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A GENERALIZATION OF q-MITTAG-LEFFLER FUNCTION

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Abstract. The aim of this paper is to introduce and study some elementary properties of new q-exponential functions with three parameters, which leads to q-analogue of the generalized Mittag-Leffler function. Some q-integral representations for these q-Mittag-Leffler functions are derived. Special cases of main results are pointed out briefly.

1. Introduction

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In 1903, Mittag-Leffler [12] introduced the following function, in terms of the power series

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)} \; (\Re(\alpha) > 0, z \in \mathbb{C}). \tag{1.1}$$

A two-index generalization of this function was given by Wiman [22] as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)} \; (\Re(\alpha) > 0, z, \beta \in \mathbb{C}). \tag{1.2}$$

Both are entire functions of order $\rho = 1/\alpha$ and type $\sigma = 1$. A detailed account of these functions is available in the monographs of Erdélyi et al. [6], Dzrbashjan [5] and Podlubny [13].

By means of the series representation a generalization of Mittag-Leffler function (1.2) is introduced by Prabhakar [14] as:

$$E^{\delta}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_n \; z^k}{\Gamma(k\alpha + \beta) \; n!} \; , \tag{1.3}$$

where $z, \alpha, \beta, \delta \in \mathbb{C}, \Re(\alpha) > 0$. It is an entire function of order $[\Re(\alpha)]^{-1}$ (see [14, p.7]) and for $\delta = 1$, reduces to Mittag-Leffler function $E_{\alpha,\beta}(z)$.

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The Mittag-Leffler function provides solutions to certain problems formulated in terms of fractional order differential, integral and difference equations, therefore, it has recently become a subject of interest for many authors in the field of fractional calculus and its applications. Motivated by these avenues of applications, a large amount of research in the theory of Mittag-Leffler functions has been published in the literature (for details, see [10], [13], [14], and [18]-[21]).

The q-calculus is the q-extension of the ordinary calculus. The theory of q-calculus operators in recent past have been applied in the areas of ordinary fractional calculus, optimal control problems, in finding solutions of the q-difference (differential) and q-integral equations, and in q-transform analysis. One may refer to [8] and recent papers [2], [4], [7], [11] and [15]-[17] on the subject.

Recently, Rajković et al. [16]-[17], have generalized the concept of fractional q-integrals with the parametric lower limit of integration and hence introduced the fractional q-derivative of Caputo type, generalized q-Leibniz formula and the following q-analogues of the Mittag-Leffler function (1.2):

$$e_{q;\alpha,\beta}(z;c) = \sum_{k=0}^{\infty} \frac{z^{\alpha k+\beta-1}(c/z;q)_{\alpha k+\beta-1}}{(q;q)_{\alpha k+\beta-1}} \ (|c| < |z|), \tag{1.4}$$

$$E_{q;\alpha,\beta}(z;c) = \sum_{k=0}^{\infty} \frac{q^{(\alpha k+\beta-1)(\alpha k+\beta-2)/2}}{(-c;q)_{\alpha k+\beta-1}} \frac{z^{\alpha k+\beta-1}(c/z;q)_{\alpha k+\beta-1}}{(q;q)_{\alpha k+\beta-1}} \ , \tag{1.5}$$

where

$$(q,z,c,\alpha,\beta \in \mathbb{C};\ \Re(\alpha) > 0, \Re(\beta) > 0, |q| < 1). \tag{1.6}$$

The q-special functions $e_{q;\alpha,\beta}(z;c)$ and $E_{q;\alpha,\beta}(z;c)$ are called as the small q-Mittag-Leffler and big q-Mittag-Leffler functions respectively.

On the other hand, Mansour [11] has introduced an another q-analogue of the Mittag-Leffler function, and derived a fundamental set of solutions for the homogeneous linear sequential q-difference equations with constant coefficients and a general solution for the corresponding non homogeneous equations. The q-Mittag-Leffler function due to Mansour [11], is given by

$$e_{\alpha,\beta}(z;q) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(\alpha k+\beta)},\ |z| < (1-q)^{-\alpha}, \tag{1.7}$$

where $\alpha > 0, \beta \in \mathbb{C}$. For further studies on the q-Mittag-Leffler functions and their applications, see [3], [11], [16] and [17].

In this paper, our purpose is to introduce new q-exponential functions with three parameters, whcih lead to q-analogues of the generalized Mittag-Leffler function (1.3) and to derive some elementary properties. Some q-integral representations for these functions are established. Special cases of the main results are given in the concluding section.

2. Preliminaries

In the theory of q-calculus (see [8]), the q-shifted factorial is defined for $a,\ q\in\mathbb{C}$ as a product of n factors by

$$(a\,;q)_n=\begin{cases}1 & ;\quad n=0\\ (1-a)\,(1-a\,q)\,\cdots(1-a\,q^{n-1}) & ;\quad n\in\mathbb{N}\,,\end{cases} \tag{2.1}$$

and its natural extension is

$$(a\,;q)_\alpha=\frac{(a\,;q)_\infty}{(aq^\alpha\,;q)_\infty},\ \ \alpha\in\mathbb{C}. \tag{2.2}$$

If $|q|<1$, the definition (2.1) remains meaningful for $n=\infty$ as a convergent infinite product:

$$(a\,;q)_\infty=\prod_{j=0}^{\infty}(1-a\,q^j)\ . \tag{2.3}$$

The q-analogue of the power (binomial) function $(x\pm y)^n$ cf. Ernst [7], is given by (see also [15])

$$(x\pm y)^{(n)}\equiv(x\pm y)_n\equiv x^n(\mp y/x;q)_n=x^n\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(k-1)/2}\,(\pm y/x)^k\ \ , \tag{2.4}$$

such that

$$\underset{q\to1^-}{Lt}\,(x\pm y)^{(n)}=(x\pm y)^n,$$

where the q-binomial coefficient is defined as:

$$\begin{bmatrix}\alpha\\k\end{bmatrix}_q=\frac{(q^{-\alpha};q)_k}{(q;q)_k}(-q^\alpha)^k q^{-k(k-1)/2}\ (k\in\mathbb{N},\alpha\in\mathbb{C}). \tag{2.5}$$

For a bounded sequence of real or complex numbers, let $f(x)=\sum_{n=-\infty}^{+\infty}A_nx^n$ be a power series in x, (see for instance, [7, p. 502, eqn. (3.18)], then we have

$$f\,[x\pm y]_q=\sum_{n=-\infty}^{+\infty}A_n\ x^n(\mp y/x;q)_n. \tag{2.6}$$

The q-gamma and the q-beta functions (cf. [8] and [11]) are defined by

$$\Gamma_q(z)=\frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}\ (z\in\mathbb{C},z\notin\{0,-1,-2,\cdots\},0<q<1), \tag{2.7}$$

and

$$B_q(\alpha,\beta)=\int_0^1 z^{\alpha-1}\ (zq;q)_{\beta-1}\ d_qz\ =\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}\ (\Re(\alpha),\Re(\beta)>0). \tag{2.8}$$

Also, the q-difference operator and q-integration of a function $f(z)$ defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [8, pp. 19-22])

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)} \ (z \neq 0,\ q \neq 1), \tag{2.9}$$

and

$$\int_0^z f(t)\, d(t;q) = z\ (1-q) \sum_{k=0}^{\infty} q^k f(zq^k)\ . \tag{2.10}$$

3. Generalized q-Mittag-Leffler functions and their elementary properties

In the theory of q-series, two q-analogues of the classical exponential function are defined by (see [8])

$$e_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} \ (|z| < 1), \tag{3.1}$$

and

$$E_q(z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}\ z^k}{(q;q)_k} \ (z \in \mathbb{C}). \tag{3.2}$$

In this section, we introduce two new q-exponential functions with three parameters, that is, q-analogues of the Mittag-Leffler function due to Prabhakar [14], which may be regarded as generalizations of the q-Mittag-Leffler function (1.7).

Definition. *Let $q, z, \alpha, \beta, \delta \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\delta) > 0$ and $|q| < 1$, then the function*

$$e^{\delta}_{\alpha,\beta}(z;q) = \sum_{k=0}^{\infty} \frac{(q^{\delta};q)_k\ z^k}{\Gamma_q(\alpha k + \beta)\ (q;q)_k},\ |z| < (1-q)^{-\alpha}, \tag{3.3}$$

is called as the generalized small q-Mittag-Leffler function. Similarly, the generalized big q-Mittag-Leffler function is introduced as

$$E^{\delta}_{\alpha,\beta}(z;q) = \sum_{k=0}^{\infty} \frac{(q^{\delta};q)_k\ q^{k(k-1)/2}\ z^k}{\Gamma_q(\alpha k + \beta)\ (q;q)_k},\ |z| < (1-q)^{-\alpha}. \tag{3.4}$$

Some important special cases of of these functions are enumerated below:

(i) $e_{\alpha,\beta}(z;q) = e^{1}_{\alpha,\beta}(z;q)$, where the left-hand side q-Mittag-Leffler function is given by (1.7).

(ii) $e_{\alpha}(z;q) = e^{1}_{\alpha,1}(z;q)$, where the function $e_{\alpha}(z;q)$ is q-analogue of the function (1.1).

(iii) $E_{\alpha}(z;q) = E^{1}_{\alpha,1}(z;q)$, where the function $E_{\alpha}(z;q)$ is another q-analogue of $E_{\alpha}(z)$.

(iv) $e_q((1-q)z) = e^1_{1,1}(z;q)$.

(v) $E_q((1-q)z) = E^1_{1,1}(z;q)$.

(vi) $Z^\alpha_n(z;m,q) = \frac{\Gamma_q(mn+\alpha+1)}{(q;q)_n} E^{-n}_{m,\alpha+1}(q^n z^m;q)$ $(\forall\ n \in \mathbb{Z}^+)$,
where the function $Z^\alpha_n(z;m,q)$ denotes the q-Konhouser polynomials due to Yadav and Singh [23], namely

$$Z^\alpha_n(z;m,q) = \frac{\Gamma_q(mn+\alpha+1)}{(q;q)_n} \sum_{k=0}^{n} (-1)^k \left[\begin{array}{c} n \\ k \end{array} \right]_q \frac{z^{mk} q^{k(k-1)}}{\Gamma_q(mk+\alpha+1)} \tag{3.5}$$

$$(m,n \in \mathbb{Z}^+; \Re(\alpha) > -1).$$

(vii) Finally, in view of the relations

$$\underset{q\to 1^-}{Lt} \frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n\ , \tag{3.6}$$

and

$$\underset{q\to 1^-}{Lt} \Gamma_q(z) = \Gamma(z)\ , \tag{3.7}$$

we observe that

$$\underset{q\to 1^-}{Lt} e^\delta_{\alpha,\beta}(z;q) = \underset{q\to 1^-}{Lt} E^\delta_{\alpha,\beta}(z;q) = E^\delta_{\alpha,\beta}(z). \tag{3.8}$$

Now we prove the following theorems, that provides elemantary properties of the q-Mittag-Leffler functions (3.3) and (3.4).

Theorem 1. *Let $a \in \mathbb{R}$ and $\Re(\beta) > \Re(\alpha) > 0$, then there holds the formulas*

$$a\ z^\alpha\ q^{\delta-1}\ e^\delta_{\alpha,\beta}(a\ z^\alpha;q) = e^\delta_{\alpha,\beta-\alpha}(a\ z^\alpha;q) - e^{\delta-1}_{\alpha,\beta-\alpha}(a\ z^\alpha;q), \tag{3.9}$$

and

$$a\ z^\alpha\ q^{\delta-2}\ E^\delta_{\alpha,\beta}(a\ z^\alpha;q) = E^\delta_{\alpha,\beta-\alpha}(a\ z^\alpha/q\ ;q) - E^{\delta-1}_{\alpha,\beta-\alpha}(a\ z^\alpha/q\ ;q). \tag{3.10}$$

Proof. To prove the result (3.9), we consider the left-hand side (say L) of (3.9) and make use of the definition (3.3), to obtain

$$L = q^{\delta-1} \sum_{k=0}^{\infty} \frac{(q^\delta;q)_k\ (az^\alpha)^{k+1}}{\Gamma_q(\alpha k+\beta)\ (q;q)_k}.$$

On using the q-identity (which can easily be obtain from definition (2.1)), namely

$$q^{\delta-1}\ (1-q^{k+1})\ (q^\delta;q)_k = (q^\delta;q)_{k+1} - (q^{\delta-1};q)_{k+1}, \tag{3.11}$$

and the q-identity given in [8, p. 6, No. (1.2.33)], we obtain

$$L = \sum_{k=0}^{\infty} \frac{\left[(q^\delta;q)_{k+1} - (q^{\delta-1};q)_{k+1}\right]\ (az^\alpha)^{k+1}}{\Gamma_q(\alpha k+\beta)\ (q;q)_{k+1}}.$$

Now, for $\Re(\beta) > \Re(\alpha) > 0$ the above series also exists for $k = -1$ and corresponding value is zero, therefore, we can write

$$L = \sum_{k=-1}^{\infty} \frac{\left[(q^{\delta};q)_{k+1} - (q^{\delta-1};q)_{k+1}\right]\ (az^{\alpha})^{k+1}}{\Gamma_q(\alpha k+\beta)\ (q;q)_{k+1}}\ . \tag{3.12}$$

After replacing k by $k-1$ and making use of (3.3), we arrive at the right-hand side of (3.9). Similarly, in view of the definition (3.4), one can easily prove the result (3.10).

Theorem 2. *Consider $\Re(\alpha) > 0$ and $x, y, \beta \in \mathbb{C}$. Then, for generalized q-Mittag-Leffler functions we have*

$$\sum_{r=0}^{\infty} (x+y)^{(r)}\ e^{r+1}_{2\alpha, r\alpha+\beta}(-xy;q) = \sum_{k=0}^{\infty} (-xy)^k\ e^{k+1}_{\alpha,2k\alpha+\beta}(x+y;q), \tag{3.13}$$

and

$$\sum_{r=0}^{\infty} q^{r(r-1)/2}(x+y)^{(r)}\ E^{r+1}_{2\alpha, r\alpha+\beta}(-xy;q) = \sum_{k=0}^{\infty} q^{k(k-1)/2}(-xy)^k\ E^{k+1}_{\alpha,2k\alpha+\beta}(x+y;q). \tag{3.14}$$

Proof. Substituting definition (3.3) in the left-hand side (say L) of (3.13) and changing the order of summations, which is valid under the given conditions, we have

$$L = \sum_{k=0}^{\infty} \frac{(-xy)^k}{(q;q)_k} \sum_{r=0}^{\infty} \frac{(q^{r+1};q)_k}{\Gamma_q(2\alpha k + r\alpha + \beta)}\ (x+y)^{(r)}. \tag{3.15}$$

Using the well-known q-identity [8, p. 234, I.18], we have

$$\frac{(q^{r+1};q)_k}{(q;q)_k} = \frac{(q^{k+1};q)_r}{(q;q)_r}. \tag{3.16}$$

Hence, we can write

$$L = \sum_{k=0}^{\infty} (-xy)^k \sum_{r=0}^{\infty} \frac{(q^{k+1};q)_r}{\Gamma_q(r\alpha + 2\alpha k + \beta)\ (q;q)_r}\ (x+y)^{(r)}. \tag{3.17}$$

Interpreting the inner series in view of the definition (3.3) the above equation leads to the right-hand side of (3.13).
Similarly, one can easily prove the result (3.14) by taking definition (3.4) into account.

An immediate consequence of the Theorem 2 is contained in:

Corollary 1. *For $\Re(\alpha) > 0$, $\Re(\gamma) > 0$ and $x, y, \beta \in \mathbb{C}$, we have*

$$\sum_{r=0}^{\infty} x^r\ e^{r+1}_{\alpha, r\gamma+\beta}(y;q) = \sum_{k=0}^{\infty} y^k\ e^{k+1}_{\gamma,k\alpha+\beta}(x;q), \tag{3.18}$$

and

$$\sum_{r=0}^{\infty} q^{r(r-1)/2}x^r \; E^{r+1}_{\alpha,r\gamma+\beta}(y;q) = \sum_{k=0}^{\infty} q^{k(k-1)/2}y^k \; E^{k+1}_{\gamma,k\alpha+\beta}(x;q). \tag{3.19}$$

4. Some q-integral representations of $e^{\delta}_{\alpha,\beta}(z;q)$ and $E^{\delta}_{\alpha,\beta}(z;q)$

In this section, we establish the following theorems in terms of the q-integral representations of the generalized q-Mittag-Leffler functions.

Theorem 3. *If $z, \alpha, \beta, \delta \in \mathbb{C}$; $\Re(\alpha) > 0, \Re(\delta) > 0$ and $\Re(\beta) > \Re(\alpha) > 0$, then*

$$e^{\delta}_{\alpha,\beta}(z;q) = \frac{z^{\alpha-\beta}}{(1-q^{1/m})} \int_0^{\infty} e_q(-t^m/z^m)\; t^{\beta-\alpha-1} \sum_{k=0}^{\infty} \frac{(q^{\delta};q)_k \; t^k \; q^{\sigma(\sigma-1)/2}}{\Gamma_q(\alpha k+\beta)\; (q;q)_k (q;q)_{\sigma-1}} \; d_qt, \tag{4.1}$$

and

$$E^{\delta}_{\alpha,\beta}(z;q) = \frac{z^{\alpha-\beta}}{(1-q^{1/m})} \int_0^{\infty} e_q(-t^m/z^m)\; t^{\beta-\alpha-1} \sum_{k=0}^{\infty} \frac{(q^{\delta};q)_k \; t^k \; q^{k(k-1)/2+\sigma(\sigma-1)/2}}{\Gamma_q(\alpha k+\beta)\; (q;q)_k (q;q)_{\sigma-1}} \; d_qt, \tag{4.2}$$

where

$$\sigma = \frac{\beta-\alpha+k}{m} \tag{4.3}$$

and m is any non zero positive number.

Proof. To prove the result (4.1), we consider the right-hand side (say R) of (4.1)

$$R = \frac{z^{\alpha-\beta}}{(1-q^{1/m})} \int_0^{\infty} e_q(-t^m/z^m)\; t^{\beta-\alpha-1} \sum_{k=0}^{\infty} \frac{(q^{\delta};q)_k \; t^k \; q^{\sigma(\sigma-1)/2}}{\Gamma_q(\alpha k+\beta)\; (q;q)_k (q;q)_{\sigma-1}} \; d_qt. \tag{4.4}$$

Substituting $t^m/z^m = u$, then in view of the q-difference operator (2.9), we get

$$d_qt = \frac{(1-q^{1/m})}{(1-q)} \; z \; u^{1/m-1} \; d_qu.$$

Hence, we can write

$$R = \frac{1}{(1-q)} \int_0^{\infty} e_q(-u)\; u^{(\beta-\alpha)/m-1} \sum_{k=0}^{\infty} \frac{(q^{\delta};q)_k \; (zu^{1/m})^k \; q^{\sigma(\sigma-1)/2}}{\Gamma_q(\alpha k+\beta)\; (q;q)_k (q;q)_{\sigma-1}} \; d_qu. \tag{4.5}$$

On interchanging the order of integration and summation, under the valid conditions given with (4.1), we obtain

$$R = \frac{1}{(1-q)} \sum_{k=0}^{\infty} \frac{(q^{\delta};q)_k \; z^k \; q^{\sigma(\sigma-1)/2}}{\Gamma_q(\alpha k+\beta)\; (q;q)_k (q;q)_{\sigma-1}} \; \int_0^{\infty} e_q(-u)\; u^{\sigma-1} \; d_qu.$$

$$R = \sum_{k=0}^{\infty} \frac{(q^\delta;q)_k \; z^k \; q^{\sigma(\sigma-1)/2}}{\Gamma_q(\alpha k+\beta) \; (q;q)_k (q;q)_{\sigma-1}} \; \mathcal{L}_q \left\{ u^{\sigma-1}; 1 \right\}, \tag{4.6}$$

where $\mathcal{L}_q \{f(u);s\}$ denotes the q-Laplace transform of $f(u)$, introduced by Hahn [9] and defined by

$$\mathcal{L}_q \{f(u);s\} = \frac{1}{(1-q)} \int_0^\infty \; e_q(-su) \; f(u) \; d_q u. \tag{4.7}$$

On using the known result due to Abdi [1], namely

$$\mathcal{L}_q \left\{ u^{\sigma-1}; s \right\} = \frac{(q;q)_{\sigma-1} \; q^{-\sigma(\sigma-1)/2}}{s^\sigma} \; (\Re(\sigma) > 0), \tag{4.8}$$

we have

$$\mathcal{L}_q \left\{ u^{\sigma-1}; 1 \right\} = (q;q)_{\sigma-1} \; q^{-\sigma(\sigma-1)/2},$$

and hence, the result (4.6) leads to the left-hand side of (4.1). This completes the proof of (4.1). On using (3.4) one can easily prove the result (4.2) of Theorem 3.

Theorem 4. *If* $z,\alpha,\beta,\delta \in \mathbb{C}$*;* $\Re(\alpha) > 0, \Re(\delta) > 0$ *and* $\Re(\beta) > \Re(\alpha) > 0$*, then*

$$e^\delta_{\alpha,\beta}(z;q) = \frac{(1-q)}{(1-q^\alpha)\Gamma_q(\beta-\alpha)} \int_0^1 \; (qt^{1/\alpha};q)_{\beta-\alpha-1} \; e^\delta_{\alpha,\alpha}(zt;q) \; d_q t, \tag{4.9}$$

and

$$E^\delta_{\alpha,\beta}(z;q) = \frac{(1-q)}{(1-q^\alpha)\Gamma_q(\beta-\alpha)} \int_0^1 \; (qt^{1/\alpha};q)_{\beta-\alpha-1} \; E^\delta_{\alpha,\alpha}(zt;q) \; d_q t. \tag{4.10}$$

Proof. Applying the definitions (3.3) and (3.4) in the right-hand sides of (4.9) and (4.10) respectively, it is easy to prove Theorem 4. For sake of brevity we omit the proof.

Theorem 5. *If* $z,\alpha,\beta,\delta \in \mathbb{C}$*;* $\Re(\alpha) > 0, \Re(\delta) > 0$ *and* $\Re(\beta) > \Re(\alpha) > 0$*, then*

$$e^\delta_{\alpha,\beta}(z;q) = \frac{1}{\Gamma_q(\alpha)} \int_0^1 \; t^{\alpha-1} \; (qt;q)_{\beta-\alpha-1} \; e^\delta_{\alpha,\beta-\alpha}(z(1-tq^{\beta-\alpha})^{(\alpha)};q) \; d_q t, \tag{4.11}$$

and

$$E^\delta_{\alpha,\beta}(z;q) = \frac{1}{\Gamma_q(\alpha)} \int_0^1 \; t^{\alpha-1} \; (qt;q)_{\beta-\alpha-1} \; E^\delta_{\alpha,\beta-\alpha}(z(1-tq^{\beta-\alpha})^{(\alpha)};q) \; d_q t. \tag{4.12}$$

Proof. In view of (3.3) and (2.6), the right-hand side (say R) of (4.11) reduces to

$$R = \frac{1}{\Gamma_q(\alpha)} \int_0^1 \; t^{\alpha-1} \; (qt;q)_{\beta-\alpha-1} \; \sum_{k=0}^\infty \frac{(q^\delta;q)_k \; z^k}{\Gamma_q(\alpha k+\beta-\alpha) \; (q;q)_k} (tq^{\beta-\alpha};q)_{k\alpha} \; d_q t.$$

On interchanging the order of integration and summation, and making use of the q-identity [8, p. 234, I.17], the above equation leads to

$$R = \frac{1}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{(q^{\delta};q)_k \; z^k}{\Gamma_q(\alpha k+\beta-\alpha) \; (q;q)_k} \int_0^1 t^{\alpha-1} \; (tq;q)_{k\alpha+\beta-\alpha-1} \; d_q t. \tag{4.13}$$

Using (2.8) and (3.3), equation (4.13) reduces to the left-hand side of (4.11). The proof of the result (4.12) of Theorem 5 follows similarly.

5. Concluding observations

We briefly consider some consequences of the results derived in the preceeding sections. For example, if we set $m = 1$, the results of Theorem 3 yields to the following:

Corollary 2. *If* $z, \alpha, \beta, \delta \in \mathbb{C}$*;* $\Re(\alpha) > 0, \Re(\delta) > 0$ *and* $\Re(\beta) > \Re(\alpha) > 0$*, then*

$$e^{\delta}_{\alpha,\beta}(z;q) = \frac{z^{\alpha-\beta}}{1-q} \int_0^{\infty} e_q(-t/z) \; t^{\beta-\alpha-1} \sum_{k=0}^{\infty} \frac{(q^{\delta};q)_k \; t^k \; q^{\sigma(\sigma-1)/2}}{\Gamma_q(\alpha k+\beta) \; (q;q)_k (q;q)_{\beta-\alpha+k-1}} \; d_q t, \tag{5.1}$$

and

$$E^{\delta}_{\alpha,\beta}(z;q) = \frac{z^{\alpha-\beta}}{1-q} \int_0^{\infty} e_q(-t/z) \; t^{\beta-\alpha-1} \sum_{k=0}^{\infty} \frac{(q^{\delta};q)_k \; t^k \; q^{k(k-1)/2+\sigma(\sigma-1)/2}}{\Gamma_q(\alpha k+\beta) \; (q;q)_k (q;q)_{\beta-\alpha+k-1}} \; d_q t. \tag{5.2}$$

Now, if we let $q \to 1^-$, and make use of the limit formulae (3.6)-(3.8), we observe that the results of Theorem 1 and Theorem 2 provide, respectively, the q-extensions of the known results due to Saxean and Saigo [19, p. 146, Lemma 1] and Soubhia et al. [21, p. 11, Theorem 3.1].

Similarly, for $q \to 1^-$ Corollary 2 and Theorems 4-5 gives the following results involving integral representations for the Mittag-Leffler function (1.3):

Corollary 3. *If* $z, \alpha, \beta, \delta \in \mathbb{C}$*;* $\Re(\alpha) > 0, \Re(\delta) > 0$ *and* $\Re(\beta) > \Re(\alpha) > 0$*, then*

$$E^{\delta}_{\alpha,\beta}(z) = z^{\alpha-\beta} \int_0^{\infty} exp(-t/z) \; t^{\beta-\alpha-1} \sum_{k=0}^{\infty} \frac{(\delta)_k \; t^k}{\Gamma(\alpha k+\beta) \; k! \; \Gamma(\beta-\alpha+k)} \; dt. \tag{5.3}$$

Corollary 4. *If* $z, \alpha, \beta, \delta \in \mathbb{C}$*;* $\Re(\alpha) > 0, \Re(\delta) > 0$ *and* $\Re(\beta) > \Re(\alpha) > 0$*, then*

$$E^{\delta}_{\alpha,\beta}(z) = \frac{1}{\alpha \; \Gamma(\beta-\alpha)} \int_0^1 (1-t^{1/\alpha})^{\beta-\alpha-1} \; E^{\delta}_{\alpha,\alpha}(zt) \; dt. \tag{5.4}$$

Corollary 5. *If* $z, \alpha, \beta, \delta \in \mathbb{C}$*;* $\Re(\alpha) > 0, \Re(\delta) > 0$ *and* $\Re(\beta) > \Re(\alpha) > 0$*, then*

$$E^{\delta}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}\,(1-t)^{\beta-\alpha-1}\,E^{\delta}_{\alpha,\beta-\alpha}(z(1-t)^{\alpha})\,dt. \tag{5.5}$$

We observe that the Corollaries 3-5 are also special cases of the known results due to Shukla and Prajapati [20, pp. 32-33, Theorems 3-5].

We conclude with the remark that the q-Mittag-Leffler functions and their properties derived in this paper, can be used to obtained results involving q-exponential functions, q-Mittag-Leffler functions and q-Konhouser polynomials and likely to find certain applications in investigating solutions for several fractional q-integral and q-difference equations.

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ГЕНЕРАЛИЗАЦИЈА НА q-MITTAG-LEFFLER ФУНКЦИЈА

S. D. Purohit и S. L. Kalla

Р е з и м е

Целта на овој труд е да воведеме и проучиме некои основни својства на нова q-експоненцијална функција со три параметри, која доведува до q-аналогија на обопштената Mittag-Leffler функција. Изведени се некои q-интегрални репрезентации на овие q-Mittag-Leffler функции. На кратко се посочени и специјални случаи на главните резултати.

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