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ESTIMATING THE FAILURE INTENSITIES OF MULTI-COMPONENT SYSTEM WITH MULTI - STATE COMPONENTS

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Dedicated to Academician Ǵorǵi Čupona

Abstract. Under consideration of this paper are unrecoverable multi-state systems with independent multi state components. It is assumed that each component can fail by one level down in time, and that transitions are homogenous Markov transitions. In order to calculate the reliability of such a system, we need to estimate the one level transition intensities of each component. Here we give an estimators for this parameters, under assumption that all one level transition of the components of the system are known.

1. Introduction

Traditionally, reliability analysis of multi-component system depends upon the assumption that the system and its components can be in a binary state; either fully working conditions or complete failures [1], [2]. But, there some examples where the binary approach gives uncorrect results, [9], [10], [11], and in this case it is found that analysis using multi-state system (MSS) assumption is more appropriate. MMS system is a system such that it and its components can operate in more than one level of performance. The proper definitions of a multi-state monotone system and of multi- state coherent systems are given in [7] and [8], where we can also find the definitions for minimal path and cut vectors.

There are few points of view in reliability analysis of MMS. In some papers MMS are considered without taking care about their structure depending on the components, [5], [4]. In the papers that analyze the influence of individual components to the reliability of the whole system it is usually assumed that they have already known distribution, [3], [9] and [10]. Our approach is quite different. We are focusing on reliability data, trying to determine the unknown failure distribution, similarly as in [1]. In this paper we give estimators of one level transition failure intensities of the particular components of the systems, under assumption that one level transitions of each of the components a monotone Markov transitions.

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2. Background

Under consideration in this paper are monotone, unrecoverable multi-state systems with independent components. Our assumption for the system is a small generalization of the MMS systems given in ([7], [8]). Suppose that the k-th component of an n - component system can work in one of $M_k + 1$ levels, such that 0 is the level of total failure and M_k is the level of perfect functioning, and for $i < j$, the level j is better then the level i. Similarly, the whole system can operate in one of $M + 1$ levels, where M is a level of perfect functioning, and 0 is the level of total failure, and as it is usual in the MMS theory, we assume that the levels of work of the whole system are linearly ordered, and greater number associates with better performance. In this paper we may regard the system as a system with two levels, 1, when it is in some working level, and 0, when it is failed.

Let us denote the set state of the system by $S = \{\mathbf{x} = (x_1, \ldots, x_n) | 0 \leq x_k \leq M_k, k = \overline{1,n}\}$, and by $\phi(\mathbf{x})$ the level of work of the whole system when it is in the state $\mathbf{x}$. Monotone MMS is a system that satisfies: $\mathbf{x} < \mathbf{y} \Rightarrow \phi(\mathbf{x}) < \phi(\mathbf{y})$. We also assume that all components of the system are relevant, i.e they give influence to the work of the system. More precisely, the k-th component of a n-component system is relevant if there is a state $(x_1, \ldots, x_k, \ldots, x_n)$ such that $\phi(x_1, \ldots, x_k, \ldots, x_n) < \phi(x_1, \ldots, M_k, \ldots, x_n)$, where M_k is a maximal level of work of the k-th component. Moreover we assume that each level of each component is relevant for the system, i.e. that for $x_k > 0$, $\phi(M_1, \ldots, x_k, \ldots, M_n) > 0$. If it is not the case, we can rename the states of k-th component by setting to 0 the first i such that $\phi(M_1, \ldots, i, \ldots, M_n) = 0$, and setting to $x_k - i$ all others x_k.

The systems of our interest are the systems for wich one level failure transitions of each component are monotone Markov transitions, i.e. the transition time from state i to state $i - 1$ of the k-th component have some exponential distribution with density function $f_i^{(k)}(t) = \lambda_i^{(k)} e^{-\lambda_i^{(k)} t}$, where $\lambda_i^{(k)}$ is the failure intensity. The problem we want to solve is to estimate this unknown parameters. The estimation of the failure intensities of a unit which is a part of the system, is more complicated then when it is regarded separately from the system, [6]. The estimation depends of that if is the system is under complete control or not. Although the system may be under complete control, this procedure will be more complicated compared with one component system. This happens because the system may stopped with its work before total failure of the certain unit, and the obtained data are not complete.

3. MLE and estimator using moments for systems under complete control

In the case when during the work, the system is under complete control, we know the exact transition time between two neighbor states of each component. But, we still do not have all information about some particular unit, since the system may stop working before its total failure. For example, suppose that in the time when the system fails, some of the components of the system is at a level i. Then, for that particular component, we know the exact transition times from level j to level $j-1$, for all $j > i$. But we do not have any information about how long it will stay at levels j, for $j \leq i$. So, since we have a little bit information, the estimation of the failure intensity of the components which are at work in complex system, even the system is under complete control, is not such simple as in the case of one-component system.

When the system is under complete control, each experiment gives as much uncensored data for each component, as the number of its one level transitions. Also, when the component is not in the state of total failure, we get one censured data. This data gives information how long the component was working at the last level, before the system stoped with its work. In fact, for each unit from each sample we get one data vector $(t_M, t_{M-1}, \ldots, t_{i+1}, t)$, where $t_j,\ j = \overline{i+1, M}$ means that in time t_j the unit transits from the state j to the state $j-1$, and t is the time when the system stops with its work. Note that in the moment of total failure of the system, the component works with level i.

It is clear that $t_M < t_{M_1} < \ldots < t_i < t$, so we can do data transformation on the following way:

$$\begin{array}{rcl} \widehat{t}_M & = & t_M, \\ \widehat{t}_j & = & t_j - t_{j+1},\ j = \overline{i+1, M-1}, \\ \widehat{t}_i & = & t - t_{i+1}. \end{array}$$

The data $\widehat{t}_j$, $j \geq i+1$, gives exact time of working of the component at level j. On the other hand, the component working with level i does not transit to level $i-1$ for the time $\widehat{t}_i$.

Note that $\widehat{t}_j,\ j = \overline{i+1, M}$ are uncensored data, and the $\widehat{t}_i$ is a censured data. Let $\widehat{\mathbf{t}} = (\widehat{t}_M, \widehat{t}_{M-1}, \ldots, \widehat{t}_i)$ we be the observed data vector.

Let us form the sets S'_k, S''_k and S_k, in the following way: S''_k is the set of all times $\widehat{t}_k$ that are the last coordinates of the vectors $\widehat{\mathbf{t}}$, and S'_k is the set of all $\widehat{t}_k$, which are not last coordinates of the vectors $\widehat{\mathbf{t}}$. S_k is defined as $S_k = S'_k \cup S''_k$. Let $n'_k = |S'_k|$, $n''_k = |S''_k|$ and $n_k = |S_k|$. Now for each level k, we form likelihood function:

$$L_k = \prod_{\widehat{t}_i \in S''_k} e^{-\lambda_k \widehat{t}_i} \prod_{\widehat{t}_i \in S'_k} \lambda_k e^{-\lambda_k \widehat{t}_i} = (\lambda_k)^{n'_k} \prod_{\widehat{t}_i \in S_k} e^{-\lambda_k \widehat{t}_i} = (\lambda_k)^{n'_k} \cdot e^{-\lambda_k \sum_{\widehat{t}_i \in S_k} \widehat{t}_i}.$$

Taking logarithm of both sides we obtain:

$$l_k = n'_k ln(\lambda_k) - \lambda_k \sum_{\widehat{t}_i \in S_k} \widehat{t}_i.$$

Taking $l'_k = 0$ we get:

$$0 = \frac{n'_k}{\lambda_k} - \sum_{\widehat{t}_i \in S_k} \widehat{t}_i,$$

and we obtaine MLE for λ_k as

$$\widehat{\lambda}_k = \frac{n'_k}{\sum_{\widehat{t}_i \in S_k} \widehat{t}_i}. \tag{1}$$

All the intensities for which we have some data can be estimated using formula (1). But, some of the states are visited infrequently by the component, so for the estimation of the failure intensities associated to those states we have less data. It is clear that the sample for the transition times for all other levels less or equal to that one is even smaller, so the estimated values will have even higher error. Consequently, the estimations of the transition intensities for higher levels are more correct then that for the lower ones. Because the greater levels have a greater influence to the work of the system, the errors that will occur at the lower values has not drastically affect the reliability estimation of the whole system.

As we expect when we have an exponential distribution, the estimator obtained by the method of moments is the same with (1). The mean transit time from level k to level $k-1$ is $\dfrac{1}{\lambda_k}$. If t_i is a censured data, i.e. we know that the system in time t_i has not failed, then we know that it will fail after some time ε_i. Because the exact failure time has exponential distribution, ε_i also has exponential distribution with parameter λ_k, i.e $\overline{\varepsilon} = \dfrac{1}{\lambda_k}$. On the other hand

$$\frac{1}{\lambda_k} = \frac{\sum_{i\in S'_k} t_i + \sum_{i\in S''_k}(t_i+\varepsilon_i)}{n'_k + n''_k} = \frac{\sum_{i\in S_k} t_i + \sum_{i\in S''_k}\varepsilon_i}{n'_k + n''_k} = \frac{\sum_{i\in S_k} t_i + n''_k\frac{1}{\lambda_k}}{n'_k + n''_k}.$$

Solving this with respect to λ_k, we again obtain (1) as an estimator for λ_k .

The verification of that, how good is this estimator, was made by simulation of such a system.

Example 1. *We will give the results of three experiments in which were generated different size of data for a systems with three components, each of them can operate in four levels: 0, 1, 2 and 3. The one - level intensities of all systems are the same and they are given by following matrix* $\Lambda = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, *where* $\Lambda_{i,j}$ *is the failure intensity of the i-th component, from level j to level j-1. Systems have different minimal cut sets. The minimal cut set for the first experiment is* $M_1 = \{(3,0,1),(1,2,1),(0,1,3),(1,3,0),(3,1,0),(0,3,1),(1,0,3)\}$, *for the second one is* $M_2 = \{(1,0,1),(0,1,2),(2,1,0),(0,3,0),(3,0,0),(0,3,0)\}$ *and for the third one is* $M_3 = \{(2,0,0),(0,0,2),(0,2,0),(1,1,1)\}$. *The estimated failure intensities by (1),* $\widehat{\Lambda}$, *are given in the following table.*

$\widehat{\Lambda}$	M_1	M_2	M_3
500	$\begin{bmatrix} 0.97 & 2.17 & 3.23 \\ 1.00 & 2.03 & 1.00 \\ 0.89 & 1.89 & 1.90 \end{bmatrix}$	$\begin{bmatrix} 1.02 & 1.94 & 3.02 \\ 1.16 & 2.00 & 0.98 \\ 0.89 & 1.99 & 2.10 \end{bmatrix}$	$\begin{bmatrix} 0.96 & 1.98 & 3.04 \\ 0.93 & 1.93 & 1.04 \\ 1.00 & 1.81 & 2.02 \end{bmatrix}$
50	$\begin{bmatrix} 0.83 & 1.72 & 2.88 \\ 1.18 & 1.78 & 0.89 \\ 1.43 & 1.70 & 1.79 \end{bmatrix}$	$\begin{bmatrix} 1.08 & 2.08 & 2.81 \\ 1.75 & 2.07 & 0.88 \\ 1.01 & 2.01 & 2.32 \end{bmatrix}$	$\begin{bmatrix} 0.99 & 2.25 & 3.07 \\ 1.16 & 2.65 & 1.11 \\ 0.78 & 1.61 & 2.65 \end{bmatrix}$
10	$\begin{bmatrix} 1.79 & 1.75 & 3.38 \\ 0.49 & 1.28 & 0.88 \\ 0.50 & 1.13 & 1.33 \end{bmatrix}$	$\begin{bmatrix} 1.20 & 1.31 & 2.74 \\ 0.74 & 2.02 & 0.99 \\ 0.94 & 2.26 & 3.30 \end{bmatrix}$	$\begin{bmatrix} 0.58 & 3.11 & 2.71 \\ 1.04 & 2.88 & 1.53 \\ 0.75 & 1.50 & 4.60 \end{bmatrix}$

From this simulation we can conclude that this estimator is really good, since for big sample 500 the estimated values are very closed to the real ones. Moreover, good estimated values are also obtained for a relatively small sample, 50. In fact, for this data size, we have a greater differences for the failure intensities from level 1 to level 0. But, this is an expected error, since for this level we have the smallest amount of data, much smaller then 50. The worst estimation, as we expected, was obtained for sample size 10, but the good think is that the estimated values do not deviate much from the actual ones. It is not bed result, because some of this values are obtained from a sample of size 1 or 2.

Also it is obvious that the quality of this estimator does not depend a lot from the type of the system, since in all examples we have similar errors.

4. Ratio estimator

In this section we give another estimator, that is produced using ratios between intensities of different components. This estimator is worst compared to MLE, but it is due to the fact that it requires less information. Let us define failure path as a sequence $(M_1, M_2, \ldots, M_n) < \mathbf{x}_1 < \ldots < \mathbf{x}_{r-1} < \mathbf{x}_r$, where $\phi(\mathbf{x}_{r-1}) > 0$ and $\phi(\mathbf{x}_r) = 0$. The estimator proposed in this section does not require all failure intensities. It requires whole failure paths and only one failure time, that can be the time to total failure, the time of first one level failure of any component or some other specific failure time.

Let us have a collection of identical n-component systems. Suppose that during there work, N of the systems are found in the state $\mathbf{x}$, and N_1 of them transit to state $\mathbf{x} - \mathbf{e}_i$ and N_2 to the state $\mathbf{x} - \mathbf{e}_j$, for $i, j = \overline{1, n}$. Thus, the estimated probability that the system will transit from the state $\mathbf{x}$ to the state $\mathbf{x} - \mathbf{e}_i$ is $\dfrac{N_1}{N}$, and from the state $\mathbf{x}$ to the state $\mathbf{x} - \mathbf{e}_j$, $\dfrac{N_2}{N}$. On the other side, theoretically we have that the probability that the system will transit from the state $\mathbf{x}$ to the state $\mathbf{x}-\mathbf{e}_i$ is $\dfrac{\lambda_{x_i}}{\sum_{k=1}^n \lambda_k}$, and from the state $\mathbf{x}$ to the state $\mathbf{x}-\mathbf{e}_j$ is $\dfrac{\lambda_{x_j}}{\sum_{k=1}^n \lambda_{x_k}}$. Consequently, we can write:

$$\frac{N_1}{N_2}=\frac{\frac{N_1}{N}}{\frac{N_2}{N}}=\frac{\frac{\lambda_{x_i}}{\sum_{k=1}^n \lambda_k}}{\frac{\lambda_{x_j}}{\sum_{k=1}^n \lambda_k}}=\frac{\lambda_{x_i}}{\lambda_{x_j}}.$$

Following this idea, all of the important intensities (the intensities that have influence the work of the system) can be expressed using only one of the intensities. Now, the likelihood function will depend of one parameter only.

In order to use the whall information we have, in expressing particular failure intensity parameter by the others, we will take all vector states where it is found. Let $\lambda_{i_1;j_1}$ be the failure intensity of the i_1-th component, from level j_1 to level j_1-1. Also let $\lambda_{i_2;j_2}$ be the failure intensity of the i_2-th component, from the level j_2 to the level j_2-1, for $i_1\neq i_2$. We will use following notations: By $V_{(i_1;j_1),(i_2;j_2)}$ we will denote the set of all path vectors for which the i_1-th coordinate is j_1, and i_2-th coordinate is j_2. For each $\mathbf{x}\in V_{(i_1;j_1),(i_2;j_2)}$, $N_{\mathbf{x}}$ is the number of the systems that visit the state $\mathbf{x}$ during their work, and $N_{\mathbf{x};i}$ the number of the systems that visit the state $\mathbf{x}$, and in the next step its i-th component failed for one level. Now we have:

$$\frac{\widehat{\lambda}_{i_1;j_1}}{\widehat{\lambda}_{i_2;j_2}}=\frac{1}{\sum\limits_{\mathbf{x}\in V_{(i_1;j_1),(i_2;j_2)}} N_{\mathbf{x}}}\cdot\sum_{\mathbf{x}\in V_{(i_1;j_1),(i_2;j_2)}} N_{\mathbf{x}}\frac{N_{\mathbf{x};i_1}}{N_{\mathbf{x};i_2}},\quad i_1\neq i_2 \tag{2}$$

Instead taking all elements from the set $V_{(i_1;j_1),(i_2;j_2)}$, we can take only the most frequent one. Let us note it by $\mathbf{v}_{(i_1;j_1),(i_2;j_2)}$. Sometimes this is a better choice, since in some of that states the system is found very rarely, and that small samples can give results with high error rate.

As a first step in the process of estimation we can choose one of the failure intensities, for example λ_{i,M_i}, and we will express the other parameters trough this one. Note that it is the best to choose the parameter of this type, because the probability that the system will be found in a higher state is always greater then the probability that the system will be found in lower state. Without lost of generality we may choose λ_{1,M_1}. Now all intensities can be expressed with this one, since by our assumption that each level of each component is relevant for the system we have that $V_{(1;M_1),(i;j)}\neq\emptyset$. Let as set $\lambda=\lambda_{1,M_1}$. For each intensity $\lambda_{i;j}$, using (2) we can calculate a real number $\widehat{\alpha}_{i;j}$ such that $\lambda_{i;j}=\widehat{\alpha}_{i;j}\lambda$. It remains to estimate λ. It can be done using the failure time of the whole system on different ways. We may use all transitions, but for simplification, it is sufficient to take only the first transition the system does, i.e. the random variable T_1 - "the time to the first transition of the system". The intensity of that transition is $\sum\limits_{i=1}^n \lambda_{i,M_i}$, and

$$\overline{t_1}=\frac{1}{\sum_{i=1}^n \lambda_{i;M_i}}=\frac{1}{\lambda\sum_{i=1}^n \alpha_{i;M_i}}.$$

At the end we can estimate λ by

$$\widehat{\lambda} = \frac{1}{\overline{t_1} \sum_{i=1}^{n} \alpha_{i;M_i}}.$$

Example 2. *We use the same data of size 500 as in the Example 1, for the system with* $\Lambda = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ *and minimal cut set* M_1*. The parameters estimated by using the ratio estimator and the time of the first faiure transition, we obtain:* $\widehat{\Lambda} = \begin{bmatrix} 0.68 & 2.06 & 2.60 \\ 0.94 & 2.05 & 0.93 \\ 1.17 & 2.13 & 1.81 \end{bmatrix}$.

Reasonably, the second estimator gives worst results, since it uses less information. In fact, for ML estimator we need to know all transitions and all one level transition times, and for ratio estimator, only the time of the first transition and the order of the falling components. So when all transition times of the system are measured, it is better to use the ML estimator. But, when we only know the order of the components falling and the time to total failure, or some other failure time, we need to use ratio estimator.

The ratio estimator can be used in a situation where instead of controlling the system all time, it is done by inspection at specific time intervals. During such inspections we will collect information about the failure path of the system, while information about the exact time of each failure transition will be incomplete.

5. Conclusion

This paper deals with multi state, multi component monotone systems with independent components. The objective is to provide methods for evaluating the reliability of such systems, based on statistical data about times of work of the systems and transition states during its work.

There are presented two types of estimators of the one level failure intensities of the components. The first estimator requires all information about failure times of each of the components of the system and it is found that it gives very good results even for a small sample. The second estimator can be used in the situation when we do not have all one level transition times of each component, but we know the failure path of the systems. Reasonable, it gives worst results, but the estimated values do not vary too much from the values estimated by MLE.

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