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ON A SHAPKAREV'S METHOD OF DIFFERENTIATION AND TRANSFORMATION
(To the memory of Prof. Ilija A. Shapkarev)

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Abstract. This paper emphasises the contribution of Prof. Dr. Ilija A. Shapkarev in the field of differential equations, particularly in the existence and construction of polynomial solution of the ordinary differential equation (ODE) of *n*-th order, or equivalently, of system of *n* equations of first order. The method that he uses is called the method of differentiation and transformation (MDT). With this paper we give a review of his work and extension of the application of his method.

1. INTRODUCTION

One of the most common problems in the theory of ordinary differential equations is the existence and uniqueness of the solution. When the ODE is linear, it might be of help to reduce it to a system of ODE's.

An important problem is to find a polynomial solution. Its importance can be treated from many aspects. For instance, it is widely known that the orthogonal polynomials as a solutions of differential equations, are of great importance in numerical mathematics. Great number of numerical methods is based on polynomial solutions of differential equations. Also, it is well-known how eigenvalues and eigenfunctions are connected to polynomial solutions of differential equations.

Existence and construction of polynomial solution need lot of methods from mathematical analysis. The existence conditions always contain positive integer.

Prof. D-r Ilija A. Shapkarev, doyen in this field in Macedonia, had been worked on this problem in many of his scientific papers. The papers, in which he transforms the differential equations in the equations of the same type, are of special importance. These results allowed wide generalization of the classes of solvable equations. In the most of his papers he uses method of differentiation and transformation (MDT).

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2. OVERVIEW OF PROF. SHAPKAREV'S WORK

In this section we will focus only on one class linear differential equations of third order, reducible to a system.

Let's consider the differential equation of the following type:

$$(x-a)(x-b)(x-c)y'''+(\beta_2 x^2+\beta_1 x+\beta_0)y''+(\gamma_1 x+\gamma_0)y'+\delta y=0\,, \qquad (2.1)$$

where $a,b,c,\beta_2,\beta_1,\beta_0,\gamma_1,\gamma_0,\delta$ are real constants and $a\neq b\neq c, y=y(x)$.

In [6], the conditions for existence and construction of polynomial solutions of (2.1), of orders *m*, *n*, *p*, are obtained as well as the reduction to a system of equations is shown. In [2], [3] and many other papers based on MDT, a special class of differential equations is treated and the corresponding conditions for existence of polynomial solutions are obtained.

In [1], [4], equation of the type (2.1) is also treated from the existence and construction of polynomial solutions point of view and some special cases for its reducibility to a system of linear equations of first order is obtained:

$$\begin{aligned}&(x-a)y'+A_1y+A_2z=0,\\&(x-b)z'+B_2z+B_3w=0,\\&(x-c)w'+C_3w=0,\end{aligned} \qquad (2.2)$$

where $a,b,c,A_1,A_2,B_2,B_3,C_3$ are real constants and $a\neq b\neq c, y=y(x), z=z(x)$, $w=w(x)$.

The method of successive differentiations and method of elimination performed over the system (1.2), will yield to the following differential equation of third order

$$\begin{aligned}&(x-a)(x-b)(x-c)y'''+\\&+[C_3(x-a)(x-b)+(A_1+2)(x-b)(x-c)+(B_2+1)(x-a)(x-c)]y''\\&+[C_3B_2(x-a)+\ (A_1+1+B_2+A_1B_2)(x-c)]y'+C_3A_1B_2y=\ 0\end{aligned} \qquad (2.3)$$

and the following differential equation of second order

$$(x-b)(x-c)z''+[(B_2+1)(x-c)+C_3(x-b)]z'+C_3B_2z=0. \qquad (2.4)$$

The equations (1.3) and (1.4) and the equation

$$(x-c)w'+C_3w=0 \qquad (2.5)$$

correspond to the system (1.2).

Successive solutions of the system (2.2) are

$$\begin{aligned}&w=K_1(x-c)^{-C_3}\\&z=(x-b)^{-B_2}[K_2+B_3K_1\int(x-c)^{-C_3}(x-b)^{B_2-1}dx]\\&y=(x-a)^{-A_1}\{K_3+A_2\int(x-a)^{A_1-1}(x-b)^{-B_2}[K_2+B_3K_1\int(x-c)^{-C_3}(x-b)^{B_2-1}dx]\,dx\}\end{aligned} \qquad (2.6)$$

where K_1,K_2,K_3 are arbitrary constants.

Actually ([4]), these are the formulas for the general solutions of the equations (2.3), (1.4) and (1.5).

The system (2.2) is important, because it is equivalent to a differential equation of third order, whose general solution is polynomial with three particular polynomial solutions. That is the equation

$$\begin{aligned}&(x-a)(x-b)(x-c)y'''+\{(-3n-2m-p+3)x^2+[n(2a+2b+2c)+m(a+b+2c)\\&+p(b+c)-(a+2b+3c)]x-n(ab+bc+ac)-mc(a+b)-pbc+c(a+2b)\}y''\\&+\{[(n+m)(3n+m+p-2)+(p-1)(n-1)]x+(n+m)[-n(a+b)+(2-n-m-p)c]\\&+(p-1)(c-nb)\}y'-n(n+m)(n+m+p)y=0,\end{aligned} \tag{2.7}$$

whose general solution is given by the following formula:

$$y=(x-a)^{n+m+p}\{K_3+\int(x-a)^{-(n+m+p+1)}(x-b)^{n+m}[K_2+K_1\int(x-c)^n(x-b)^{-(n+m+1)}dx]\,dx\} \tag{2.8}$$

and the corresponding system is ([4], [12])

$$\begin{aligned}&(x-a)y'-(n+m+p)y+z=0,\\&(x-b)z'-(n+m)z+mw=0,\\&(x-c)w'-nw=0.\end{aligned} \tag{2.9}$$

3. APPLICATION OF MDT

In [3,13] the authors treat the problem of reducibility of a class of a linear differential equations of second order with polynomial coefficients of the following type

$$(x-a)(x-b)y''+(b_1x+b_0)y'+c_0y=0, \tag{3.1}$$

where a,b,b_1,b_0,c_0 are real constants and $a\neq b, y=y(x)$.

Using the conditions for existence of the general polynomial solution of the differential equation (3.1) a class of linear differential equations of second order of the type

$$(x-a)(x-b)y''-[(2n+m-1)x-(r+n)b-(m+n-r-1)a]y'+n(n+m)y=0, \tag{3.2}$$

where $m,n,r\in\mathbb{Z}^+,\ r\in\{1,2,...,m-1\}$ has general solution given by formula

$$\begin{aligned}y=&C_1(x-a)^{n+r+1}(x-b)^{n+m-r}[(x-a)^{-r-1}(x-b)^{-m+r}]^{(n)}\\&+C_2(x-a)^{n+r+1}(x-b)^{n+m-r}[(x-a)^{-r-1}(x-b)^{-m+r}\int(x-a)^r(x-b)^{m-r-1}dx]^{(n)}\end{aligned} \tag{3.3}$$

Moreover, it is shown that the equation (3.2) can be reduced to the following system of differential equations:

$$\begin{aligned}&(x-a)y'-(n+r+1)y+(r+1)z=0,\\&(x-b)z'-(n+m-r-1)z+(m-r-1)y=0.\end{aligned} \tag{3.4}$$

Let's now consider the system

$$\begin{aligned}&(x-a)y'-(n+m+p)y+z=0,\\&(x-b)z'-(n+r+1)z+(r+1)w=0,\\&(x-c)w'-(n+m-r-1)w+(m-r-1)z=0,\end{aligned} \tag{3.5}$$

where $m,n,p,r\in\mathbb{Z}^+,\ r\in\{1,2,...,m-1\}$.

By method of differentiation and transformation the last system can be transformed in the following differential equation of third order:

$$\begin{aligned}(x-a)(x-b)(x-c)y''' + \{(-3n-2m-p+3)x^2 + [a(2n+m-1)+b(2n+2m+p-r-3)\\ +c(2n+m+p+r-2)]x-(n+m+p-2)bc-(n+r)ac-(n+m-r-1)ab\}y''\\ +\{[(n+m+p-1)(2n+m-1)+n(n+m)]x-(n+m+p-1)(n+r)c\\ -(n+m+p-1)(n+m-r-1)b-n(n+m)a\}y'-n(n+m)(n+m+p)y=0.\end{aligned} \quad (3.6)$$

The second and third equations of the system (3.5) are of the type (3.4). In accordance with the formula (3.3) and using the first equation, the general solution of the equation (3.6) will be given by the formula

$$\begin{aligned}y=(x-a)^{n+m+p}\{C_1+\\ +C_2\int\{(x-a)^{-(n+m+p)}(x-b)^{n+r+1}(x-c)^{n+m-r}\{C_2[(x-b)^{-r-1}(x-c)^{-m+r}]^{(n)}\\ +C_3[(x-b)^{-r-1}(x-c)^{-m+r}\int(x-b)^r(x-c)^{m-r-1}dx\,]^{(n)}\}\}dx\}\end{aligned} \quad (3.7)$$

By this, the following theorem is proved:

Theorem: *The general solution of the differential equation of third order with polynomial coefficients* (3.6) *is a polynomial, given with the formula* (3.7). *Also, it can be reduced on a system of differential equations* (3.5).

Remark. The equation (1.7) is a special case of the equation (3.6), for $r=m-1$, while the system (1.9) is a special case of the system (3.5).

4. CONCLUSION

Prof. Dr. Ilija A. Shapkarev has a great contribution in development and research in the field of ordinary differential equation in Macedonia (and wider), especially with his method of differentiation and transformation. In this paper we made an overview of a part of his work and we gave an example of a differential equation of third order, where his method can be applied for obtaining the polynomial solution.

References

[1] Б. М. Пиперевски, *Полиномни решенија на една класа линеарни диференцијални равенки и нивна примена, докторска теза*, Математички Факултет, Скопје (1982).

[2] И. А. Шапкарев, *За една редуктибилна хомогена линеарна диференцијална равенка чиј општ интеграл е полином*, Седми македонски симпозиум по диференцијални равенки, Зборник на трудови, ISBN: 9989 - 630 - 37 – 2, Охрид, Македонија, (2003), стр. 73 - 84, cim.feit.ukim.edu.mk.

[3] И. А. Шапкарев, Б. М. Пиперевски, Е. Хаџиева, Н. Серафимова, К. Митковска Трендова, *За една класа линеарни диференцијални равенки од втор ред чие општо*

решение е полином, Седми македонски симпозиум по диференцијални равенки, Зборник на трудови, ISBN: 9989 - 630 - 37 – 2, Охрид, Македонија, (2003), стр. 27-39, cim.feit.ukim.edu.mk.

[4] Е. Хаџиева, В. М. Пиперевски, *За редуктибилноста на една класа линеарни диференцијални равенки од трет ред со полиномни коефициенти*, Математички билтен, 27 (LIII), (2002), 107-112.

[5] И. А. Шапкарев, *За некои послаби специјални услови за редуктибилност на една линеарна диференцијална равенка од трет ред, чиј општ интеграл е полином*, Осми македонски симпозиум по диференцијални равенки, Зборник на трудови, ISBN: 9989-630-49-6, 2004, Охрид, Македонија, стр. 1-14, cim.feit.ukim.edu.mk.

[6] И. А. Шапкарев, *За една итеративна метода за добивање полиномни решенија на една линеарна диференцијална равенка од трет ред со полиномни коефициенти*, Зборник на трудови од третиот конгрес на математичарите на Македонија, Струга, Македонија, 411-420 (2005)

[7] Б. М. Пиперевски, Егзистенција и конструкција на полиномно решение на една класа диференцијални равенки од трет ред, Зборник на трудови, Електротехнички Факултет, Скопје, год. 19, бр. 1-2, стр. 47-48, (1996).

[8] I. A. Shapkarev, *Uber eine reduzierbare homogene lineare differentialgleichung derenallgemeines integral ein polynomdarstellt*, Седми симпозиум по диференцијални равенки, Зборник на трудови, ISBN: 9989 - 630 - 37 - 2, стр. 73-84, 2002, Охрид, Македонија cim.feit.ukim.edu.mk.

[9] И. А. Шапкарев, *Егзистенција и конструкција на полиномни решенија на една класа линеарни диференцијални равенки од втор ред*, Математички билтен 11-12 (XXXVII-XXXVIII), (1987-1988), 5-11.

[10] B.M. Piperevski, *Sur des equations differentielles lineaires du duxieme ordre qui solution generale est polynome*, University "Kiril and Metodij" Department of electrical engineering, Proceedings, No 4, Year 9, 13-17, Skopje, (1986).

[11] I. A. Shapkarev, B. Jolevska, *Invariant non-homogeneons tirad order linear differential equations in respect to function substitution*, Зборник на трудови од првиот конгрес на математичарите и информатичарите на Македонија, 93-100, Охрид, Македонија (1996).

[12] Илија А. Шапкарев, Полином како општо решение на една хомогена линеарна диференцијална равенка од трет ред, Зборник на трудови од вториот конгрес на математичарите и информатичарите на Македонија, Охрид, Македонија, (2002) 105-110.

[13] N. Serafimova, K. Mitkovska Trendova, B. Piperevski, *On a class of second order differential equations' systems, whose general solution is polynomial*, Зборник на трудови од вториот конгрес на математичарите и информатичарите на Македонија, Охрид, Македонија, (2002), 101-104.

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