$

$$\cdot \frac{1}{\left|\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k} \\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{kk} \end{vmatrix}\right|^{n-k}} \|b_{k1}, b_{k2}, \ldots, b_{kn}\|$$

where

$$b_{k1} = \sum_{i=1}^{n} a_{1i}x_i\,, \quad b_{k,2} = \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{21} & a_{2i} \end{vmatrix} x_i, \ldots,$$

$$b_{kk-1} = \sum_{i=k-1}^{n} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-2} & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2k-2} & a_{2i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-21} & a_{k-22} & \cdots & a_{k-2k-2} & a_{k-2i} \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-2} & a_{k-1i} \end{vmatrix} x_i\,,$$

$$b_{kk} = \sum_{i=k}^{n} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1i} \\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{ki} \end{vmatrix} x_i\,,$$

$$b_{kk+1} = \sum_{i=k+1}^{n} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2k} & a_{2i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} & a_{ki} \\ a_{k+11} & a_{k+12} & \cdots & a_{k+1k} & a_{k+1i} \end{vmatrix} x_i, \ldots,$$

$$b_{kn} = \sum_{i=k+1}^{n} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2k} & a_{2i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} & a_{ki} \\ a_{n1} & a_{n2} & \cdots & a_{nk} & a_{ni} \end{vmatrix} x_i\,.$$

After finite number of steps, we get the vectors

$$c_1 = \sum_{i=1}^{n} a_{1i}x_i, \qquad c_2 = \sum_{i=2}^{n} \begin{vmatrix} a_{11} & a_{1i} \\ a_{21} & a_{2i} \end{vmatrix} x_i, \ldots,$$

$$c_{k-1}=\sum_{i=k-1}^{n}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-2} & a_{1i}\\ a_{21} & a_{22} & \cdots & a_{2k-2} & a_{2i}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-21} & a_{k-22} & \cdots & a_{k-2k-2} & a_{k-2k-1}\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-2} & a_{k-1k-1}\end{vmatrix} x_i\,,$$

$$c_{k}=\sum_{i=k}^{n}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1i}\\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2i}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1i}\\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{ki}\end{vmatrix} x_i\,,$$

$$c_{k+1}=\sum_{i=k+1}^{n}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} & a_{1i}\\ a_{21} & a_{22} & \cdots & a_{2k} & a_{2i}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{k1} & a_{k2} & \cdots & a_{kk} & a_{ki}\\ a_{k+11} & a_{k+12} & \cdots & a_{k+1k} & a_{k+1i}\end{vmatrix} x_i, \ldots,$$

$$c_{n-1}=\sum_{i=n-1}^{n}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n-2} & a_{1i}\\ a_{21} & a_{22} & \cdots & a_{2n-2} & a_{2i}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{n-21} & a_{n-22} & \cdots & a_{n-2n-2} & a_{n-2i}\\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n-2} & a_{n-1i}\end{vmatrix} x_i\,, \quad c_n=(\det A)x_n\,.$$

and

$$\|A(x_1,x_2,\ldots,x_n)^T\| = \frac{1}{|a_{11}|}\,\frac{1}{\left\|\begin{matrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{matrix}\right\|}\cdots$$

$$\cdot\;\frac{1}{\left\|\begin{matrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k}\\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{n-21} & a_{n-22} & \cdots & a_{n-2n-2} & a_{n-2n-1}\\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n-2} & a_{n-1n-1}\end{matrix}\right\|}\|c_1,c_2,c_3,\ldots,c_{n-1},c_n\|\,.$$

Finally, using Lemma 3, we get

$$\|A(x_1,x_2,\ldots,x_n)^T\| = |\det A|\;\|x_1,x_2,\ldots,x_n\|\,.$$

**Case 2.** Let $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}$ and $\det A=0$. Then the set of vectors $\left\{\sum_{i=1}^{n} a_{1i}x_i, \sum_{i=1}^{n} a_{2i}x_i, \ldots, \sum_{i=1}^{n} a_{ni}x_i\right\}$ is linearly dependent set, so

$$\|A(x_1,x_2,\ldots,x_n)^T\| = \left\|\sum_{i=1}^{n} a_{1i}x_i, \sum_{i=1}^{n} a_{2i}x_i, \ldots, \sum_{i=1}^{n} a_{ni}x_i\right\| = 0$$
$$= \det A\|x_1,x_2,\ldots,x_n\| = |\det A|\,\|x_1,x_2,\ldots,x_n\|\,. \qquad \square$$

According to the definition of n-norm of a vector space $X$ over the field $\mathbf{R}$, any n-norm satisfies the following conditions:

(P1) If $\|x_1,x_2,\ldots,x_n\|=0$, then the set $\{x_1,\ldots,x_n\}$ is linearly dependant;

(P2) For every matrix $A\in M_n(\mathbf{R})$ and for any $x_1,x_2,\ldots,x_n\in X$,

$$\|A(x_1,x_2,\ldots,x_n)^T\| = |\det A|\,\|x_1,x_2,\ldots,x_n\|\,;$$

(P3) For any $x_1,x_1',x_2,\ldots,x_n\in X$ ,

$$\|x_1+x_1',x_2,\ldots,x_n\| \le \|x_1,x_2,\ldots,x_n\| + \|x_1',x_2,\ldots,x_n\|\,.$$

Let us assume that $\|\cdot,\ldots,\cdot\|\colon X^n\to\mathbf{R}$ is a function which satisfies the conditions (P1)–(P3).

(i) Let us suppose that the set $\{x_1,\ldots,x_n\}$ is linearly dependant. Then there exists $i\in(1,2,\ldots,n\}$ such that

$$x_i = \alpha_1x_1+\alpha_2x_2+\cdots+\alpha_{i-1}x_{i-1}+\alpha_{i+1}x_{i+1}+\cdots+\alpha_nx_n\,.$$

So,

$$\begin{aligned}(x_1,\ldots,x_n) &= (x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n)\\ &= \Big(x_1,\ldots,x_{x-i},\sum_{j=1,j\neq i}^{n}\alpha_jx_j,x_{i+1},\ldots,x_n\Big)\\ &= A(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)^T\,,\end{aligned}$$

where

$$A=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0\\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{i-1} & 0 & \alpha_{i+1} & \cdots & \alpha_n\\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1\end{bmatrix}$$

According to (P2) and using that $\det A = 0$, we have

$$\begin{aligned}\|x_1,\dots,x_n\| &= \|x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n\| \\ &= \Big\|x_1,\dots,x_{i-1},\sum_{j=1,j\neq 1}^{n}\alpha_j x_j, x_{i+1},\dots,x_n\Big\| \\ &= \|A(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)^T\| \\ &= |\det A|\,\|x_1,x_2,\dots,x_{i-1},0,x_{i+1},\dots,x_n\| \\ &= 0\cdot\|x_1,x_2,\dots,x_{i-1},0,x_{i+1},\dots,x_n\| = 0\,.\end{aligned}$$

We proved that if $\|\cdot,\dots,\cdot\|\colon X^n \to \mathbf{R}$ is a function which satisfies the conditions (P1)–(P3), then $\|x_1,\dots,x_n\| \geq 0$ and $\|x_1,\dots,x_n\| = 0$ if and only if the set $\{x_1,\dots,x_n\}$ is linearly dependant, i.e. we proved that the axiom (N1) holds.

(ii) Let $x_1,\dots,x_n \in X$ and let $\pi\colon\{1,\dots,n\} \to \{1,\dots,n\}$ be arbitrary permutation. Then, because of the fact that any permutation matrix has determinant 1 or $-1$, we have

$$\begin{aligned}\|x_{\pi(1)},\dots,x_{\pi(n)} &= \|A(x_1,\dots,x_n)^T\| = |\det A|\,\|x_1,\dots,x_n\| \\ &= 1\cdot\|x_1,\dots,x_n\| = \|x_1,\dots,x_n\|\,.\end{aligned}$$

We proved that if $\|\cdot,\dots,\cdot\|\colon X^n \to \mathbf{R}$ is a function which satisfies the conditions (P1)–(P3), then $\|x_{\pi(1)},\dots,x_{\pi(n)}\| = \|x_1,\dots,x_n\|$, i.e. we proved that the axiom (N2) holds.

(iii) Suppose that $\alpha \in \mathbf{R}$ and $x_1,\dots,x_n \in X$ are arbitrarily chosen. Then

$$\begin{aligned}&\|\alpha x_1,x_2\dots,x_n\| = \|A(x_1,x_2,\dots,x_n)^T\| \\ &= |\det A|\,\|x_1,x_2,\dots,x_n\| = |\alpha|\,\|x_1,x_2,\dots,x_n\|\end{aligned}$$

where

$$A = \begin{bmatrix}\alpha & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1\end{bmatrix} \quad\text{and}\quad \det A = \alpha\,.$$

So, we proved Theorem 3.

**Theorem 3.** The conditions (N1)–(N4) are equivalent to the conditions (P1)–(P3). □

The notion of $n$-seminorm is generalization of the notion of 2-seminorm.

**Definition 2.** Let $X$ be a vector space over the field $\mathbf{R}$ and let $p\colon X^n \to \mathbf{R}$ be a mapping which satisfies the conditions:

a) $p(x_1, x_2, \dots, x_n) = 0$, for every linearly dependant set $\{x_1, \dots, x_n\}$;
b) $p(x_1, \dots, x_n) = p(x_{\pi(1)}, \dots, x_{\pi(n)})$, for every permutation $\pi: \{1, \dots, n\} \to \{1, \dots, n\}$;
c) $p(\alpha x_1, x_2, \dots, x_n) = |\alpha| p(x_1, x_2, \dots, x_n)$, for any scalar $\alpha$ and arbitrary $x_1, \dots, x_n \in X$;
d) $p(x_1 + x_1', x_2, \dots, x_n) \leq p(x_1, x_2, \dots, x_n) + p(x_1', x_2, \dots, x_n)$, for arbitrary $x_1, x_1', x_2, \dots, x_n \in X$.

The function $p: X^n \to \mathbf{R}$ which satisfies the conditions a)–d) is called n-seminorm, and the ordered pair $(X, p)$ is called real n-seminormed space.

**Theorem 4.** If $p: X^n \to \mathbf{R}$ is n-seminorm of the vector space $X$ over the field $\mathbf{R}$ and $(x_1, \dots, x_n), (x_1', \dots, x_n) \in M$, $A \in M_n(\mathbf{R})$, where $M = \{(x_1, \dots, x_n) | p(x_1, \dots, x_n) = 0\}$, then $(x_1, +x_1', \dots, x_n) \in M$ and $A(x_1, \dots, x_n)^T \in M$.

**Proof.** Let $(x_1, \dots, x_n), (x_1', \dots, x_n) \in M$ be arbitrary elements and let $A \in M_n(\mathbf{R})$ be arbitrary matrix. Then

$$0 \leq p(x_1 + x_1', \dots, x_n) \leq p(x_1, \dots, x_n) + p(x_1', \dots, x_n) = 0 + 0 = 0$$

i.e. $(x_1 + x_1', \dots, x_n) \in M$ and

$$p(A(x_1, \dots, x_n)^T) = |\det A| p(x_1, \dots, x_n) = |\det A| \cdot 0 = 0$$

i.e. $A(x_1, \dots, x_n)^T \in M$. □

With similar technique as it was used in the proves on n-norm, we can prove the next theorem.

**Theorem 5.** The conditions a), b), c) i d) in the definition 2 are equivalent to the conditions:
(S1) $p(A(x_1, x_2, \dots, x_n)^T) = |\det A| p(x_1, x_2, \dots, x_n)$, for any $x_1, \dots, x_n \in X$ and for every matrix $A \in M_n(\mathbf{R})$;
(S2) $p(x_1 + x_1', x_2, \dots, x_n) \leq p(x_1, x_2, \dots, x_n) + p(x_1', x_2, \dots, x_n)$. □

It can easily be seen the analogy between n-norm and n-seminorm with norm and seminorm, respectively, and that the role of the constant is now played by the determinants of order $n$.

So, it is clear that some properties of seminorm will immediately hold on n-seminorm.

## 2.2. The kernel of n-seminorm

The above work is motivation for the following definition.

**Definition 3.** Let $X$ be a vector space over the field $\mathbf{R}$ and $X^n$ denote the Cartesian product. For each $i \in \{1, 2, \dots, n\}$, we define

$$(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) + (x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n) =$$
$$= (x_1, \dots, x_{i-1}, x_i + x_i', x_{i+1}, \dots, x_n),$$

$x_1, \ldots, x_{i-1}, x_i', x_{i+1} \ldots, x_n \in X$ and we call it $i$-coordinative adding.

We, also, define $A \cdot (x_1, x_2, \ldots, x_n) = A(x_1, x_2, \ldots, x_n)^T$, $(x_1, \ldots, x_n) \in X^n$, $A \in M_n(\mathbf{R})$ and we call it multiplication with a matrix of order $n$ in $X^n$.

The set $X^n$ with the defined operations, we call it, **$n$-vector space.**

**Note.** The previous definitions, properties and notes show that the theory of $n$-normed spaces can be considered as a theory of "special algebraic structure". Thus this theory is analogous to the theory of normed and seminormed spaces, but with some specialties, some of them being the subject of our consideration.

**Definition 4.** Let $X^n$ be $n$-vector space. The subset $S \subset X^n$ such that

$$(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) + (x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \in S,$$
$$A(x_1, x_2, \ldots, x_n)^T \in S$$

for arbitrary $(x_1, \ldots, x_n)$, $(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$, $(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \in S$ and for every $A \in M_n(\mathbf{R})$, we call it **$n$-vector subspace** of the $n$-vector space $X^n$.

So, the set $M = \{(x_1, \ldots, x_n) \mid p(x_1, \ldots, x_n) = 0\}$, where $p\colon X^n \to \mathbf{R}$ is $n$-seminorm, is $n$-vector subspace.

For further considerations, we need the next theorem.

**Theorem 6.** The intersection of arbitrary subfamily of $n$-vector subspaces of given $n$-vector space $X^n$ is $n$-vector subspace of $X^n$.

**Proof.** Let $\{S_\gamma \mid \gamma \in \Gamma\}$ be an arbitrary family of $n$-subspaces of given $n$-vector space $X^n$. Then for $(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$, $(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \in \bigcap_{\gamma \in \Gamma} S_\gamma = S$, we have that

$$(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n), (x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \in S_\gamma,$$

for every $\gamma \in \Gamma$. Since $S_\gamma, \gamma \in \Gamma$ are $n$-vector subspaces, we get that

$$(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) + (x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \in S_\gamma,$$

for every $\gamma \in \Gamma$ i.e. $(x_1, \ldots, x_{i-1}, x_i + x_i', x_{i+1}, \ldots, x_n) \in \bigcap_{\gamma \in \Gamma} S_\gamma = S$.

Now let $A \in M_n(\mathbf{R})$ and $(x_1, \ldots, x_n) \in S$. Then $(x_1, \ldots, x_n) \in S_\gamma$, for every $\gamma \in \Gamma$. Since $S_\gamma$, $\gamma \in \Gamma$ are $n$-vector subspaces, we get that $A(x_1, \ldots, x_n)^T \in S_\gamma$, for every $\gamma \in \Gamma$. Therefore $A(x_1, \ldots, x_n)^T \in \bigcap_{\gamma \in \Gamma} S_\gamma = S$.

We proved that $S = \bigcap_{\gamma \in \Gamma} S_\gamma$ is $n$-vector subspace of the given $n$-vector space $X^n$. □

**Theorem 7.** Let $K$ be the kernel of n-seminorm $p$, i.e. $K$ is the set of elements of $X^n$ where $p$ is zero. Then $K$ is polygonal connected in $X^n$.

**Proof.** We will give the proof for $n = 2$. The proof for arbitrary $n$ is similar.

Let $(x,y),(x',y') \in K$ and let $t \in [0,1]$. Then $p(\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}(x,y)) = |\det \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}| p(x,y) = t^2 \cdot 0 = 0$, i.e. $(tx,ty) \in K$, and $p(\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}(x',y')) = |\det \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}| p(x',y') = t^2 \cdot 0 = 0$, i.e. $(tx',ty') \in K$.

So, in the ordinary sense of direct product of vector space, it holds $(1-t)(0,0)+t(x,y) \in K$, $t \in [0,1]$ and $(1-t)(0,0)+t(x',y') \in K$,$t \in [0,1]$.

Therefore there exists a polygonal line in $K$ which connects the points $(x,y)$,$(x',y') \in K$. □

If $A$ is an arbitrary subset of the $n$-vector space $X^n$ over the field $\mathbf{R}$, then the class $S_A$, $S_A = \{S \mid A \subseteq S, S \text{ is } n\text{-subspace of } X^n\}$ is not empty since $X^n \in S_A$. Therefore $\bigcap_{S \in S_A} S = M_A$ is $n$-vector subspace of $X^n$ and, at the same time, it is the smallest $n$-vector subspace which contains the set $A$. This subset is called the $n$-vector space generated by the set $A$. The set $A$ is called the generator set of the 2-vector subspace $M_A$.

### 2.3. Examples of 2-subspace of 2-space

**Example 1.** We will consider the real vector space $\mathbf{R}^3(\mathbf{R})$. Let $\mathbf{x},\mathbf{y},\mathbf{z}$ be arbitrary elements such that $L(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{R}^3$, i.e. $\mathbf{x},\mathbf{y},\mathbf{z}$ are linearly independent vectors. Let $S(\mathbf{a},\mathbf{b})$ be the 2-subspace generated by $\mathbf{a} = (\mathbf{x},\mathbf{z})$ and $\mathbf{b} = (\mathbf{y},\mathbf{z})$. We will prove that the 2-subspace $S(\mathbf{a},\mathbf{b})$ consists of all elements $(\mathbf{u},\mathbf{v})$, where $\mathbf{u},\mathbf{v} \in L(\alpha\mathbf{x}+\beta\mathbf{y}+\gamma\mathbf{z}, \alpha_1\mathbf{x}+\beta_1\mathbf{y}+\gamma_1\mathbf{z})$, and $L = L(\alpha\mathbf{x}+\beta\mathbf{y}+\gamma\mathbf{z}, \alpha_1\mathbf{x}+\beta_1\mathbf{y}+\gamma_1\mathbf{z})$ denotes the two dimensional subspace of $\mathbf{R}^3$ generated by the elements $\alpha\mathbf{x}+\beta\mathbf{y}+\gamma\mathbf{z}$ and $\alpha_1\mathbf{x}+\beta_1\mathbf{y}+y_1\mathbf{z}$.

We denote with $\pi_{\mathbf{x},\mathbf{z}}$ the plane which passes through the origin and it is parallel to the vectors $\mathbf{x}$ and $\mathbf{z}$, and analogously $\pi_{\mathbf{y},\mathbf{z}}$ denotes the plane which passes through the origin and it is parallel to the vectors $\mathbf{y}$ and $\mathbf{z}$. $\pi(\mathbf{x},\mathbf{y};\mathbf{z})$ denotes the pencil of the planes $\pi_{\mathbf{x},\mathbf{z}}$ and $\pi_{\mathbf{y},\mathbf{z}}$. We will prove that

$$S(\mathbf{a},\mathbf{b}) = \bigcup_{\sigma \in \pi(\mathbf{x},\mathbf{y};\mathbf{z})} \sigma \times \sigma$$

The two basic planes from the pencil of the planes are

$$\pi_{\mathbf{x},\mathbf{z}} = \{a\mathbf{x}+b\mathbf{z} \mid a,b \in \mathbf{R}\} \quad \text{and} \quad \pi_{\mathbf{y},\mathbf{z}} = \{c\mathbf{y}+d\mathbf{z} \mid c,d \in \mathbf{R}\}.$$

Any element of $\pi_{\mathbf{x},\mathbf{z}} \times \pi_{\mathbf{x},\mathbf{z}}$, $(a\mathbf{x}+b\mathbf{z}, a_1\mathbf{x}+b_1\mathbf{z})$, can be written in the form

$$\begin{bmatrix} a & b \\ a_1 & b_1 \end{bmatrix} (\mathbf{x},\mathbf{z})^T$$

i.e. it belongs to the 2-subspace generated by $\mathbf{a} = (\mathbf{x}, \mathbf{z})$ and $\mathbf{b} = (\mathbf{y}, \mathbf{z})$. Similarly, any element of $\pi_{\mathbf{y},\mathbf{z}} \times \pi_{\mathbf{y},\mathbf{z}}$, $(c\mathbf{y} + d\mathbf{z}, c_1\mathbf{y} + d_1\mathbf{z})$, is of the form

$$\begin{bmatrix} c & d \\ c_1 & d_1 \end{bmatrix} (\mathbf{y}, \mathbf{z})^T$$

i.e. it belongs to the 2-subspace generated by $\mathbf{a} = (\mathbf{x}, \mathbf{z})$ and $\mathbf{b} = (\mathbf{y}, \mathbf{z})$.

Each plane $\sigma$ of the the pencil of the planes is of the form $\sigma = \{\lambda(a\mathbf{x} + b\mathbf{y}) + \mu\mathbf{z} \mid \lambda, \mu \in \mathbf{R}\}$, where $a, b \in \mathbf{R}$ are fixed real numbers. Arbitrary element of $\sigma \times \sigma$ is of the form

$$(\lambda(a\mathbf{x} + b\mathbf{y}) + \mu\mathbf{z}, \lambda_1(a\mathbf{x} + b\mathbf{y}) + \mu_1\mathbf{z}),$$

where $\lambda, \mu, \lambda_1, \mu_1$ are arbitrary given, but fixed numbers. Therefore

$$(\lambda(a\mathbf{x} + b\mathbf{y}) + \mu\mathbf{z}, \lambda_1(a\mathbf{x} + b\mathbf{y}) + \mu_1\mathbf{z}) = \begin{bmatrix} \lambda & \mu \\ \lambda_1 & \mu_1 \end{bmatrix} (a\mathbf{x} + b\mathbf{y}, \mathbf{z})$$

$$= \begin{bmatrix} \lambda & \mu \\ \lambda_1 & \mu_1 \end{bmatrix} \{(a\mathbf{x}, \mathbf{z}) + (b\mathbf{y}, \mathbf{z})\} = \begin{bmatrix} \lambda & \mu \\ \lambda_1 & \mu_1 \end{bmatrix} \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} (\mathbf{x}, \mathbf{z}) + \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} (\mathbf{y}, \mathbf{z}) \right\}.$$

So, $\sigma \times \sigma \subseteq S(\mathbf{a}, \mathbf{b})$.

Let $\sigma$ and $\sigma_1$ be arbitrary planes from the the pencil of the planes such that $\sigma_1 \neq \sigma$. Then

$$\sigma_1 = \{\lambda_1(a_1\mathbf{x} + b_1\mathbf{y}) + \mu_1\mathbf{z} \mid \lambda_1, \mu_1 \in \mathbf{R}\}$$

and

$$\sigma = \{\lambda(a\mathbf{x} + b\mathbf{y}) + \mu\mathbf{z} \mid \lambda, \mu \in \mathbf{R}\}.$$

It is obvious that $a_1 \neq 0$ or $b_1 \neq 0$, and $a \neq 0$ or $b \neq 0$. (If $a_1 = b_1 = 0$ then $\sigma_1$ is not a plane i.e. it is a line, and analogously if $a = b = 0$, then $\sigma$ is not a plain but it is a line, which contradicts the assumption)

We will prove that $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \neq 0$ . Suppose the contrary, i.e. that $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$. We will consider four cases:

1. If $a_1 \neq 0$, then $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$ implies that $b = \dfrac{b_1}{a_1} a$. So

$$a\mathbf{x} + b\mathbf{y} = a\mathbf{x} + \frac{b_1}{a_1} a\mathbf{y} = a\Big(\mathbf{x} + \frac{b_1}{a_1}\mathbf{y}\Big), \; a_1\mathbf{x} + b_1\mathbf{y} = a_1\Big(\mathbf{x} + \frac{b_1}{a_1}\mathbf{y}\Big)$$

which is not possible because, in that case, we get that $\sigma = \sigma_1$.

2. If $b_1 \neq 0$, then $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$ implies that $a = \frac{a_1}{b_1}b$. So

$$a\mathbf{x} + b\mathbf{y} = a\mathbf{x} + \frac{b_1}{a_1}a\mathbf{y} = b\Big(\frac{b_1}{a_1}\mathbf{x} + \mathbf{y}\Big) \quad \text{and} \quad a_1\mathbf{x} + b_1\mathbf{y} = b_1\Big(\mathbf{y} + \frac{a_1}{b_1}\mathbf{x}\Big)$$

which is not possible because, in that case, we get that $\sigma_1 = \sigma$.
Similarly, in the left cases

3. $a \neq 0$ and $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$ and
4. $b \neq 0$ and $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$

we get that $\sigma_1 = \sigma$, which contradicts the assumption.

In the 2-subspaces, the adding of two elements
$\mathbf{a} = (\lambda_1(a_1\mathbf{x} + b_1\mathbf{y}) + \mu_1\mathbf{z}, \lambda_2(a_1\mathbf{x} + b_1\mathbf{y}) + \mu_2\mathbf{z}) \in \sigma_1 \times \sigma_1$ and
$\mathbf{b} = (\alpha_1(a\mathbf{x} + b\mathbf{y}) + \beta_1\mathbf{z}, \alpha_2(a\mathbf{x} + b\mathbf{y}) + \beta_2\mathbf{z}) \in \sigma \times \sigma$ is possible only if

1. $\lambda_1(a_1\mathbf{x} + b_1\mathbf{y}) + \mu_1\mathbf{z} = \alpha_1(a\mathbf{x} + b\mathbf{y}) + \beta_1\mathbf{z}$
2. $\lambda_2(a_1\mathbf{x} + b_1\mathbf{y}) + \mu_2\mathbf{z} = \alpha_2(a\mathbf{x} + b\mathbf{y}) + \beta_2\mathbf{z}$.

We will consider only the first case, since the cases are similar. Because of the linear independence of the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ the equality is possible if and only if

$$\begin{aligned} \lambda_1 a_1 - \alpha_1 a &= 0 \\ \lambda_1 b_1 - \alpha_1 b &= 0 \\ \mu_1 &= \beta_1 . \end{aligned}$$

So, we get four subcases:

1. $\lambda_1 = 0$, $\alpha_1 \neq 0$. Since $a_1 \neq 0$ or $b_1 \neq 0$, and $a \neq 0$ or $b \neq 0$, this case is impossible.
2. $\lambda_1 \neq 0$, $\alpha_1 = 0$. Since $a_1 \neq 0$ or $b_1 \neq 0$, and $a \neq 0$ or $b \neq 0$, this case is also impossible.
3. $\lambda_1 \neq 0$, $\alpha_1 \neq 0$. Then $a_1 = \frac{\alpha_1}{\lambda_1}a$ and $b_1 = \frac{\alpha_1}{\lambda_1}b$, so we have that $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$, which is not possible.
4. $\lambda_1 = \alpha_1 = 0$. Then the adding is possible and the sum is

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= (\mu_1\mathbf{z}, \lambda_2(a_1\mathbf{x} + b_1\mathbf{y}) + \mu_2\mathbf{z}) + (\mu_1\mathbf{z}, \alpha_2(a\mathbf{x} + b\mathbf{y}) + \beta_2\mathbf{z}) \\ &= (\mu_1\mathbf{z}, (\lambda_2 a_1 + \alpha_2 a)\mathbf{x} + (\lambda_2 b_1 + \alpha_2 b)\mathbf{y} + (\beta_2 + \mu_2)\mathbf{z}) \\ &= \begin{bmatrix} \mu_1 & 0 \\ 1 & \beta_2 + \mu_2 \end{bmatrix} (\mathbf{z}, (\lambda_2 a_1 + \alpha_2 a)\mathbf{x} + (\lambda_2 b_1 + \alpha_2 b)\mathbf{y})^T \\ &= \begin{bmatrix} \mu_1 & 0 \\ 1 & \beta_2 + \mu_2 \end{bmatrix} \{(\mathbf{z}, (\lambda_2 a_1 + \alpha_2 a)\mathbf{x}) + (\mathbf{z}, (\lambda_2 b_1 + \alpha_2 b)\mathbf{y})\}^T \\ &= \begin{bmatrix} \mu_1 & 0 \\ 1 & \beta_2 + \mu_2 \end{bmatrix} \left\{ \begin{bmatrix} 0 & 1 \\ \lambda_2 a_1 + \alpha_2 a & 0 \end{bmatrix} (\mathbf{x}, \mathbf{z})^T + \begin{bmatrix} 0 & 1 \\ \lambda_2 b_1 + \alpha_2 b & 0 \end{bmatrix} (\mathbf{y}, \mathbf{z})^T \right\}^T \end{aligned}$$

According to this $\mathbf{a}+\mathbf{b} \in \sigma_0 \times \sigma_0$, where

$$\sigma_0 = \{\delta\mathbf{z} + \eta((\lambda_2 a_1 + \alpha_2 a)\mathbf{x} + (\lambda_2 b_1 + \alpha_2 b)\mathbf{y}) \mid \delta,\ \eta \in \mathbf{R}\}.$$

So $\dot{\bigcup\limits_{\sigma\in\pi(\mathbf{x},\mathbf{y};\mathbf{z})}} \sigma\times\sigma$ is 2-subspace of $\mathbf{R}^3 \times \mathbf{R}^3$ which contains the elements $\mathbf{a} = (\mathbf{x}, \mathbf{z})$ and $\mathbf{b} = (\mathbf{y}, \mathbf{z})$, and according to the construction, it is a subspace of any subspace $S$ which contains the same elements. Therefore

$$S(\mathbf{a},\mathbf{b}) = \bigcup_{\sigma\in\pi(\mathbf{x},\mathbf{y};\mathbf{z})} \sigma \times \sigma\,.$$

**Example 2.** Let $X$ be a vector space over a field $\Phi$ and let $\dim X > 2$. Let $x, y, z \in X$ be arbitrary vectors of $X$, such that $\dim L(x, y, z) = 3$. We will consider the 2-subspace $S$ generated by the vectors $\mathbf{a} = (x, z)$, $\mathbf{b} = (y, z)$, i.e. $S = S(\mathbf{a}, \mathbf{b})$. Let $M = M(a_1x + a_2y + a_3z, b_1x + b_2y + b_3z)$ be the two dimensional vector subspace of $X$, which contains, as its own subspace, the one dimensional vector subspace $M_0 = M_0(z)$. Any element of the form $(u, v)$, where $u, v \in M$, belongs to $S(\mathbf{a}, \mathbf{b})$. The subspace $M$ is usually denoted by $M(z)$. The class of two dimensional subspaces of this form is denoted with $P(x, y; z)$, and we call it pencil of the subspaces with the base $M_0 = M_0(z)$. It can be proven (as in the previous example) that

$$S(\mathbf{a},\mathbf{b}) = \dot{\bigcup_{L\in P(x,y;z)}} L \times L\,.$$

**Problem.** It remains to characterize the n- subspaces of $X^n$, where $X$ is a vector space over an arbitrary field $\Phi$.

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# АЛГЕБАРСКА СТРУКТУРА НА ЈАДРОТО НА $n$-ПОЛУНОРМА

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## Резиме

Поимот $n$-норма е воведен во [4] како обопштување на поимот 2-норма, воведен во [1]. Поимот $n$-полунорма е воведен во [3] како обопштување на поимот 2-полунорма. Еквивалентна дефиниција на 2-норма е дадена во [2]. Во првиот дел од овој труд даваме еквивалентна дефиниција на $n$-полунорма. Во вториот дел од трудот правиме карактеризација на јадрото на $n$-полунорма како класа на подмножества од $X^n$ и конструираме два примери на 2-подпростор од 2-векторски простор.

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