Математички Билтен
31 (LVII)
2007 (21–32)
Скопје, Македонија

ISSN 0351–336X

CERTAIN EXPANSIONS INVOLVING GENERALIZED BASIC HYPERGEOMETRIC FUNCTIONS

S.D. Purohit[1] and S.L. Kalla[2]

Abstract

Some fundamental operators of fractional q-calculus are used to prove a theorem on an expansion formula for the generalized basic hypergeometric functions. Certain interesting consequences of the theorem are also discussed. This general theorem is then applied to derive a number of expansion formulae for the familiar q-analogue of various basic hypergeometric functions.

1. Introduction

Agarwal [1] introduced the fractional q-derivative operator as under:

$$D^{\beta}_{z,q}\{f(z)\} = \frac{1}{\Gamma_q(-\beta)} \int_0^z (z-tq)_{-\beta-1} f(t)\, d(t;q), \tag{1.1}$$

valid for all values of β.

In view of the definition of the basic integral cf. Agarwal [1] (see also Gasper and Rahman [4]), namely

$$\int_0^z f(t)d(t;q) = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k), \tag{1.2}$$

the operator (1.1) can be expressed as:

$$D^{\beta}_{z,q}\{f(z)\} = \frac{z^{-\beta}(1-q)}{\Gamma_q(-\beta)} \sum_{k=0}^{\infty} q^k(1-q^{k+1})_{-\beta-1} f(zq^k). \qquad (1.3)$$

For $f(z) = z^{\alpha-1}$, the equation (1.3) yields to

$$D^{\beta}_{z,q}(z^{\alpha-1}) = \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}\, z^{\alpha-\beta-1}, \qquad Re(\alpha) > 0, \qquad (1.4)$$

valid for all values of β.

Further, Agarwal [1] defined the q-extension of the Leibniz rule for the fractional q-derivatives for a product of two functions in terms of a series involving fractional q-derivatives of the individual functions in the following manner:

$$D^{\beta}_{z,q}\{U(z)V(z)\} = \qquad (1.5)$$
$$= \sum_{n=0}^{\beta} \frac{(-1)^n q^{n(n+1)/2}\, (q^{-\beta};q)_n}{(q;q)_n}\, D^{\beta-n}_{z,q}\{U(zq^n)\}\, D^{n}_{z,q}\{V(z)\},$$

where $U(z)$ and $V(z)$ are two regular functions such that

$$U(z) = \sum_{r=0}^{\infty} a_r z^r, \quad |z| < R_1; \qquad V(z) = \sum_{r=0}^{\infty} b_r z^r, \quad |z| < R_2,$$

then for the result (1.5), $|z| < R = \min(R_1, R_2)$.

On using q-identity [4, I.42, p. 235], the equation (1.5) can also be written as

$$D^{\beta}_{z,q}\{U(z)V(z)\} = \sum_{n=0}^{\beta} \begin{bmatrix}\beta\\ n\end{bmatrix}_q q^{n^2-\beta n}\, D^{\beta-n}_{z,q}\{U(zq^n)\}\, D^{n}_{z,q}\{V(z)\}, \quad (1.6)$$

where the q-Binomial coefficient is defined as:

$$\begin{bmatrix}m\\ n\end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_{m-n}(q;q)_n}. \qquad (1.7)$$

We shall make use of the following notations and definitions in the sequel: For real or complex a, $|q| < 1$, the q-shifted factorial is defined as:

$$(a;q)_n = \begin{cases} 1; & \text{if} \quad n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}); & \text{if} \quad n \in N. \end{cases} \qquad (1.8)$$

In terms of the q-gamma function, the equation (1.8) can be expressed as

$$(a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0, \tag{1.9}$$

where the q-gamma function (cf. Gasper and Rahman [4]), is given by

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty(1-q)^{a-1}}. \tag{1.10}$$

Indeed, it is easy to verify that

$$\lim_{q\to 1^-} \Gamma_q(a) = \Gamma(a) \quad \text{and} \quad \lim_{q\to 1^-} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n, \tag{1.11}$$

where

$$(a)_n = a(a+1)\cdots(a+n-1). \tag{1.12}$$

The abnormal type of generalized basic hypergeometric series ${}_r\Phi_s(.)$ is defined as:

$${}_r\Phi_s\left[\begin{matrix} a_1,\cdots,a_r;q;z \\ b_1,\cdots,b_s;q^\lambda \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n} z^n \, q^{\lambda n(n+1)/2}, \tag{1.13}$$

where $\lambda > 0$.

For $\lambda = 0$, series (1.13) reduces to the generalized basic hypergeometric series ${}_r\Phi_s(.)$ (see Slater [9]), as under:

$${}_r\Phi_s\left[\begin{matrix} a_1,\cdots,a_r; \\ b_1,\cdots,b_s; \end{matrix} \; q,z\right] = \sum_{n=0}^{\infty} \frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n} z^n, \tag{1.14}$$

where for the convergence of the series (1.14), we have $|q| < 1$, for all z if $r \le s$ and $|z| < 1$ if $r = s+1$.

Recently, in a series of papers Yadav and Purohit (see [11], [12] and [6]) investigated certain applications of q-Leibniz rule given by equation (1.5) and deduced several interesting transformations and expansions involving various basic hypergeometric functions of one and more variables including the basic analogue of Foxs H-function due to Saxena, Modi and Kalla [7]. Earlier Agarwal [1], Denis [2] and Shukla [8] have applied the formula (1.5) to derive certain interesting transformations involving basic hypergeometric functions of one variable.

The motive of the present paper is to explore the possibility for the derivation of an expansion formula for the basic hypergeometric function by making use of certain fundamental operators of fractional q-calculus. Some interesting special cases and applications of the main result are also stated in the concluding section.

2. Main Rezult

In this section, we shall establish a theorem, which is an immediate consequence of the q-Leibniz rule.

Firstly, we mention below the results, which are needed in the sequel.

Lemma 1. If $Re(\mu) > Re(\lambda) > 0$, $\sigma \geq 0$ and ρ being any complex quantity, then

$$D_{z,q}^{\lambda-\mu}\left\{ z^{\lambda-1}\ {}_r\Phi_s\left[\begin{matrix}\alpha_1,\cdots,\alpha_r;q,\rho z\\ \beta_1,\cdots,\beta_s;\ q^{\sigma}\end{matrix}\right]\right\} = \tag{2.1}$$
$$= \frac{\Gamma_q(\lambda)}{\Gamma_q(\mu)}\, z^{\mu-1}\ {}_{r+1}\Phi_{s+1}\left[\begin{matrix}\alpha_1,\cdots,\alpha_r,q^{\lambda};q,\rho z\\ \beta_1,\cdots,\beta_s,q^{\mu};\ q^{\sigma}\end{matrix}\right].$$

Lemma 2. If $\sigma \geq 0$ and ρ being any complex quantity, then

$$D_{z,q}^{n}\left\{ {}_r\Phi_s\left[\begin{matrix}\alpha_1,\cdots,\alpha_r;q,\rho z\\ \beta_1,\cdots,\beta_s;\ q^{\sigma}\end{matrix}\right]\right\} = \tag{2.2}$$
$$= \frac{(\alpha_1;q)_n\cdots(\alpha_r;q)_n}{(\beta_1;q)_n\cdots(\beta_s;q)_n}(1-q)^{-n}\ \rho^n q^{\sigma n(n+1)/2}\ {}_r\Phi_s\left[\begin{matrix}\alpha_1 q^n,\cdots,\alpha_r q^n;q,\rho z q^{\sigma n}\\ \beta_1 q^n,\cdots,\beta_s q^n;\ q^{\sigma}\end{matrix}\right].$$

For the sake of brevity, we avoid the detailed proof of lemmas. One can easily prove the above mentioned lemmas, by making use of the definition (1.13) and fractional q-derivative formula (1.4).

Theorem. *If*

$${}_r\Phi_s\left[\begin{matrix}\alpha_1,\cdots,\alpha_r;\ q,\rho z\\ \beta_1,\cdots,\beta_s;\ q^{\sigma}\end{matrix}\right] = \tag{2.3}$$
$$= {}_k\Phi_l\left[\begin{matrix}a_1,\cdots,a_k;\ q,\omega z\\ b_1,\cdots,b_l;\ q^{\gamma}\end{matrix}\right]\ {}_m\Phi_p\left[\begin{matrix}c_1,\cdots,c_m;\ q,\eta z\\ d_1,\cdots,d_p;\ q^{\delta}\end{matrix}\right],$$

where $\sigma,\gamma,\delta \geq 0$ and ρ,ω,η being any complex quantities then, with due regard to convergence,

$$
{}_{r+1}\Phi_{s+1}\left[\begin{matrix}\alpha_1,\cdots,\alpha_r,q^{\lambda};\ q,\rho z\\ \beta_1,\cdots,\beta_s,q^{\mu};\ q^{\sigma}\end{matrix}\right]=
$$

$$
=\sum_{n\geq 0}\left[\begin{matrix}\lambda-\mu\\ n\end{matrix}\right]_q \eta^n\ z^n\ q^{n^2+\mu n-n+\delta n(n+1)/2} \tag{2.4}
$$

$$
\times\ \frac{(c_1;q)_n\cdots(c_m;q)_n}{(q^{\mu};q)_n\ (d_1;q)_n\cdots(d_p;q)_n}\ {}_{k+1}\Phi_{l+1}\left[\begin{matrix}a_1,\cdots,a_k,q^{\lambda}\ ;\ q,\omega z q^n\\ b_1,\cdots,b_l,q^{\mu+n};\ q^{\gamma}\end{matrix}\right]
$$

$$
\times\ {}_m\Phi_p\left[\begin{matrix}c_1q^n,\cdots,c_mq^n;\ q,\eta z q^{\delta n}\\ d_1q^n,\cdots,d_pq^n;\ q^{\delta}\end{matrix}\right].
$$

Proof. To prove the theorem (2.4), we multiplying $z^{\lambda-1}$ both the sides of the relation (2.3), and then on operating $D_{z,q}^{\lambda-\mu}$ to both the sides, we obtain

$$
D_{z,q}^{\lambda-\mu}\left\{z^{\lambda-1}{}_r\Phi_s\left[\begin{matrix}\alpha_1,\cdots,\alpha_r;\ q,\rho z\\ \beta_1,\cdots,\beta_s;\ q^{\sigma}\end{matrix}\right]\right\}= \tag{2.5}
$$

$$
=D_{z,q}^{\lambda-\mu}\left\{z^{\lambda-1}\ {}_k\Phi_l\left[\begin{matrix}a_1,\cdots,a_k;\ q,\omega z\\ b_1,\cdots,b_l;\ q^{\gamma}\end{matrix}\right]\ {}_m\Phi_p\left[\begin{matrix}c_1,\cdots,c_m;\ q,\eta z\\ d_1,\cdots,d_p;\ q^{\delta}\end{matrix}\right]\right\}.
$$

Now, on setting

$$
U(z)=z^{\lambda-1}\ {}_k\Phi_l\left[\begin{matrix}a_1,\cdots,a_k;\ q,\omega z\\ b_1,\cdots,b_l;\ q^{\gamma}\end{matrix}\right]
$$

and

$$
V(z)={}_m\Phi_p\left[\begin{matrix}c_1,\cdots,c_m;\ q,\eta z\\ d_1,\cdots,d_p;\ q^{\delta}\end{matrix}\right]
$$

and then using the q-Leibniz rule (1.6) in the right hand side of equation (2.5), it yields to

$$
D_{z,q}^{\lambda-\mu}\left\{z^{\lambda-1}{}_r\Phi_s\left[\begin{matrix}\alpha_1,\cdots,\alpha_r;\ q,\rho z\\ \beta_1,\cdots,\beta_s;\ q^{\sigma}\end{matrix}\right]\right\}= \tag{2.6}
$$

$$
=\sum_{n\geq 0}\left[\begin{matrix}\lambda-\mu\\ n\end{matrix}\right]_q q^{n^2-(\lambda-\mu)n}\ D_{z,q}^{\lambda-\mu-n}\left\{(zq^n)^{\lambda-1}\ {}_k\Phi_l\left[\begin{matrix}a_1,\cdots,a_k;\ q,\omega z q^n\\ b_1,\cdots,b_l;\ q^{\gamma}\end{matrix}\right]\right\}
$$

$$
\times\ D_{z,q}^{n}\left\{{}_m\Phi_p\left[\begin{matrix}c_1,\cdots,c_m;\ q,\eta z\\ d_1,\cdots,d_p;\ q^{\delta}\end{matrix}\right]\right\}.
$$

With the help of lemmas, one can easily obtain the following fractional q-derivative formulae for the abnormal type of generalized basic hypergeometric function ${}_r\Phi_s(.)$, involved in the equation (2.6), namely

$$D_{z,q}^{\lambda-\mu-n}\left\{z^{\lambda-1}\ {}_k\Phi_l\left[\begin{matrix}a_1,\cdots,a_k;q,\omega zq^n\\ b_1,\cdots,b_l;\ q^\gamma\end{matrix}\right]\right\}=$$
$$=\frac{\Gamma_q(\lambda)}{\Gamma_q(\mu+n)}z^{\mu+n-1}\ {}_{k+1}\Phi_{l+1}\left[\begin{matrix}a_1,\cdots,a_k,q^\lambda\ ;q,\omega zq^n\\ b_1,\cdots,b_l,q^{\mu+n};q^\gamma\end{matrix}\right], \tag{2.7}$$

and

$$D_{z,q}^{n}\left\{{}_m\Phi_p\left[\begin{matrix}c_1,\cdots,c_m;q,\eta z\\ d_1,\cdots,d_p;\ q^\delta\end{matrix}\right]\right\}= \tag{2.8}$$
$$=\frac{(c_1;q)_n\cdots(c_m;q)_n}{(d_1;q)_n\cdots(d_p;q)_n}(1-q)^{-n}\ \eta^n q^{\delta n(n+1)/2}\ {}_m\Phi_p\left[\begin{matrix}c_1q^n,\cdots,c_mq^n;q,\eta zq^{\delta n}\\ d_1q^n,\cdots,d_pq^n;\ q^\delta\end{matrix}\right].$$

On substituting the values of the various expressions involved in the equation (2.6), from equations (2.1), (2.7) and (2.8), we arrive at the main theorem (2.4).

3. Special Cases

In this section, we discuss some of the important special cases of the main theorem proved in the previous section:

(i) On setting $\sigma=\gamma=\delta=0$, the theorem (2.4) leads to an expansion formula for generalized basic hypergeometric function ${}_r\Phi_s(.)$, namely

$${}_{r+1}\Phi_{s+1}\left[\begin{matrix}\alpha_1,\cdots,\alpha_r,q^\lambda;\\ \beta_1,\cdots,\beta_s,q^\mu;\end{matrix}\ q,\rho z\right]=$$
$$=\sum_{n\geq 0}\begin{bmatrix}\lambda-\mu\\ n\end{bmatrix}_q\ \eta^n\ z^n\ q^{n^2+\mu n-n} \tag{3.1}$$
$$\times\ \frac{(c_1;q)_n\cdots(c_m;q)_n}{(q^\mu;q)_n\ (d_1;q)_n\cdots(d_p;q)_n}\ {}_{k+1}\Phi_{l+1}\left[\begin{matrix}a_1,\cdots,a_k,q^\lambda\ ;\\ b_1,\cdots,b_l,q^{\mu+n};\end{matrix}\ q,\omega zq^n\right]$$
$$\times\ {}_m\Phi_p\left[\begin{matrix}c_1q^n,\cdots,c_mq^n;\\ d_1q^n,\cdots,d_pq^n;\end{matrix}\ q,\eta z\right],$$

where

$$
{}_r\Phi_s\left[\begin{matrix}\alpha_1,\cdots,\alpha_r;\\ \beta_1,\cdots,\beta_s;\end{matrix}\; q,\rho z\right] = \\
= {}_k\Phi_l\left[\begin{matrix}a_1,\cdots,a_k;\\ b_1,\cdots,b_l;\end{matrix}\; q,\omega z\right]\; {}_m\Phi_p\left[\begin{matrix}c_1,\cdots,c_m;\\ d_1,\cdots,d_p;\end{matrix}\; q,\eta z\right]. \tag{3.2}
$$

Interestingly, for $\rho = \omega = \eta = 1$ and in view of the limit formula (1.11), one can easily prove that the result (3.1) is a q-extension of the known result due to Manocha and Sharma [5, p. 382, eqn (6.1)].

(ii) Again, if we put $\omega = 0$ in the result (3.1), it yields to

$$
{}_{r+1}\Phi_{s+1}\left[\begin{matrix}\alpha_1,\cdots,\alpha_r,q^\lambda;\\ \beta_1,\cdots,\beta_s,q^\mu;\end{matrix}\; q,\rho z\right] = \\
= \sum_{n\geq 0}\begin{bmatrix}\lambda-\mu\\ n\end{bmatrix}_q \rho^n\, z^n\, q^{n^2+\mu n-n} \tag{3.3} \\
\times \frac{(\alpha_1;q)_n\cdots(\alpha_r;q)_n}{(q^\mu;q)_n\,(\beta_1;q)_n\cdots(\beta_s;q)_n}\; {}_r\Phi_s\left[\begin{matrix}\alpha_1 q^n,\cdots,\alpha_r q^n;\\ \beta_1 q^n,\cdots,\beta_s q^n;\end{matrix}\; q,\rho z\right],
$$

where

$$
{}_r\Phi_s\left[\begin{matrix}\alpha_1,\cdots,\alpha_r;\\ \beta_1,\cdots,\beta_s;\end{matrix}\; q,\rho z\right] = {}_m\Phi_p\left[\begin{matrix}c_1,\cdots,c_m;\\ d_1,\cdots,d_p;\end{matrix}\; q,\eta z\right] \tag{3.4}
$$

Further, it is interesting to observe that, for $\rho = 1$ the result (3.3) reduces to a known result due to Yadav and Purohit [11, p. 43, eqn. (3.3)].

4. Applications of the Main Result

We shall now illustrate the applications of the main theorem introduced by means of the equation (2.4) to derive a number of expansion formulae involving basic hypergeometric functions. The results are obtained by considering the well-known transformations [similar as the result (2.1)] involving basic hypergeometric functions. For example, if we consider the following transformation (cf. Gasper and Rahmann [4], Yadav and Purohit [11], Srivastava and Agarwal [10] and Denis, Singh and Singh [3]), namely

$$
{}_1\Phi_0\left[\begin{matrix}ab\,;\\ -;\end{matrix}\; q,z\right] = {}_1\Phi_0\left[\begin{matrix}a\,;\\ -;\end{matrix}\; q,z\right]{}_1\Phi_0\left[\begin{matrix}b\,;\\ -;\end{matrix}\; q,az\right], \tag{4.1}
$$

$$ {}_2\Phi_1\left[\begin{matrix} a,b; \\ c; \end{matrix}\; q,z\right] = {}_1\Phi_0\left[\begin{matrix} ab/c; \\ -; \end{matrix}\; q,z\right] {}_2\Phi_1\left[\begin{matrix} c/a,c/b; \\ c; \end{matrix}\; q,abz/c\right], \tag{4.2} $$

$$ {}_1\Phi_1\left[\begin{matrix} q^{\alpha}; \\ q^{\beta}; \end{matrix}\; q,z\right] = {}_0\Phi_0\left[\begin{matrix} -; \\ -; \end{matrix}\; q,-z\right] {}_1\Phi_1\left[\begin{matrix} q^{\beta-\alpha};\; q, -zq^{\alpha-1} \\ q^{\beta};\; q^1 \end{matrix}\right], \tag{4.3} $$

$$ {}_2\Phi_1\left[\begin{matrix} a,0; \\ b; \end{matrix}\; q,z\right] = {}_1\Phi_0\left[\begin{matrix} 0; \\ -; \end{matrix}\; q,z\right] {}_1\Phi_1\left[\begin{matrix} b/a; \\ b; \end{matrix}\; q,az\right], \tag{4.4} $$

$$ \begin{aligned} &{}_4\Phi_3\left[\begin{matrix} \sqrt{\omega}, -\sqrt{\omega}, \sqrt{\omega q}, -\sqrt{\omega q}; \\ \omega, -\omega/a, -aq; \end{matrix}\; q,z\right] = \\ &= {}_2\Phi_1\left[\begin{matrix} a, aq; \\ a^2q^2; \end{matrix}\; q^2, zq\right] {}_2\Phi_1\left[\begin{matrix} \omega q/a, \omega/a; \\ \omega^2/a^2; \end{matrix}\; q^2, z\right], \end{aligned} \tag{4.5} $$

which on using q-identity [4, I.28, p.234] can also be written as

$$ \begin{aligned} &{}_4\Phi_3\left[\begin{matrix} \sqrt{\omega}, -\sqrt{\omega}, \sqrt{\omega q}, -\sqrt{\omega q}; \\ \omega, -\omega/a, -aq; \end{matrix}\; q,z\right] = \\ &= {}_4\Phi_3\left[\begin{matrix} \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; \\ aq, -aq, -q; \end{matrix}\; q, zq\right] \\ &\times\; {}_4\Phi_3\left[\begin{matrix} \sqrt{\omega q/a}, -\sqrt{\omega q/a}, \sqrt{\omega/a}, -\sqrt{\omega/a}; \\ \omega/a, -\omega/a, -q; \end{matrix}\; q,z\right], \end{aligned} \tag{4.6} $$

and then applying the theorem (2.4), we obtain the respective expansion formulae involving basic hypergeometric functions as under:

$$ \begin{aligned} &{}_2\Phi_1\left[\begin{matrix} ab\,, q^{\lambda}; \\ q^{\mu}; \end{matrix}\; q,z\right] = \\ &= \sum_{n\geq 0} \left[\begin{matrix} \lambda-\mu \\ n \end{matrix}\right]_q a^n\, z^n\, q^{n^2+\mu n-n} \\ &\times\; \frac{(b;q)_n}{(q^{\mu};q)_n}\, {}_2\Phi_1\left[\begin{matrix} a, q^{\lambda}; \\ q^{\mu+n}; \end{matrix}\; q, zq^n\right] {}_1\Phi_0\left[\begin{matrix} bq^n; \\ -; \end{matrix}\; q, az\right], \end{aligned} \tag{4.7} $$

$$
\begin{aligned}
&{}_3\Phi_2\left[\begin{matrix} a, b, q^{\lambda}; \\ c, q^{\mu}; \end{matrix}\; q, z\right] = \\
&= \sum_{n\geq 0} \begin{bmatrix} \lambda-\mu \\ n \end{bmatrix}_q (ab/c)^n \; z^n \; q^{n^2+\mu n-n} \\
&\times \; \frac{(c/a;q)_n (c/b;q)_n}{(q^{\mu};q)_n (c;q)_n} \; {}_2\Phi_1\left[\begin{matrix} ab/c, q^{\lambda}; \\ q^{\mu+n}; \end{matrix}\; q, zq^n\right] \\
&\times \; {}_2\Phi_1\left[\begin{matrix} cq^n/a, cq^n/b; \\ cq^n; \end{matrix}\; q, abz/c\right],
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
&{}_2\Phi_2\left[\begin{matrix} q^{\alpha}, q^{\lambda}; \\ q^{\beta}, q^{\mu}; \end{matrix}\; q, z\right] = \\
&= \sum_{n\geq 0} \begin{bmatrix} \lambda-\mu \\ n \end{bmatrix}_q (-1)^n \; z^n \; q^{3n(n-1)/2+n(\mu+\alpha)} \\
&\times \; \frac{(q^{\beta-\alpha};q)_n}{(q^{\mu};q)_n (q^{\beta};q)_n} \; {}_1\Phi_1\left[\begin{matrix} q^{\lambda}; \\ q^{\mu+n}; \end{matrix}\; q, -zq^n\right] \\
&\times \; {}_1\Phi_1\left[\begin{matrix} q^{\beta-\alpha+n}; q, -zq^{n+\alpha-1} \\ q^{\beta+n}; q^1 \end{matrix}\right],
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
&{}_3\Phi_2\left[\begin{matrix} a, 0, q^{\lambda}; \\ b, q^{\mu}; \end{matrix}\; q, z\right] = \\
&= \sum_{n\geq 0} \begin{bmatrix} \lambda-\mu \\ n \end{bmatrix}_q (a)^n \; z^n \; q^{n^2+\mu n-n} \\
&\times \; \frac{(b/a;q)_n}{(q^{\mu};q)_n (b;q)_n} \; {}_2\Phi_1\left[\begin{matrix} 0, q^{\lambda}; \\ q^{\mu+n}; \end{matrix}\; q, zq^n\right] \\
&\times \; {}_1\Phi_1\left[\begin{matrix} bq^n/a; \\ bq^n; \end{matrix}\; q, az\right],
\end{aligned}
\tag{4.10}
$$

and

$$
{}_5\Phi_4\left[\begin{matrix}\sqrt{\omega},-\sqrt{\omega},\sqrt{\omega q},-\sqrt{\omega q},q^{\lambda}; \\ \omega,-\omega/a,-aq,q^{\mu};\end{matrix}\ q,z\right]=
$$

$$
=\sum_{n\geq 0}\begin{bmatrix}\lambda-\mu\\ n\end{bmatrix}_q z^n\, q^{n^2+\mu n-n}
$$

$$
\times\ \frac{(\sqrt{\omega q/a};q)_n(-\sqrt{\omega q/a};q)_n(\sqrt{\omega/a};q)_n(-\sqrt{\omega/a};q)_n}{(q^{\mu};q)_n\ (\omega/a;q)_n(-\omega/a;q)_n(-q;q)_n} \tag{4.11}
$$

$$
\times\ {}_5\Phi_4\left[\begin{matrix}\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},q^{\lambda}; \\ aq,-aq,-q,q^{\mu+n};\end{matrix}\ q,zq^{n+1}\right]
$$

$$
\times\ {}_4\Phi_3\left[\begin{matrix}\sqrt{\omega q/a}q^n,-\sqrt{\omega q/a}q^n,\sqrt{\omega/a}q^n,-\sqrt{\omega/a}q^n; \\ \omega q^n/a,-\omega q^n/a,-q^{n+1};\end{matrix}\ q,z\right].
$$

We conclude with the remark that, the theorem introduced by means of the equation (2.4), is an elegant technique for deriving numerous transformations, and expansions involving various basic hypergeometric functions. The results thus derived in this paper are general in character and give some contributions to the theory of the q-series.

References

[1] R. P. Agarwal: *Fractional q-derivatives and q-integrals and certain hypergeometric transformations*, Ganita **27**(1976), 25-32.

[2] R.Y. Denis: *On certain special transformations of poly-basic hypergeometric functions*, The Math. Student, **51**(1983), no.1-4, 121-125.

[3] R.Y. Denis, S.N. Singh and S.P. Singh: *On reducibility of certain q-double hypergeometric series and clausen type identities*, South East Asian J. Math. and Math. Sc., **1(1)** (2002), 1-17.

[4] G. Gasper and M. Rahman: *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, (1990).

[5] H.L. Manocha and B.L. Sharma: *Fractional derivatives and summation*, J. Indian Math. Soc. **38** (1974), 371-382.

[6] S.D. Purohit and R.K. Yadav: *Applications of q-Leibniz rule and transformations involving generalized basic multiple hypergeometric functions*, (communicated).

[7] R.K. Saxena, G.C. Modi and S.L. Kalla: *A basic analogue of Fox's H-function*, Rev. Tec. Ing. Univ. Zulia, **6**(1983), 139-143.

[8] H.L. Shukla: *Certain results involving basic hypergeometric functions and fractional q-derivative*, The Math. Student, **61**(1992), no. 1-4, 107-112.

[9] L.J. Slater: *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, London and New York, (1966).

[10] H.M. Srivastava and A.K. Agarwal: *Generating functions for a class of q-polynolmials*, Annali di Matematica Pura ed applicata **IV. CLIV**(1989), 99-109.

[11] R.K. Yadav and S.D. Purohit: *Fractional q-derivatives and certain basic hypergeometric transformations*, South East Asian J. Math. and Math. Sc., **2(2)** (2004), 37-46.

[12] R.K. Yadav and S.D. Purohit: *On fractional q-derivatives and transformations of the generalized basic hypergeometric functions*, J. Indian Acad. Math., **28(2)**(2006), 321-326.

ОДРЕДЕНИ ПРОШИРУВАЊА КОИ ОПФАЌААТ ОПШТИ ОСНОВНИ ХИПЕРГЕОМЕТРИСКИ ФУНКЦИИ

S.D. Purohit[1] and S.L. Kalla[2]

Р е з и м е

Некои фундаментални операции на фракционите q-сметки се употребени да се докаже теоремата на формулата за проширување за општите основни хипергеометриски функции. Исто така се разгледуваат одредени интересни последици на теоремата. Потоа оваа општа теорема се применува за да се изведат бројни формули за проширување за познати q-аналогии на различни хипергеометриски функции.

[1]Department of Basic-Sciences (Mathematics),
College of Technology and Engineering,
M.P. University of Agriculture and Technology, Udaipur-313001,
India.

e-mail: sunil_a_purohit@yahoo.com

[2]Department of Mathematics and Computer Science,
P.O. Box-5969, Safat 13060
Kuwait.

e-mail: shyamkalla@yahoo.com