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ON A DECOMPOSITION OF $T_{1/2}$- SPACES

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Abstract. The aim of this paper is to introduce $T_{\tilde{g}}$-spaces, ${}_gT_{\tilde{g}}$-spaces and ${}_{\alpha}T_{\tilde{g}}$-spaces. Moreover, we obtain a decomposition of $T_{1/2}$-spaces and we investigate properties of these spaces.

1. Introduction

Levine [8] introduced the notion of $T_{1/2}$-spaces which properly lie between T_1-spaces and T_0-spaces. Many authors studied properties of $T_{1/2}$-spaces: Dunham [6], Arenas et al. [2] etc. In this paper, we introduce the notions called $T_{\tilde{g}}$-spaces, ${}_gT_{\tilde{g}}$-spaces and ${}_{\alpha}T_{\tilde{g}}$-spaces. Also, by using these spaces, we obtain a decomposition of $T_{1/2}$-spaces.

Throughout this paper, (X, τ), (Y, σ), and (Z, η) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A, respectively.

A subset A is said to be α-open [10] (resp. semi-open [7], semi-preopen [1]) if A⊆int(cl(int(A))) (resp. A⊆cl(int(A)), A⊆cl(int(cl(A)))). The complement of α-open (resp. semi-open, semi-preopen) set is said to be α-closed (resp. semi-closed, semi-preclosed). The intersection of all α-closed sets of X containing A is called α-closure of A and denoted by α-cl(A) [10]. Similarly, scl(A) and spcl(A) are defined in [7] and [1], respectively.

A subset A of a space X is called a generalized closed (briefly g-closed) set [8] if cl(A)⊆U whenever A⊆U and U is open in (X, τ), α-generalized closed (briefly αg-closed) set [9] if αcl(A)⊆U whenever A⊆U and U is open in (X, τ), generalized semi-pre-closed (briefly gsp-closed) set [5] if spcl(A)⊆U whenever A⊆U and U is open in (X, τ), ω-closed set [13] if cl(A)⊆U whenever A⊆U and U is semi-open in

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(X,τ), g^*-closed set [14] if cl(A)⊆U whenever A⊆U and U is g-open in (X,τ),*g-closed set [15] if cl(A)⊆U whenever A⊆U and U is ω-open in (X,τ), $^\#$g-closed set [16] if cl(A)⊆U whenever A⊆U and U is *g-open in (X,τ), $^\#$g-semi-closed (briefly $^\#$gs-closed) set [17] if scl(A)⊆U whenever A⊆U and U is *g-open in (X,τ), $\widetilde{g}$-closed set [11] if cl(A)⊆U whenever A⊆U and U is $^\#$gs-open in (X,τ) and $\widetilde{g}$-semi-closed (briefly $\widetilde{g}$s-closed) set [12] if scl(A)⊆U whenever A⊆U and U is $^\#$gs-open in (X, τ).The complements of the above mentioned sets are called their respective open sets. The family of all g-open (resp. ω-open and $\widetilde{g}$-closed) sets in (X,τ) denoted by (resp. GO(X, τ) (resp. τ^ω and $\widetilde{G}C(X,\tau)$)).

A space (X, τ) is called a $T_{1/2}$-space [8] if every g-closed set is closed, a semi-pre-$T_{1/2}$-space [8] if every gsp-closed set is semi-preclosed, T_b-space[4] if every gs-closed set is closed, ${}_\alpha T_b$-space [3] if every αg-closed set is closed, ${}_\alpha T_d$-space [3] if every αg-closed set is g-closed, T_ω-space [13] if every ω-closed set is closed, $T^*_{1/2}$-space [14] if every g*-closed set is closed, $^*T_{1/2}$-space [15] if every *g-closed set is closed, ${}_{gs}T^\#_{1/2}$-space [17] if every $^\#$g-semi-closed set is closed, $T_{\widetilde{g}s}$-space [12] if every $\widetilde{g}$-semi-closed set is closed, α-space [10] if every α-closed set is closed.

2. $T_{\widetilde{g}}$ SPACES

We introduce the following definition

Definition 2.1. *A space (X,τ) is called a $T_{\widetilde{g}}$-space if every $\widetilde{g}$-closed set in it is closed.*

Example 2.2. *Let $X=\{a,b,c\}$ and $\tau=\{\varnothing,\{a\},X\}$. $\widetilde{G}C(X,\tau)=\{\varnothing,\{b,c\},X\}$. Thus (X, τ) is a $T_{\widetilde{g}}$-space.*

Example 2.3. *Let $X=\{a,b,c\}$ and $\tau=\{\varnothing,\{a,b\},X\}$.
$\widetilde{G}C(X,\tau)=\{\varnothing,\{c\},\{b,c\},\{a,c\},X\}$. Thus (X, τ) is not a $T_{\widetilde{g}}$-space.*

Proposition 2.4. *Every $T_{1/2}$-space is $T_{\widetilde{g}}$-space but not conversely.*

Proof. Follows from Theorem 3.4 [11]. □

The converse of the above Proposition need not be true as seen from the following example.

Example 2.5. *Let X and τ as in the example 2.2,
$GO(X,\tau)=\{\varnothing,\{b\},\{c\},\{a,b\},\{b,c\},X\}$. Thus (X, τ) is not a $T_{1/2}$-space.*

Proposition 2.6. *Every T_ω-space is $T_{\widetilde{g}}$-space but not conversely.*

Proof. Follows from Theorem 2.4 [11]. □

The converse of the above Proposition 2.6 need not be true as seen from the following example.

Example 2.7. *Let* $X=\{a,b,c\}$ *and* $\tau = \{\varnothing, \{a\}, \{b,c\}, X\}$. *Then* $\tau^{\omega} = P(X)$ *and* $\widetilde{G}C(X,\tau)=\{\varnothing, \{a\}, \{b,c\}, X\}$. *Thus the space* (X,τ) *is* $T_{\widetilde{g}}$*-space but not a* T_{ω}*-space.*

Proposition 2.8. *Every* $_{gs}T^{\#}_{1/2}$*-space is* $T_{\widetilde{g}}$*-space but not conversely.*

Proof. Follows from Theorem 3.9 [11]. □

The converse of the above Proposition 2.8 need not be true as seen from the following example.

Example 2.9. *The space* (X, τ) *in the Example 2.2 is a* $T_{\widetilde{g}}$*-space but not a* $_{gs}T^{\#}_{1/2}$*-space.*

Proposition 2.10. *Every* $T_{\widetilde{g}s}$*-space is* $T_{\widetilde{g}}$*-space but not conversely.*

Proof. Follows from Theorem 3.7 [11]. □

The converse of the above Proposition 2.10 need not be true as seen from the following example.

Example 2.11. *The space* (X, τ) *in the example 2.2 is a* $T_{\widetilde{g}}$*-space but not a* $T_{\widetilde{g}s}$*-space.*

Proposition 2.12. *Every* T_b*-space is* $T_{\widetilde{g}}$*-space but not conversely.*

Proof. Follows from Theorem 3.11 [11]. □

Example 2.13. *The space* (X, τ) *in the example 2.2 is* $T_{\widetilde{g}}$*-space but not a* T_b*-space.*

Remark 2.14. $T_{\widetilde{g}}$*-space and* α*-space are independent.*

Example 2.15. *The space* (X, τ) *in the Example 2.2 is a* $T_{\widetilde{g}}$*-space but not a* α*-space and space* (X, τ) *in the Example 2.3 is an* α*-space but not a* $T_{\widetilde{g}}$*-space.*

Remark 2.16. $T_{\widetilde{g}}$*-space and semi-pre-*$T_{1/2}$*-space are independent.*

Example 2.17. *The space* (X, τ) *in the example 2.2 is a* $T_{\widetilde{g}}$*-space but not a semi-pre-*$T_{1/2}$*-space and the space* (X, τ) *in the example 2.3 is a semi-pre-*$T_{1/2}$*-space but not* $T_{\widetilde{g}}$*-space.*

Remark 2.18. $T_{\widetilde{g}}$*-space and* $^{*}T_{1/2}$*-space are independent.*

Example 2.19. *The space* (X, τ) *in the example 2.7 is a* $T_{\widetilde{g}}$*-space but not* $^{*}T_{1/2}$*-space. The space* (X, τ) *in the example 2.3 is a* $^{*}T_{1/2}$*-space but not a* $T_{\widetilde{g}}$*-space.*

Theorem 2.20. *For a space* (X, τ) *the following properties are equivalent:*
(i). (X, τ) *is a* $T_{\widetilde{g}}$*-space,*
(ii). Every singleton subset of (X, τ) *is either* $^{\#}g$*-semi-closed or open.*

Proof. (i)⇒(ii): Assume that for some x∈X, the set $\{x\}$ is not a $^{\#}$gs-closed in (X, τ). Then the only $^{\#}$gs-open set containing $\{x\}^c$ is X and so $\{x\}^c$ is g-closed in (X, τ).By assumption $\{x\}^c$ is closed in (X, τ) or equivalently $\{x\}$ is open.

(ii)⇒(i): Let A be a g-closed subset of (X, τ) and let x∈Cl(A). By assumption $\{x\}$ is either $^{\#}$gs-closed or open. □

Case (i): Suppose $\{x\}$ is $^{\#}$gs-closed. If x∉A, then Cl(A)−A contains a non-empty $^{\#}$gs-closed set $\{x\}$,which is a contradiction to Theorem 3.21 [11]. Therefore x∈A.

Case (ii): Suppose $\{x\}$ is open. Since x∈Cl(A), $\{x\}\bigcap$A≠ ∅ and therefore Cl(A)⊆A or equivalently A is a closed set of (X, τ).

Definition 2.21. *A topological space (X, τ) is said to be*
(1) $^{\#}gs$-T_0 if for x, y∈X such that x≠y there exists a $^{\#}$gs-open set U of X containing x but not y or a $^{\#}$gs-open set V of X containing y but not x,

(2) $^{\#}gs$-T_1 if for distinct points x, y∈X, there exists a $^{\#}$gs-open set U of X containing x but not y and a $^{\#}$gs-open set V of X containing y but not x.

Lemma 2.22. *Let (X, τ) be a topological space. X is $^{\#}gs$-T_1 if and only if for each x∈X, the singleton $\{x\}$ is $^{\#}$gs-closed.*

Theorem 2.23. *For a topological space (X, τ), the following properties hold:*
(1) if (X, τ) is $^{\#}gs$-T_1, then it is $T_{\tilde{g}}$,
(2) if (X, τ) is $T_{\tilde{g}}$, then it is $^{\#}gs$-T_0.

Proof. (1) The proof is obvious from Lemma 2.22.
(2) Let x and y be two distinct elements of X. Since the space (X, τ) is $T_{\tilde{g}}$, we have that $\{x\}$ is $^{\#}$gs-closed or open. Suppose that $\{x\}$ is open. Then the singleton $\{x\}$ is a $^{\#}$gs-open set such that x∈ $\{x\}$ and y∉ $\{x\}$. Also, if $\{x\}$ is $^{\#}$gs-closed, then X\\$\{x\}$ is $^{\#}$gs-open such that y∈X\\$\{x\}$ and x∉X\\$\{x\}$. Thus, in the above two cases, there exists a $^{\#}$gs-open set U of X such that x∈U and y∉U or x∉U and y∈U. Thus, the space (X, τ) is $^{\#}$gs-T_0. □

Definition 2.24. *Let (X, τ) be a topological space and A⊆X. We define the $^{\#}$gs-closure of A (briefly $^{\#}$gs-cl(A)) to be the intersection of all $^{\#}$gs-closed sets containing A.*

Definition 2.25. *A topological space (X, τ) is said to be $^{\#}gs$-R_0 if every $^{\#}$gs-open set contains the $^{\#}$gs-closure of each of its singletons.*

Theorem 2.26. *For a $^{\#}gs$-R_0 topological space (X, τ), the following properties are equivalent:*
(1) (X, τ) is $^{\#}gs$-T_0,
(2) (X, τ) is $T_{\tilde{g}}$,
(3) (X, τ) is $^{\#}gs$-T_1.

Proof. It suffices to prove only (1)⇒(3).
Let x≠y and since (X, τ) is $^{\#}$gs-T_0, we may assume that x∈U⊆X\{y} for some $^{\#}$gs-open set U. Then x∈X\$^{\#}$gs-cl({$y$}) and X\$^{\#}$gs-cl({$y$}) is $^{\#}$gs-open. Since (X, τ) is $^{\#}$gs-R_0, we have $^{\#}$gs-cl({x}) ⊆X\$^{\#}$gs-cl({$y$})⊆X\{$y$} and hence y∉$^{\#}$gs-cl({$x$}). There exists $^{\#}gs$-open set V such that y∈V⊆X\{x} and (X, τ) is a $^{\#}$gs-T_1-space. □

3. $_g T_{\tilde{g}}$-SPACES

Definition 3.1. *A space (X, τ) is called a $_g T_{\tilde{g}}$-space if every g-closed set of (X, τ) is a $\tilde{g}$-closed set in (X, τ).*

Example 3.2. *The space (X, τ) in the Example 2.3 is a $_g T_{\tilde{g}}$-space and the space (X, τ) in the Example 2.2 is not a $_g T_{\tilde{g}}$-space.*

Proposition 3.3. *Every $T_{1/2}$-space is $_g T_{\tilde{g}}$-space but not conversely.*

Proof. Follows from Theorem 3.2 [11]. □

Example 3.4. *The space (X, τ) in the Example 2.3 is a $_g T_{\tilde{g}}$-space but not a $T_{1/2}$-space.*

Remark 3.5. *$T_{\tilde{g}}$-space and a $_g T_{\tilde{g}}$-space are independent.*

Example 3.6. *The space (X, τ) in the Example 2.3 is a $_g T_{\tilde{g}}$-space but not $T_{\tilde{g}}$-space and the space (X, τ) in the Example 2.2 is $T_{\tilde{g}}$-space but not a $_g T_{\tilde{g}}$-space.*

Remark 3.7. *$T^*_{1/2}$-space and a $_g T_{\tilde{g}}$-space are independent.*

Example 3.8. *The space (X, τ) in the Example 2.3 is a $_g T_{\tilde{g}}$-space but not $T^*_{1/2}$-space and the space (X, τ) in the Example 2.2 is $T^*_{1/2}$-space but not a $_g T_{\tilde{g}}$-space.*

Remark 3.9. *$^*T_{1/2}$-space and a $_g T_{\tilde{g}}$-space are independent.*

Example 3.10. *The space (X, τ) in the Example 2.3 is a $_g T_{\tilde{g}}$-space but not $^*T_{1/2}$-space and the space (X, τ) in the Example 2.7 is $^*T_{1/2}$-space but not a $_g T_{\tilde{g}}$-space.*

Theorem 3.11. *If (X, τ) is a $_g T_{\tilde{g}}$-space, then every singleton subset of (X, τ) is either g-closed or $\tilde{g}$-open.*

Proof. Assume that for some x∈X, the set {x} is not a g-closed in (X, τ). Then the only open set containing $\{x\}^c$ is X itself and so $\{x\}^c$ is g-closed in (X, τ). By assumption, $\{x\}^c$ is a $\tilde{g}$-closed set in (X, τ) or equivalently {x} is $\tilde{g}$-open. □

The converse of the above Theorem 3.11 need not be true as seen from the following example.

Example 3.12. *Let X and τ be as in the Example 2.2. The sets {b} and {c} are g-closed in (X, τ) and the set {a} is $\tilde{g}$-open. But the space (X, τ) is not a $_g T_{\tilde{g}}$-space.*

Theorem 3.13. *A space (X, τ) is $T_{1/2}$ if and only if it is both $T_{\tilde{g}}$ and $_g T_{\tilde{g}}$.*

Proof. Necessity follows from Propositions 2.3 and 3.3. □

Sufficiency: Assume that (X, τ) is both $T_{\widetilde{g}}$ and ${}_gT_{\widetilde{g}}$. Let A be a g-closed set of (X, τ). Then A is $\widetilde{g}$-closed again by assumption A is closed in (X, τ). Therefore (X, τ) is a $T_{1/2}$-space.

4. ${}_\alpha T_{\widetilde{g}}$-SPACES

Definition 4.1. *A space (X, τ) is called a ${}_\alpha T_{\widetilde{g}}$-space if every αg-closed set of (X, τ) is a $\widetilde{g}$-closed set in (X, τ).*

Example 4.2. *The space (X, τ) in the Example 2.3 is a ${}_\alpha T_{\widetilde{g}}$-space and the space (X, τ) in the Example 2.2 is not a ${}_\alpha T_{\widetilde{g}}$-space.*

Proposition 4.3. *Every ${}_\alpha T_b$-space is ${}_\alpha T_{\widetilde{g}}$-space but not conversely.*

Proof. Follows from Theorem 3.2 [11]. □

Example 4.4. *The space (X, τ) in the Example 2.3 is a ${}_\alpha T_{\widetilde{g}}$-space but not a ${}_\alpha T_b$-space.*

Proposition 4.5. *Every ${}_\alpha T_{\widetilde{g}}$-space is ${}_\alpha T_d$-space but not conversely.*

Proof. Let (X, τ) be an ${}_\alpha T_{\widetilde{g}}$-space and let A be an αg-closed set of (X, τ). Then A is a $\widetilde{g}$-closed subset of (X, τ) and by Theorem 3.4 [11], A is g-closed. Therefore (X, τ) is an ${}_\alpha T_d$-space. □

The converse of the above Proposition 4.5 need not be true as seen from the following example.

Example 4.6. *The space (X, τ) in the Example 2.2 is a ${}_\alpha T_d$-space but not a ${}_\alpha T_{\widetilde{g}}$-space.*

Theorem 4.7. *If (X, τ) is a ${}_\alpha T_{\widetilde{g}}$-space, then every singleton subset of (X, τ) is either αg-closed or $\widetilde{g}$-open.*

Proof. Similar to Theorem 3.11. □

The converse of the above Theorem 4.7 need not be true as seen from the following example.

Example 4.8. *Let X and τ be as in the Example 2.2. Then the sets $\{b\}$ and $\{c\}$ are g-closed in (X, τ) and the set $\{a\}$ is $\widetilde{g}$-open. But the space (X, τ) is not a ${}_g T_{\widetilde{g}}$-space.*

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