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BAND DECOMPOSITIONS OF SEMIGROUPS (A-SURVEY)*

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Dedicated to Academician Ǵorǵi Čupona

Abstract. Semigroups having a decomposition into a band of semigroups have been studied in many papers. In the present paper, using various types of bands congruences, we give a survey of structural characterizations of bands of Archimedean, left Archimedean, π-groups, k-Archimedean, $\mathcal{J}_k$-simple, $\mathcal{L}_k$-simple, $\mathcal{H}_k$-simple, λ-simple and η-simple semigroups. At the end are formulated some open problems in the band decomposition theory.

1. Introduction

Band decompositions of semigroups play very important role in semigroup theory. The existence of the greatest band decomposition of semigroups was established by T. Tamura and N. Kimura [74], 1955. After that, several authors have worked on this important topic.

Note that bands (semilattices) of Archimedean semigroups have been studied by a number of authours. M. S. Putcha, [54], gave the first complete description of such semigroups. Another characterizations of bands (semilattices) of Archimedean semigroups are given by T. Tamura, [73], S. Bogdanović and M. Ćirić, [11], M. Ćirić and S. Bogdanović, [26]. Semilettices of some special classes of Archimedean semigroups are considered by M. Mitrović, [47], [48].

M. S. Putcha in [55] and [56] gave a general characterization of a band of left Archimedean semigroups. Some special decompositions of this type have been also treated in a number of papers. P. Protić in [58], [59], [60], S. Bogdanović and M. Ćirić in [13], and S. Bogdanović, M. Cirić and B. Novikov in [16] studied bands of left Archimedean semigroups whose related band homomorphic images belong to several very important varieties of bands. L. N. Shevrin investigated in [69] bands of nil-extensions of left groups, and S. Bogdanović and M. Ćirić investigated in [8] bands of nil-extensions of groups. S. Bogdanović investigated in [1], [2] bands of power-joined semigroups. Bands of left simple semigroups, in the general and

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some special cases, were investigated by P. Protić in [59] and M. Ćirić and S. Bogdanović in [28].

A rectangular band of Archimedean semigroups is an Archimedean semigroup and by well known Clifford's result from 1954, [24], we have that a band of Archimedean semigroups is a semilattice of Archimedean semigroups. As M. S. Putcha [55] mentioned this is not the case if the word "Archimedean" is replaced by "left Archimedean". The question is now: Let $\mathcal{C}$ be a class of semigroups. Under which conditions a band of semigroups from $\mathcal{C}$ coincide with a semilattice of semigroups from $\mathcal{C}$? The answer on this question is given in the section two.

L. N. Shevrin [68] proved that in a completely π-regular semigroup $R(\mathcal{D})$ is transitive if and only if it is a semilattice congruence. A more general result has been obtained by M. S. Putcha [53] who proved that in a completely π-regular semigroup the transitive closure of $R(\mathcal{J})$ is the smallest semilattice congruence. Since $\mathcal{D} = \mathcal{J}$ on any completely π-regular semigroup, the above Shevrin's result can be also derived from the one of M. S. Putcha.

Various characterizations of semigroups in which the radical $R(\varrho)$ $(T(\varrho))$, where $\varrho \in \{\mathcal{J}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{H}\}$, is a band (semilattice) congruence have been investigated by S. Bogdanović and M. Ćirić in [14], [15], and by S. Bogdanović, M. Ćirić and Ž. Popović in [17].

In fourth section is defined one new radical ϱ_k, $k \in \mathbf{Z}^+$, of a relation ρ on a semigroup S by

$$(a,b) \in \varrho_k \;\Leftrightarrow\; (a^k, b^k) \in \varrho.$$

Using this radical we describe the structure of a semigroup in which this radical is a band (semilattice) congruence for some Green's relation. For these descriptions of the structure of semigroups we consider some new types of k-regularity of semigroups and also some new types of k-Archimedness of semigroups. These descriptions are given by S. Bogdanović, Ž. Popović and M. Ćirić in [20].

Various special types of bands of λ- semigroups and semilattices of matrices of λ-semigroups are treated in the section five. Some characterizations in the general case are given by S. Bogdanović, M. Ćirić and Ž. Popović [17]. Some special decompositions of this type have been also treated in a number of papers. By the well-known result of A. H. Clifford [24], any band of λ-simple semigroups is a semillatice of matrices of λ-simple semigroups. These semigroups are characterized by S. Bogdanović, Ž. Popović and M. Ćirić in [20]. All band decompositions are special cases of semilattice-matrix decompositions. The general lattice theoretical properties of semilattice-matrix decompositions of semigroups are investigated by M. Ćirić and S. Bogdanović [29]. A semilattice of matrix of left Archimedean semigroups were studied by S. Bogdanović and M. Ćirić [13]. Semilattices of λ-simple semigroups are described in [16], [20] and [26].

In sixth section are considered bands decompositions of semigroups induced by identities. Here are described identities which induce decompositions of π-regular semigroups into a band of nil-extensions of groups. Decompositions into a band of nil-extensions of groups are considered by L. N. Shevrin [67], J. L. Galbiati and L. M. Veronesi [32], B. L. Madison, K. T. Mukherjee and M. K. Sen

[43], S. Bogdanović and M. Ćirić [8], and so one. Identities which induced such decompositions are described by M. Ćirić and S. Bogdanović in [26]. The main result of this section is Theorem 30 in which are characterized all identities which induce decompositions of π-regular semigroups into a band of π-groups. Also, in this section are described identities which induce decompositins of union of groups into a band of groups. For a conection of these results with semigroup varieties we refer to [61] and [70].

Very interesting decompositions are band decompositions in which components are power joined, periodic and both power joined and periodic semigroups. These decompositions was studied by T. Tamura [72], T. Nordahl [49], K. Iseki [38] and S. Bogdanović [1], [2]. T. Tamura [72] considered commutative Archimedean semigroups which have a finite number of power joined components. Bands of power joined semigroups are studied by T. Nordahl [49], in medial case, and by S. Bogdanović [1], in the general case. K Iseki [38] considered periodic semigroup which is the disjoint union of semigroup, each containing only one idempotent. In [2] S. Bogdanović considered bands of periodic power joined semigroups.

In seventh section, on a semigroup S, for $k \in \mathbf{Z}^+$, are introduced some new equivalence relations η, η_k and τ. If these equivalences are band congruences then they makes band decompositions of η-simple (power joined) semigroups, and band decompositions of two types of periodic power joined semigroups (η_k-simple and τ-simple semigroups). The obtained results generalize the results of above mentioned authors. Also, on a semigroup S, for $k, m, n \in \mathbf{Z}^+$, are defined the following relations $\overline{\eta}_{(m,n)}$, $\overline{\eta}_{(k;m,n)}$ and $\overline{\tau}_{(m,n)}$. These relations are congruences and they are generalizations of system of congruences defined by S. J. L. Kopamu in [40]. Some characterizations of semigroups, by congruences which are more general then ones introduced by S. J. L. Kopamu in [40], are considered by S. Bogdanović, Ž. Popović and M. Ćirić in [19]. In this paper are given some equivalent statements in the case when these relations are band congruences.

For undefined notions and notations we refer [10], [36], [37] and [51].

2. Bands and semilattices of semigroups

In this section are given conditions under which the term "band" can be replaced by term "semilattice".

By $\mathcal{B}$ ($\mathcal{S}$, $\mathcal{RB}$, $\mathcal{O}$) we denote the class of all bands (semilattices, rectangular bands, one-element bands).

For two classes $\mathcal{X}_1$ and $\mathcal{X}_2$ of semigroups, $\mathcal{X}_1 \circ \mathcal{X}_2$ will denote the *Mal'cev product* [45] of $\mathcal{X}_1$ and $\mathcal{X}_2$, i.e. the class of all semigroups S on which there exists a congruence ϱ such that S/ϱ belongs to $\mathcal{X}_2$ and each ϱ-class of S which is a subsemigroup of S belongs to $\mathcal{X}_1$. Also, in such a case, by $\mathcal{X}_1 \circledast \mathcal{X}_2$ we denote a class of all semigroups which are retract extensions of semigroups from $\mathcal{X}_1$ by semigroups from $\mathcal{X}_2$.

The following lemma was proved by S. Bogdanović, M. Ćirić and B. Novikov in [16], 1998.

Lemma 1. [16] *Let $\mathcal{C}$ be a class of semigroups and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two classes of bands. Then*

$$\mathcal{C} \circ (\mathcal{B}_1 \circ \mathcal{B}_2) \subseteq (\mathcal{C} \circ \mathcal{B}_1) \circ \mathcal{B}_2.$$

For the class $\mathcal{G}$ of all groups, $\mathcal{G} \circ \mathcal{B} = \mathcal{G} \circ (\mathcal{RB} \circ \mathcal{S})$ is the class of all semigroups that are bands of groups, and $(\mathcal{G} \circ \mathcal{RB}) \circ \mathcal{S}$ is the class of all semigroups that are union of groups. As known, these classes are different, so $\mathcal{G} \circ (\mathcal{RB} \circ \mathcal{S}) \subsetneq (\mathcal{G} \circ \mathcal{RB}) \circ \mathcal{S}$. This proves that the inclusion in Lemma 1 can be proper.

The following theorem is very important result. It gives the conditions under which a band of semigroups from any class of semigroups coincide with a semilattice of semigroups from the same class. This theorem was proved by S. Bogdanović, Ž. Popović and M. Ćirić in [19].

Theorem 1. [19] *Let $\mathcal{C}$ be a class of semigroups. Then*

$$\mathcal{C} \circ \mathcal{RB} \subseteq \mathcal{C} \quad \Leftrightarrow \quad \mathcal{C} \circ \mathcal{B} = \mathcal{C} \circ \mathcal{S}.$$

3. Bands of Archimedean semigroups

In this section are given results which generalized results obtained by M. S. Putcha [55], P. Protić [59], S. Bogdanović and M. Ćirić [13], S. Bogdanović, M. Ćirić and B. Novikov [16] and M. Mitrović [47].

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers and $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{D}$ and $\mathcal{H}$ will denote known Green's relations. By $Reg(S)$ $(Gr(S), E(S))$ we denote the set of all *regular* (*completely regular, idempotent*) elements of a semigroup S. A semigroup in which all elements are idempotents is a *band*. A commutative band is a *semilattice*. If e is an idempotent of a semigroup S, then by G_e we denote the maximal subgroup of S with e as its identity. It is known that $Gr(S) = \cup\{G_e \mid e \in E(S)\}$. A nonzero idempotent e of a semigroup S is *primitive* if for every nonzero $f \in E(S)$, $f = ef = fe \Rightarrow f = e$, i.e. if e is minimal in the set of all nonzero idempotents of S relative to the partial order on this set.

By a *radical* of the subset A of a semigroup S we mean the set $\sqrt{A}$ defined by $\sqrt{A} = \{a \in S \mid (\exists n \in \mathbf{Z}^+)\ a^n \in A\}$. By $S = S^1$ we denote that S is a semigroup with the identity 1. By $S = S^0$ we denote that S is a semigroup with the zero 0 and in this case $S^* = S - \{0\}$. If $S = S^0$, then element from the set $Nil(S) = \sqrt{\{0\}}$ are *nilpotent elements* (*nilpotents*). A semigroup $S = S^0$ is a *nil-semigroup* if $S = Nil(S)$. A semigroup $S = S^0$ is *n-nilpotent* if $S^n = \{0\}$, $n \in \mathbf{Z}^+$. An ideal extension S of T is a *nil-extension* if S/T is a nil-semigroup (i.e. $S = \sqrt{T}$). An ideal extension S of a semigroup K is a *n-nilpotent extension* if S/K is a n-nilpotent semigroup.

Let $a, b \in S$. Then $a \mid b$ if $b \in S^1 a S^1$, $a \mid_l b$ if $b \in S^1 a$, $a \mid_r b$ if $b \in a S^1$, $a \mid_t b$ if $a \mid_l b$ and $a \mid_r b$, $a \longrightarrow b$ if $a \mid b^n$ for some $n \in \mathbf{Z}^+$, $a \xrightarrow{h} b$ if $a \mid_h b^n$, for some $n \in \mathbf{Z}^+$, where h is l, r or t, $a \overset{p}{—} b$ if $a^m = b^n$ for some $m, n \in \mathbf{Z}^+$, $a — b$ if $a \longrightarrow b \longrightarrow a$, $a \overset{h}{—} b$ if $a \xrightarrow{h} b \xrightarrow{h} a$, where h is l, r or t.

A semigroup S is: *Archimedean* if $S = \sqrt{SaS}$, *left Archimedean* if $S = \sqrt{Sa}$, *right Archimedean* if $S = \sqrt{aS}$, *t-Archimedean* if it is left Archimedean and right Archimedean, *power joined* if $\xrightarrow{p} = S \times S$, *completely Archimedean* if it is Archimedean and has a primitive idempotent, *left π-regular* if $(\forall a \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in Sa^{n+1}$, *$\pi$-regular* if $(\forall a \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in a^nSa^n$, *completely π-regular* if $(\forall a \in S)(\exists n \in \mathbf{Z}^+)(\exists x \in S)\ a^n = a^nxa^n,\ a^nx = xa^n$ (equivalently if $S = \sqrt{Gr(S)}$). On a completely π-regular semigroup can be introduced two unary operations $x \mapsto \overline{x}$ and $x \mapsto x^0$ by: $\overline{x} = (xe_x)^{-1}$, where $e_x \in E(S)$ such that $x^n \in G_{e_x}$ for some $n \in \mathbf{Z}^+$ and $^{-1}$ is the inversion in G_{e_x}, and $x^0 = x\overline{x}$, [65].

An element a of a semigroup S is *periodic* if there are $m, n \in \mathbf{Z}^+$, such that $a^m = a^{m+n}$. A semigroup S is *periodic* if every its element is periodic.

Let S be a semigroup and let $a, b \in S$. By a *sequence between a and b* we mean a (possibly empty) finite sequence $(x_i)_{i=1}^n$ in S such that $a \longrelbar x_1,\ x_i \longrelbar x_{i+1} (i = 1, \dots, n-1),\ x_n \longrelbar b$. We call n the *length* of $(x_i)_{i=1}^n$. By $n = 0$ (or $(x_i)_{i=1}^n$ empty) we mean $a \longrelbar b$. We say $(x_i)_{i=1}^n$ is *minimal* if it is nonempty and there is no sequence of smaller length (including the empty sequence) between a and b. By a *sequence from a to b* we mean a (possibly empty) finite sequence $(x_i)_{i=1}^n$ in S such that $a \longrightarrow x_1,\ x_i \longrightarrow x_{i+1} (i = 1, \dots, n-1),\ x_n \longrightarrow b$. Again n is the *length* of $(x_i)_{i=1}^n$ and by $n = 0$ (or $(x_i)_{i=1}^n$ empty) we mean $a \longrightarrow b$. We say $(x_i)_{i=1}^n$ is *minimal* if it is nonempty and there is no sequence of smaller length (including the empty sequence) from a to b. The *rank* $\rho_1(S)$ of a semigroup S is a zero if there is no minimal sequence between any two points. Otherwise $\rho_1(S)$ is the supremum of the lengths of the minimal sequences between points in S. The *semirank* $\rho_2(S)$ of a semigroup S is a zero if there is no minimal sequence from a point to another in S. Otherwise $\rho_2(S)$ is the supremum of the lengths of the minimal sequences from one point to another in S, [56].

A subset A of a semigroup S is *consistent* if $xy \in A \Rightarrow x, y \in A,\ x, y \in S$. A subsemigroup A of a semigroup S is a *filter* if A is consistent. By $N(a)$ we denote the least filter of S containing an element a of S (i.e. the intersection of all filters of S containing a).

3.1. **The general case.** In this part, using radicals, are given descriptions of bands of Archimedean semigroups.

Let ϱ be an arbitrary relation on a semigroup S. Then the *radical* $R(\varrho)$ of ϱ is a relation on S defined by:

$$(a, b) \in R(\varrho) \Leftrightarrow (\exists p, q \in \mathbf{Z}^+)\,(a^p, b^q) \in \varrho.$$

The radical $R(\varrho)$ was introduced by L. N. Shevrin in [68].

An equivalence relation ξ is a *left* (*right*) *congruence* if for all $a, b \in S$, $a\,\xi\,b$ implies $ca\,\xi\,cb$ ($ac\,\xi\,bc$). An equivalence ξ is a congruence if it is both left and right congruence. A congruence relation ξ is a *band congruence* on S if S/ξ is a band, i.e. if $a\,\xi\,a^2$, for all $a \in S$. A congruence relation ξ is a *semilattice congruence* on S if S/ξ is a semilattice, i.e. if $a\,\xi\,a^2$ and $ab\,\xi\,ba$, for all $a, b \in S$.

Let ξ be an equivalence on a semigroup S. By $\xi^\flat$ we define the largest congruence relation on S contained in ξ. It is well-known that

$$\xi^\flat = \{(a, b) \in S \times S \,|\, (\forall x, y \in S^1)\ (xay, xby) \in \xi\}.$$

Let $m,n\in\mathbf{Z}^+$. On a semigroup S we define a relation $\rho_{(m,n)}$ by

$$(a,b)\in\rho_{(m,n)}\Leftrightarrow(\forall x\in S^m)(\forall y\in S^n)\ xay\!\longrightarrow\!\!\!\!\longleftarrow xby,$$

i.e.

$$(a,b)\in\rho_{(m,n)}\Leftrightarrow(\forall x\in S^m)(\forall y\in S^n)(\exists i,j\in\mathbf{Z}^+)\ (xay)^i\in SxbyS\wedge(xby)^j\in SxayS.$$

If instead of the relation — we assume the equality relation, then we obtain the relation which was introduced and discussed by S. J. L. Kopamu in [40], 1995. So, the relation $\rho_{(m,n)}$ is a generalization of Kopamu's relation.

By the following theorem we give a very important characteristic of the $\rho_{(m,n)}$ relation.

Theorem 2. [22] *Let $m,n\in\mathbf{Z}^+$. On a semigroup S the relation $\rho_{(m,n)}$ is a congruence relation.*

The following two lemmas ware proved by S. Bogdanović, Ž. Popović and M. Ćirić in [19].

Lemma 2. [19] *Let ξ be an equivalence on a semigroup S. Then ξ is a congruence relation on S if and only if $\xi=\xi^\flat$.*

Lemma 3. [19] *Let ξ be an equivalence relation on a semigroup S. Then $\xi^\flat$ is a band congruence if and only if*

$$(\forall a\in S)(\forall x,y\in S^1)\ (xay,xa^2y)\in\xi.$$

The main result of this subsection is the following theorem.

Theorem 3. [11] *The following conditions on a semigroup S are equivalent:*

(i) *S is a band of Archimedean semigroups;*
(ii) *S is a semilattice of Archimedean semigroups;*
(iii) $(\forall a,b\in S)\ a\,|\,b\ \Rightarrow\ a^2\longrightarrow b$;
(iv) $(\forall a,b\in S)\ a^2\longrightarrow ab$;
(v) $(\forall a,b\in S)(\forall k\in\mathbf{Z}^+)\ a^k\longrightarrow ab$;
(vi) *$\sqrt{A}$ is an ideal of S, for every ideal A of S;*
(vii) *$\sqrt{SaS}$ is an ideal of S, for every $a\in S$;*
(viii) *in every homomorphic image with zero of S, the set of all nilpotent elements is an ideal;*
(ix) *$N(x)=\{y\in S\mid y\longrightarrow x\}$, for all $x\in S$;*
(x) $(\forall a,b,c\in S)\ a\longrightarrow b\wedge b\longrightarrow c\ \Rightarrow\ a\longrightarrow c$;
(xi) $(\forall a,b,c\in S)\ a\longrightarrow c\wedge b\longrightarrow c\ \Rightarrow\ ab\longrightarrow c$;
(xii) $\rho_1(S)=0$;
(xiii) $\rho_2(S)=0$;
(xiv) $(\forall a,b\in S)\ \sqrt{SabS}=\sqrt{SaS}\cap\sqrt{SbS}$;
(xv) *$\rho_{(m,n)}$ is a band congruence;*
(xvi) $(\forall a\in S)(\forall x\in S^m)(\forall y\in S^n)\ xay\!\longrightarrow\!\!\!\!\longleftarrow xa^2y$;

(xvii) $R(\rho_{(m,n)}) = \rho_{(m,n)}$;

(xviii) $\rho^\flat_{(m,n)}$ *is a band congruence;*

(xix) $(\forall a \in S)(\forall u, v \in S^1)\ (uav, ua^2v) \in \rho_{(m,n)}$.

The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ are from M. S. Putcha [54], $(ii) \Leftrightarrow (xii) \Leftrightarrow (xiii)$ is, also, due by M. S. Putcha [56]. The equivalences $(ii) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi)$ are from M. Ćirić and S. Bogdanović [26], the conditions (vii) and $(viii)$ are from S. Bogdanović and M. Ćirić [7] and the conditions (x) and (xi) are from T. Tamura [73]. The equivalences $(ii) \Leftrightarrow (xv) \Leftrightarrow (xvi) \Leftrightarrow (xvii) \Leftrightarrow (xviii) \Leftrightarrow (xix)$ were proved by S. Bogdanović, Ž. Popović and M. Ćirić in [22]. For some related results we refer to P. Protić [57]. For some more general results we refer to M. Ćirić and S. Bogdanović [27] and S. Bogdanović, M. Ćirić and Ž. Popović [17].

3.2. **Bands of left Archimedean semigroups.** By Theorem 3 we have that bands of Archimedean semigroups are semilattices of Archimedean semigroups, but the class of bands of left (or right or twosided) Archimedean semigroups is not equal to the class of semilattices of left (or right or twosided) Archimedean semigroups.

Theorem 4. [55] *A semigroup S is a band of left Archimedean semigroups if and only if $xay \overset{l}{\text{---}} xa^2y$, for all $a \in S,\ x, y \in S^1$.*

Theorem 5. [55] *A semigroup S is a band of t-Archimedean semigroups if and only if $xay \overset{t}{\text{---}} xa^2y$, for all $a \in S,\ x, y \in S^1$.*

A band B is *normal* (*left normal*) if it satisfies the identity $axya = ayxa$ $(axy = ayx)$.

Theorem 6. [8] *A semigroup S is a normal band of t-Archimedean semigroups if and only if $ac \xrightarrow{t} abc$, for all $a, b, c \in S$.*

Theorem 7. [8] *The following conditions on a semigroup S are equivalent:*

(i) *S is a left normal band on t-Archimedean semigroups;*

(ii) $(\forall a, b, c \in S)\ ac \xrightarrow{r} abc \wedge a \xrightarrow{l} abc$;

(iii) $(\forall a, b, c \in S)\ ac \xrightarrow{r} abc \wedge b \xrightarrow{l} abc$.

Theorem 8. [1] *The following conditions on a semigroup S are equivalent:*

(i) *S is a band of power joined semigroups;*

(ii) $(\forall a, b \in S)\ ab \overset{p}{\text{---}} a^2b \overset{p}{\text{---}} ab^2$;

(iii) $(\forall a, b \in S)(\forall m, n \in \mathbf{Z}^+)\ ab \overset{p}{\text{---}} a^m b^n$.

Bands of power joined semigroups are studied by T. Nordahl [49] in the medial case $(xaby = xbay)$. For the related results in the periodic case see M. Yamada [75].

A band B is *right zero* (*left zero*) if it satisfies the identity $ax = a$ $(xa = a)$. A band B is *rectangular* if it is isomorphic to a direct product of a left zero and a right zero band.

Theorem 9. [1] *A semigroup S is a rectangular band of power joined semigroups if and only if $abc \overset{p}{—} ac$, for all $a, b, c \in S$.*

Corollary 9.1. [1] *A semigroup S is a left zero band of power joined semigroups if and only if $ab \overset{p}{—} a$, for all $a, b \in S$.*

Theorem 10. [2] *A semigroup S is a band of periodic power joined semigroups if and only if for every $a, b \in S$ and $n \in \mathbf{Z}^+$ there exists $r \in \mathbf{Z}^+$ such that $(ab)^r = (a^n b^n)^r$.*

Lemma 4. [18] *A semigroup S is a union of nil-semigroups if and only if for every $a \in S$ there exists $r \in \mathbf{Z}^+$ such that $a^r = a^{r+1}$.*

Theorem 11. *The following conditions on a semigroup S are equivalent:*

(i) *S is a band of nil-semigroups;*

(ii) *S is a union of nil-semigroups and S is a band of power joined semigroups;*

(iii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^{3n+1} = (a^2b)^{2n+1} = (ab^2)^{2n+1}$.

For the related results see also D. W. Miller [46].

3.3. **Bands of π-groups.** In this part are given general result about decompositions of a semigroup into a band of π-groups.

A subsemigroup K of a semigroup S is a *retract* of S if there exists a homomorphism φ of S onto K such that $\varphi(a) = a$ for all $a \in K$. An ideal extension S of K is a *retract extension* (or *retractive extension*) of K if K is a retract of S.

In a π-regular semigroup S we consider the equivalence relations $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{J}^*$ and $\mathcal{H}^*$ defined by:

$$a\mathcal{L}^*b \Leftrightarrow Sa^p = Sb^q,\ a\mathcal{R}^*b \Leftrightarrow a^pS = b^qS,\ a\mathcal{J}^*b \Leftrightarrow Sa^pS = Sb^qS,\ \mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*,$$

where p, q are the smallest positive integers such that $a^p, b^q \in Reg(S)$ (J. L. Galbiati and M. L. Veronesi [33]). If $e \in E(S)$, then by G_e we denote the maximal subgroup of S with e as its identity and $T_e = \sqrt{G_e}$. On a semigroup S we denote the relation τ by $a\ \tau\ b \iff (\exists e \in E(S))\ a, b \in T_e$. The relation τ is an equivalence on S if and only if S is completely π-regular. A semigroup S is a *π-group* if S is a nil-extension of a group.

Proposition 3.1. [8] *Let S be a band of π-groups and let $Reg(S)$ be a subsemigroup of S. Then $Reg(S)$ is a band of groups and it is a retract of S.*

Conversely, if S contain a retract K which is a band of groups and if $S = \sqrt{K}$, then S is a band of π-groups.

Theorem 12. [11] *The following conditions on a semigroup S are equivalent:*

(i) *S is a band of π-groups;*

(ii) *S is a semilattice of completely Archimedean semigroups and $\mathcal{H}^*$ is a congruence on S;*

(iii) *S is π-regular and $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in a^2bSab^2$;*

(iv) *S is completely π-regular and $ab\ \tau\ a^2b\ \tau\ ab^2$;*

(v) *S is a band of t-Archimedean semigroups and S is completely π-regular;*

(vi) *S is completely π-regular and $xay \overset{t}{—} xa^2y$, for all $a \in S$, $x, y \in S^1$;*

(vii) *S is completely π-regular with $(xy)^0 = (x^2y)^0 = (xy^2)^0$.*

The equivalence $(i) \Leftrightarrow (ii)$ is from J. L. Galbiati and M. L. Veronesi [32]. The condition (iv) is given by B. Madison, T. K. Mukherjee and M. K. Sen [43], see also [44]. The conditions (v) and (vi) are from M. S. Putcha [54], for (vi) see also [55]. The condition (iii) is from S. Bogdanović and M. Ćirić [8].

A band S is *left regular* if $ax = axa$ for all $a, x \in S$.

Theorem 13. [8] *The following conditions on a semigroup S are equivalent:*

(i) *S is a left regular band of π-groups;*

(ii) *S is completely π-regular and for all $a, b \in S$, $ab \ \tau \ a^2b \ \tau \ aba$;*

(iii) *S is π-regular and $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ (ab)^n \in a^2bSa$;*

(iv) *S is completely π-regular with $(xy)^0 = (x^2y)^0 = (xyx)^0$.*

Theorem 14. [8] *The following conditions on a semigroup S are equivalent:*

(i) *S is a normal band of π-groups;*

(ii) *S is completely π-regular and for all $a, b, c, d \in S$, $abcd \ \tau \ acbd$;*

(iii) *S is π-regular and $(\forall a, b, c \in S)(\exists n \in \mathbf{Z}^+)\ (abc)^n \in acSac$;*

(iv) *S is completely π-regular with $(xyzu)^0 = (xzyu)^0$.*

Theorem 15. [8] *The following conditions on a semigroup S are equivalent:*

(i) *S is a left normal band of π-groups;*

(ii) *S is completely π-regular and for all $a, b, c \in S$, $abc \ \tau \ acb$;*

(iii) *S is π-regular and $(\forall a, b, c \in S)(\exists n \in \mathbf{Z}^+)\ (abc)^n \in acSa$;*

(iv) *S is completely π-regular with $(xyz)^0 = (xzy)^0$.*

Theorem 16. [12] *A semigroup S is a band of π-groups and $E^2(S) = E(S)$ if and only if S is completely π-regular and $(xy)^0 = x^0y^0$.*

3.4. **Rédei's bands of π-groups.** Here are described decompositions of a semigroup into a Rédei's band of π-groups.

In connection with a study of a lattice of subsemigroups of some semigroup the important place is captured by $\mathcal{U}$-semigroups. A semigroup S is a *$\mathcal{U}$-semigroup* if the union of every two subsemigroups of S is a subsemigroup of S, which is equivalent with $xy \in \langle x \rangle \cup \langle y \rangle$ for all $x, y \in S$. A more detailed description can be find in M. Petrich [51]. These semigroups have been considered more recently, predominantly in special cases. Here we present some general results of Rédei's bands of π-groups. Several special cases of this the reader can find in E. G. Shutov [71], N. Kimura, T. Tamura and R. Merkel [39], E. S. Lyapin and A. E. Evseev [42], A. E. Evseev [31], B. Trpenovski [63], S. Bogdanović, P. Kržovski, P. Protić and B. Trpenovski [18], J. Pelikán [50], B. Pondeliček [52], L. Rédei [62], S. Bogdanović and B. Stamenković [23], B. Trpenovski and N. Celakoski [64], S. Bogdanović and M. Ćirić [5] and M. Ćirić and S. Bogdanović [25].

A semigroup S is a *Rédei's band* if $xy = x$ or $xy = y$ for all $x, y \in S$, [62].

Theorem 17. [8] *The following conditions on a semigroup S are equivalent:*

(i) *S is a Rédei's band of π-groups;*

(ii) *S has a retract K which is a Rédei's band of groups and $\sqrt{K} = S$;*

(iii) $(\forall a, b \in S)(\exists n \in \mathbf{Z}^+)\ a^n \in (ab)^n S (ab)^n \ \vee\ b^n \in (ab)^n S (ab)^n$.

Corollary 17.1. [8] *A semigroup S is a Rédei's band of groups if and only if*

$$(\forall a, b \in S)\ a \in abSab \ \vee\ b \in abSab.$$

Let $n \in \mathbf{Z}^+$. A semigroup S is a *generalized $\mathcal{U}_{n+1}$-semigroup* or simply *$\mathcal{GU}_{n+1}$-semigroup* if S satisfies the following condition:

$$(\forall x_1, x_2, \ldots, x_{n+1})(\exists m)\ (x_1 x_2 \cdots x_{n+1})^m \in \langle x_1 \rangle \cup \langle x_2 \rangle \cup \ldots \cup \langle x_{n+1} \rangle.$$

A $\mathcal{GU}_2$-semigroup we call $\mathcal{GU}$-semigroup. A chain Y of semigroups $S_\alpha,\ \alpha \in Y$, is a *$\mathcal{GU}_{n+1}$-chain of semigroups* if for all $\alpha_1, \alpha_2, \ldots \alpha_{n+1} \in Y$ such that $\alpha_i \neq \alpha_j$ for some $i, j \in \{1, 2, \ldots, n+1\}$, and for all $x_k \in S_{\alpha_k},\ k \in \{1, 2, \ldots, n+1\}$ there exists $m \in \mathbf{Z}^+$ such that $(x_1 x_2 \cdots x_{n+1})^m \in \langle x_1 \rangle \cup \langle x_2 \rangle \cup \ldots \cup \langle x_{n+1} \rangle$, [3, 25].

Theorem 18. [25] *The following conditions on a semigroup S are equivalent:*

(i) *S is a Rédei's band of periodic π-groups;*

(ii) *S is a π-regular $\mathcal{GU}_{n+1}$-semigroup;*

(iii) *S is a periodic $\mathcal{GU}_{n+1}$-semigroup;*

(iv) *S is a $\mathcal{GU}_{n+1}$-chain of retractive nil-extensions of periodic left and right groups;*

(v) *S is a π-regular $\mathcal{GU}$-semigroup;*

(vi) *S is a periodic $\mathcal{GU}$-semigroup;*

(vii) *S contains a retract K which is a regular $\mathcal{GU}$-semigroup and $\sqrt{K} = S$.*

Theorem 19. [25] *The following conditions on a semigroup S are equivalent:*

(i) *S is a left zero band of periodic π-groups;*

(ii) *S is a π-regular $\mathcal{GU}$-semigroup and $E(S)$ is a left zero band;*

(iii) *S is a retractive nil-extension of a periodic left group.*

A semigroup S is a *$\mathcal{U}_{n+1}$-semigroup* if

$$(\forall x_1, x_2, \ldots, x_{n+1} \in S)\ x_1 x_2 \cdots x_n \in \langle x_1 \rangle \cup \langle x_2 \rangle \cup \cdots \cup \langle x_{n+1} \rangle,$$

$n \in \mathbf{Z}^+$. A band I of semigroups $S_i,\ i \in I$, is a *$\mathcal{U}_{n+1}$-band of semigroups* if $x_1 x_2 \cdots x_{n+1} \in \langle x_1 \rangle \cup \langle x_2 \rangle \cup \cdots \cup \langle x_{n+1} \rangle$, for all $x_1 \in S_{i_1},\ x_2 \in S_{i_2}, \ldots, x_{n+1} \in S_{i_{n+1}}$, such that $i_k \neq i_l$ for some $k, l \in \{1, 2, \ldots, n\}$. One defines analogously *$\mathcal{U}_{n+1}$-semilattice* and *$\mathcal{U}_{n+1}$-chain of semigroups*.

Theorem 20. [6] *The following conditions on a semigroup S are equivalent:*

(i) *S is a $\mathcal{U}_{n+1}$-semigroup;*

(ii) *S is a $\mathcal{U}_{n+1}$-chain of retract extensions of $\mathcal{U}$-groups and singular bands by $\mathcal{U}_{n+1}$-nil-semigroups;*

(iii) *S is a $\mathcal{U}_{n+1}$-bands of ideal extensions of $\mathcal{U}$-groups by $\mathcal{U}_{n+1}$-nil-semigroups.*

Theorem 21. [11] *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a completely simple semigroup;*

(ii) *S is a rectangular band of π-groups;*

(iii) *S is completely Archimedean and for all $a \in S$, $x, y \in S^1$ there exists $p, q, r, s \in \mathbf{Z}^+$ such that*

$$(xay)^p S = (xa^2y)^p S \quad \text{and} \quad S(xay)^r = S(xa^2y)^s;$$

(iv) *for all $a, b, c \in S$ there exists $n \in \mathbf{Z}^+$ such that*

$$a^n c \in ea^n bSa^n cf \quad \text{and} \quad ca^n \in fca^n bSa^n e,$$

where $a \in \sqrt{G_e}$, $c \in \sqrt{G_f}$;

(v) *S is an Archimedean semigroup with an idempotent and for every $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in a^2Sb^2$;*

(vi) *S is a subdirect product of a completely simple semigroup and a nil-semigroup.*

The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ are proved by J. L. Galbiati and M. L. Veronesi [32]. The condition (vi) is given by L. N. Shevrin [66]. The condition (iv) is from S. Bogdanović [4] and (v) is from S. Bogdanović and M. Ćirić [9].

Corollary 21.1. *A semigroup S is a retractive nil-extension of a periodic completely simple semigroup if and only if S is a rectangular band of nil-extensions of periodic groups.*

Corollary 21.2. *A semigroup S is a retractive nil-extension of a rectangular band if and only if S is a rectangular band of nil-semigroups.*

Rectangular bands of groups were studied by G. Čupona in [30].

Theorem 22. [11] *The following conditions on a semigroup S are equivalent:*

(i) *S is a retractive nil-extension of a left group;*

(ii) *S is a left zero band of π-groups;*

(iii) *S is left Archimedean with an idempotent and for all $a \in S$, $x, y \in S^1$ there exists $p, q, r, s \in \mathbf{Z}^+$ such that*

$$(xay)^p S = (xa^2y)^q S \quad \text{and} \quad S(xay)^r = S(xa^2y)^s;$$

(iv) *S is completely π-regular and for all $a, b, c \in S$ there exists $n \in \mathbf{Z}^+$ such that $ca^n \in gca^nSa^nf$, where $f, g \in E(S)$ such that $b \in \sqrt{G_f}$, $c \in \sqrt{G_g}$;*

(v) *S is Archimedean π-regular and for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $(ab)^n \in a^2Sa$.*

The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ are from J. L. Galbiati and M. L. Veronesi [32]. The condition (iv) is from S. Bogdanović [4] and (v) is from S. Bogdanović and M. Ćirić [9].

4. Bands of k-Archimedean semigroups

In this section, using radicals of some Green's relations and its properties, are described the structure of bands of $\mathcal{J}_k$-simple ($\mathcal{L}_k$-simple, $\mathcal{H}_k$-simple) semigroups. Some general characterizations of these bands are given by S. Bogdanović, Ž. Popović and M. Ćirić in [19].

Let ϱ be an arbitrary relation on a semigroup S. Recal that the *radicals* $R(\varrho)$ and $T(\varrho)$ of ϱ are relations on S defined by:

$$(a,b)\in R(\varrho)\Leftrightarrow(\exists p,q\in\mathbf{Z}^+)\ (a^p,b^q)\in\varrho,$$

and

$$(a,b)\in T(\varrho)\Leftrightarrow(\exists p\in\mathbf{Z}^+)\ (a^p,b^p)\in\varrho.$$

The radical $R(\varrho)$ was introduced by L. N. Shevrin in [68] and the radical $T(\varrho)$ was introduced by S. Bogdanović and M. Ćirić in [15]. The radical ϱ_k, $k\in\mathbf{Z}^+$ of ϱ is a relation on S defined by

$$(a,b)\in\varrho_k\Leftrightarrow(a^k,b^k)\in\varrho.$$

The radical ϱ_k was introduced by S. Bogdanović, Ž. Popović and M. Ćirić in [19]. It is clear that

$$\varrho_k\subseteq T(\varrho)\subseteq R(\varrho).$$

If $\varrho\in\{\mathcal{J},\mathcal{L},\mathcal{R},\mathcal{D},\mathcal{H}\}$, then it is easy to see that ϱ_k, $k\in\mathbf{Z}^+$ is an equivalence relation. So, in this case these equivalences are very similar to Green's equivalences and they can be consider as its generalizations. The conditions under which the relations $R(\varrho)$ and $T(\varrho)$ are transitive (i.e. are equivalences) have been discussed by L. N. Shevrin in [68] by S. Bogdanović and M. Ćirić in [14] and [15], and by S. Bogdanović, M. Ćirić and Ž. Popović in [17].

Let $k\in\mathbf{Z}^+$ be a fixed integer. A semigroup S is: *k-regular* if $(\forall a\in S)\ a^k\in a^kSa^k$, *left k-regular* if $(\forall a\in S)\ a^k\in Sa^{k+1}$, *right k-regular* if $(\forall a\in S)\ a^k\in a^{k+1}S$, *completely k-regular* if $(\forall a\in S)\ a^k\in a^{k+1}Sa^{k+1}$, *intra k-regular* if $(\forall a\in S)\ a^k\in Sa^{2k}S$, *$k$-Archimedean* if $(\forall a,b\in S)\ a^k\in S^1bS^1$, *left k-Archimedean* if $(\forall a,b\in S)\ a^k\in S^1b$, *right k-Archimedean* if $(\forall a,b\in S)\ a^k\in bS^1$ and *t-k-Archimedean* if $(\forall a,b\in S)\ a^k\in bS^1\cap S^1b$. k-regular semigroups were introduced by K. S. Harinath in [35]. The other types of semigroups were introduce for the first time by S. Bogdanović, Ž. Popović and M. Ćirić in [19].

4.1. **Bands of $\mathcal{J}_k$-simple semigroups.** Let $k\in\mathbf{Z}^+$. Let $\mathcal{J}$ be a Green's relation on a semigroup S. On S we define the following relations by

$$(a,b)\in\mathcal{J}_k\ \Leftrightarrow\ (a^k,b^k)\in\mathcal{J};$$

$$(a,b)\in\mathcal{J}_k^\flat\ \Leftrightarrow\ (\forall x,y\in S^1)\ (xay,xby)\in\mathcal{J}_k.$$

It is easy to verify that $\mathcal{J}_k$ is an equivalence relation on a semigroup S. But $R(\mathcal{J})$ and $T(\mathcal{J})$ are not equivalences (see [68]).

A semigroup S is *$\mathcal{J}_k$-simple* if

$$(\forall a,b\in S)\ (a,b)\in\mathcal{J}_k.$$

It is clear that a semigroup S is $\mathcal{J}_k$-simple if and only if S is k-Archimedean. In the further our consideration there is not distinction between these notions.

Lemma 5. [19] *Let S be a semigroup and let $k \in \mathbf{Z}^+$. If S is a rectangular band of k-Archimedean semigroups, then S is k-Archimedean.*

By the following result we describe the structure of a semigroup which can be decompose into a band (semilattice) of $\mathcal{J}_k$-simple semigroups.

Theorem 23. [19] *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *$\mathcal{J}_k$ is a band congruence;*
(ii) *$\mathcal{J}_k = \mathcal{J}_k^\flat = R(\mathcal{J}_k)$;*
(iii) *S is a band of k-Archimedean semigroups;*
(iv) *S is a semilattice of k-Archimedean semigroups;*
(v) *$\mathcal{J}_k^\flat$ is a band congruence;*
(vi) $(\forall a \in S)(\forall x, y \in S^1)$ *$xay \,\mathcal{J}_k\, xa^2y$;*
(vii) *$\mathcal{J}_k^\flat = R(\mathcal{J}_k^\flat)$;*
(viii) *S is a semilattice of Archimedan semigroups and intra k-regular semigroup.*

4.2. **Bands of $\mathcal{L}_k$-simple semigroups.** Let $k \in \mathbf{Z}^+$. Let $\mathcal{L}$ be a Green's relation on a semigroup S. On S we define the following relations by

$$(a, b) \in \mathcal{L}_k \;\Leftrightarrow\; (a^k, b^k) \in \mathcal{L};$$

$$(a, b) \in \mathcal{L}_k^\flat \;\Leftrightarrow\; (\forall x, y \in S^1)\ (xay, xby) \in \mathcal{L}_k.$$

It is easy to verify that $\mathcal{L}_k$ is an equivalence relation on a semigroup S.

A semigroup S is $\mathcal{L}_k$-*simple* or left k-Archimedean, if $a \,\mathcal{L}_k\, b$, for all $a, b \in S$. It is clear that a $\mathcal{L}_k$-simple semigroup is left π-regular and left Archimedean.

Lemma 6. [19] *Let S be a semigroup and let $k \in \mathbf{Z}^+$. If S is a band of left Archimedean semigroups and left k-regular, then S is a band of left k-Archimedean semigroups.*

By the following theorem we describe the structure of a semigroup which can be decompose into a band of $\mathcal{L}_k$-simple semigroups. Also, here should be to emphasize that a band of left k-Archimedean semigroups is not coincident with a semilattice of left k-Archimedean semigroups.

Theorem 24. [19] *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is a band of left k-Archimedean semigroups;*
(ii) $(\forall a, b \in S)$ $(\,ab \,\mathcal{L}_k\, ab^2 \;\wedge\; a \,\mathcal{L}_k\, a^2)$*;*
(iii) *$\mathcal{L}_k^\flat$ is a band congruence on S;*
(iv) $(\forall a \in S)(\forall x, y \in S^1)$ *$xay \,\mathcal{L}_k\, xa^2y$;*
(v) *$R(\mathcal{L}_k^\flat) = \mathcal{L}_k^\flat$;*
(vi) *S is a band of left Archimedean semigroups and left k-regular.*

Theorem 25. [19] *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $\mathcal{L}_k$ *is a band congruence on* S*;*
(ii) $\mathcal{L}_k = \mathcal{L}_k^\flat = R(\mathcal{L}_k)$*;*
(iii) $R(\mathcal{L}_k) = \mathcal{L}_k$ *and* $\mathcal{L}_k$ *is a congruence on* S*.*

Proposition 4.1. [19] *Let $k \in \mathbf{Z}^+$. If $\mathcal{L}_k$ is a band congruence on a semigroup S, then S is a band of left k-Archimedean semigroups.*

4.3. **Bands of $\mathcal{H}_k$-simple semigroups.** Let $k \in \mathbf{Z}^+$. Let $\mathcal{H}$ be a Green's relation on a semigroup S. On S we define the following relations by

$$(a,b) \in \mathcal{H}_k \;\Leftrightarrow\; (a^k, b^k) \in \mathcal{H};$$

$$(a,b) \in \mathcal{H}_k^\flat \;\Leftrightarrow\; (\forall x, y \in S^1)\ (xay, xby) \in \mathcal{H}_k.$$

It is easy to verify that $\mathcal{H}_k$ is an equivalence relation on a semigroup S. Also, it is evident that $\mathcal{H}_k = \mathcal{L}_k \cap \mathcal{R}_k$.

A semigroup S is $\mathcal{H}_k$-*simple* (t-k-Archimedean), if $a\,\mathcal{H}_k\,b$, for all $a, b \in S$. Also, it is easy to verify that a semigroup S is $\mathcal{H}_k$-simple if it is both $\mathcal{L}_k$-simple and $\mathcal{R}_k$-simple, and conversely.

By the following theorem we describe the structure of a semigroup which can be decompose into a band of $\mathcal{H}_k$-simple semigroups.

Theorem 26. [19] *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is a band of t-k-Archimedean semigroups;*
(ii) $\mathcal{H}_k^\flat$ *is a band congruence on* S*;*
(iii) $(\forall a \in S)(\forall x, y \in S^1)\ xay\,\mathcal{H}_k\,xa^2y$*;*
(iv) $R(\mathcal{H}_k^\flat) = \mathcal{H}_k^\flat$*;*
(v) *S is a band of t-Archimedean semigroups and completely k-regular.*

Theorem 27. [19] *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $\mathcal{H}_k$ *is a band congruence on* S*;*
(ii) $\mathcal{H}_k = \mathcal{H}_k^\flat = R(\mathcal{H}_k)$*;*
(iii) $R(\mathcal{H}_k) = \mathcal{H}_k$ *and* $\mathcal{H}_k$ *is a congruence on* S*.*

5. Bands of λ-simple semigroups

In this section are presented some results concerning decompositions into a band of λ-simple semigroups in the general and some special cases. Some new characterizations of bands of λ-simple semigroups are given by S. Bogdanović, Ž. Popović and M. Ćirić in [20].

Let ϱ be an arbitrary binary relation on a semigroup S. The intersection of all transitive relations on S containing ϱ is a transitive relation on S, denoted by ϱ^∞. It is easy to prove that $\varrho^\infty = \cup_{n \in \mathbf{Z}^+} \varrho^n$. The relation ϱ^∞ we call the *transitive closure* of ϱ.

For an element a of a semigroup S we introduce the following notation

$$\Sigma(a) = \{x \in S \mid a \longrightarrow^{\infty} x\}, \qquad \Lambda(a) = \{x \in S \mid a \xrightarrow{l}{}^{\infty} x\},$$

$$\Lambda_n(a) = \{x \in S \mid a \xrightarrow{l}{}^{n} x\}.$$

On a semigroup S we define the following equivalences by

$$a \,\sigma\, b \Leftrightarrow \Sigma(a) = \Sigma(b), \qquad a \,\lambda\, b \Leftrightarrow \Lambda(a) = \Lambda(b),$$

$$a \,\lambda_n\, b \Leftrightarrow \Lambda_n(a) = \Lambda_n(b).$$

In [27] is proved that the relation σ is the greatest semilattice congruence on a semigroup, λ is an equivalence and it is a generalization of the well-known Green's equivalence $\mathcal{L}$.

A semigroup S is λ-*simple* (σ-*simple*, λ_n-*simple*) if $a \,\lambda\, b$ ($a \,\sigma\, b$, $a \,\lambda_n\, b$), for all $a, b \in S$. We denote by Λ the class of all λ-simple semigroups.

By $\mathcal{LZ}$ ($\mathcal{RZ}$, $\mathcal{LN}$, $\mathcal{RN}$) we denote the variety of left zero (right zero, left normal, right normal) bands.

Lemma 7. [20] *Let S be a semigroup. Then*

$$\Lambda = \Lambda \circ \mathcal{LZ}.$$

The lattice **LVB** of all varieties of bands was studied by P. A. Birjukov, C. F. Fennemore, J. A. Gerhard, M. Petrich and others. Here we use the characterization of **LVB** given by J. A. Gerhard and M. Petrich in [34]. They defined inductively three systems of words as follows:

$$\begin{array}{lll} G_2 = x_2 x_1, & H_2 = x_2, & I_2 = x_2 x_1 x_2, \\ G_n = x_n \overline{G}_{n-1}, & H_n = x_n \overline{G}_{n-1} x_n \overline{H}_{n-1}, & I_n = x_n \overline{G}_{n-1} x_n \overline{I}_{n-1}, \end{array}$$

(for $n \geq 3$), and they shown that the lattice **LVB** can be represented by the graph given in Figure 1.

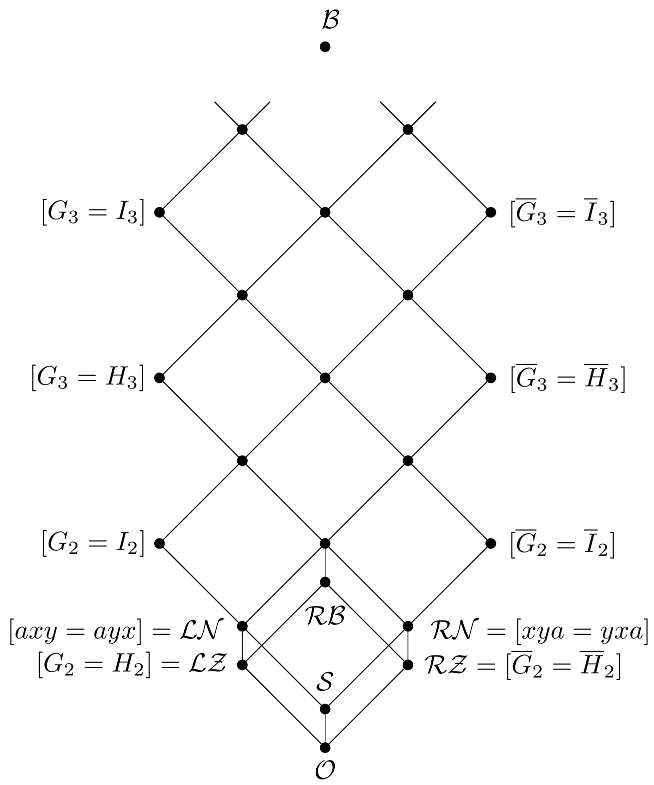

Figure 1.

Theorem 28. [16] *Let $\mathcal{V}$ be an arbitrary variety of bands. Then*

$$\mathcal{LZ} \circ \mathcal{V} = \begin{cases} \mathcal{LZ}, & \text{if } \mathcal{V} \in [\mathcal{O}, \mathcal{LZ}]; \\ \mathcal{RB}, & \text{if } \mathcal{V} \in [\mathcal{RZ}, \mathcal{RB}]; \\ [G_2 = I_2], & \text{if } \mathcal{V} \in [\mathcal{S}, [G_2 = I_2]]; \\ [G_3 = I_3], & \text{if } \mathcal{V} \in [\mathcal{RN}, [G_3 = H_3]]; \\ [G_{n+1} = I_{n+1}], & \text{if } \mathcal{V} \in \big[[\overline{G}_n = \overline{I}_n], [G_{n+1} = I_{n+1}]\big], n \geq 2; \\ [G_{n+1} = H_{n+1}], & \text{if } \mathcal{V} \in \big[[\overline{G}_n = \overline{H}_n], [G_{n+1} = H_{n+1}]\big], n \geq 3. \end{cases}$$

Our next goal is to characterize semigroups from $\Lambda \circ \mathcal{V}$, for an arbitrary variety of bands $\mathcal{V}$.

Theorem 29. [20] *Let $\mathcal{V}$ be an arbitrary variety of bands. Then*

$$\Lambda \circ \mathcal{V} = \begin{cases} \Lambda, & \text{if } \mathcal{V} \in [\mathcal{O}, \mathcal{LZ}]; \\ \Lambda \circ \mathcal{RZ}, & \text{if } \mathcal{V} \in [\mathcal{RZ}, \mathcal{RB}]; \\ \Lambda \circ \mathcal{S}, & \text{if } \mathcal{V} \in [\mathcal{S}, [G_2 = I_2]]; \\ \Lambda \circ \mathcal{RN}, & \text{if } \mathcal{V} \in [\mathcal{RN}, [G_3 = H_3]]; \\ \Lambda \circ \big[\overline{G}_n = \overline{I}_n\big], & \text{if } \mathcal{V} \in \big[[\overline{G}_n = \overline{I}_n], [G_{n+1} = I_{n+1}]\big], n \geq 2; \\ \Lambda \circ \big[\overline{G}_n = H_n\big], & \text{if } \mathcal{V} \in \big[[\overline{G}_n = \overline{H}_n], [G_{n+1} = H_{n+1}]\big], n \geq 3. \end{cases}$$

6. Identities and bands of π-groups

In this section we study identities which induce decompositions into a band of nil-extensions of groups.

Also, here we will use several semigroups given by the following presentations:

$$B_2 = \langle a, b \mid a^2 = b^2 = 0,\ aba = a,\ bab = b\rangle,$$
$$A_2 = \langle a, e \mid a^2 = 0,\ e^2 = e,\ aea = a,\ eae = e\rangle,$$
$$N_m = \langle a \mid a^{m+1} = a^{m+2},\ a^m \neq a^{m+1}\rangle,$$
$$L_{3,1} = \langle a, f \mid a^2 = a^3,\ f^2 = f,\ a^2 f = a^2,\ fa = f\rangle,$$
$$LZ(n) = \langle a, e \mid a^{n+1} = a,\ e^2 = e,\ ea = a^n e = e\rangle,$$

$m, n \in \mathbf{Z}^+,\ n \geq 2$, and $R_{3,1}$ ($RZ(n)$) will be the dual semigroup of $L_{3,1}$ ($LZ(n)$). Semigroups B_2 and A_2 are not semilattices of Archimedean semigroups. $L_{3,1}$ is a nil-extension of a union of groups, but it is not a retractive nil-extension of a union of groups. The semigroup $LZ(n)$ has $2n$ elements, it is a chain of the cyclic group $\langle a\rangle = \{a, a^2, \ldots, a^n\}$ and the left zero band $\{e, ae, \ldots, a^{n-1}e\}$, it is a union of groups and it is not a band of groups.

By A^+ we denote the *free semigroup* over an alphabet A and by A^* we denote the *free monoid* over an alphabet A. By $|u|$ we denote the *length* of a word $u \in A^+$ and by $|x|_u$ we denote the number of appearances of the letter x in u. A word $v \in A^+$ is a *subword* (*left cut, right cut*) of a word $u \in A^+$ if $v \,|\, u$ ($v \,|_l\, u,\ v \,|_r\, u$). If $u \in A^+, |u| \geq 2$, then by $h^{(2)}(u)$ ($t^{(2)}(u)$) we denote the left (right) cut of u of the length 2. By $h(u)$ ($t(u)$) we denote the *head* (*tail*), by $i(u)$ ($f(u)$) we denote the *initial* (*final*) *part*, by $l(u)$ ($r(u)$) we denote the *left* (*right*) *part*, by $\overline{u}$ we denote the *mirror image* and by $c(u)$ we denote the *content* of the word u [51]. For a word $u \in A^+$ by $\Pi(u)$ we denote the set $\Pi(u) = \{\ x \in A \mid |x|_u = 1\ \}$, by $u = u(x_1, \ldots, x_n)$ we denote that $c(u) = \{x_1, \ldots, x_n\}$.

By $[u = v]$ we denote the *variety* determined by the identity $u = v$. Identities $u = v$ and $u' = v'$ over an alphabet A_n^+ are *p-equivalent* if $u' = v'$ can be obtained from $u = v$ by some permutation of letters. It is clear that p-equivalent identities determines the same variety. If $\mathcal{X}$ is a class of semigroups, then $u = v$ is an $\mathcal{X}$-*identity* if $[u = v] \subseteq \mathcal{X}$. If $\mathcal{X}_1$ and $\mathcal{X}_2$ are classes of semigroups, then $u = v$ is a $\mathcal{X}_1 \triangleright \mathcal{X}_2$-*identity* if $[u = v] \cap \mathcal{X}_1 \subseteq \mathcal{X}_2$.

An identity $u = v$ is *homotype* if $c(u) = c(v)$ and it is *heterotype* if $c(u) \neq c(v)$. The homotype identities will be considered firstly, i.e. the identity of the form

$$u(x_1, x_2, \ldots, x_n) = v(x_1, x_2, \ldots, x_n). \tag{6.1}$$

For an identity (6.1), by p_i we denote the number

$$p_i = ||x_i|_u - |x_i|_v|\ ,$$

$i \in \{1, 2, \ldots, n\}$. An identity (6.1) is *periodic* if $p_i \neq 0$ for some $i \in \{1, 2, \ldots, n\}$. In such a case the number $p = \gcd(p_1, \ldots, p_n)$ is the *period* of the identity (6.1). Otherwise we say that (6.1) is *aperiodic* and that it is of the period $p = 0$, [26].

Periodic identities have been considered with various names, but, following the sense of the Proposition 6.1, we will use the previous name.

Proposition 6.1. [26] *The following conditions on an identity* (6.1) *are equivalent:*

(i) $[u=v]$ *consists of* π*-regular semigroups;*

(ii) $[u=v]$ *consists of completely* π*-regular semigroups;*

(iii) $[u=v]$ *consists of periodic semigroups;*

(iv) (6.1) *is a periodic identity.*

Let (6.1) be an identity for which

$$i(u) = i(v) = x_{\pi(1)}x_{\pi(2)}\dots x_{\pi(n)}$$

for some permutation π of a set $\{1,2,\dots,n\}$. For $k \in \{1,2,\dots,n-1\}$ by u_k (v_k) we denote the left cut of u (v) of the greatest length which contains exactly k letters (i.e. which not contains the letter $x_{\pi(k+1)}$). It is clear that

$$u_k = u_k(x_{\pi(1)},\dots,x_{\pi(k)})\ , \quad v_k = v_k(x_{\pi(1)},\dots,x_{\pi(k)})\ .$$

For $k \in \{1,2,\dots,n-1\}$ and $i \in \{1,\dots,k\}$ we will use the notation

$$l_{k,i} = ||x_{\pi(i)}|_{u_k} - |x_{\pi(i)}|_{v_k}|\ .$$

An identity (6.1) is an identity *with left distortion* if $i(u) \neq i(v)$. Otherwise, (6.1) is *without left distortion*. Similarly we define identities *with* (*without*) *right distortion*, [26].

We define the *left characteristic* l of an identity (6.1) in the following way:

(*i*) $l = 1$, if (6.1) is an identity with left distortion;

(*ii*) l is the greatest common divisor of integers

$$p,\ l_{k,i},\ 1 \le k \le n-1,\ 1 \le i \le k,$$

if (6.1) is without left distortion and some of integers p and $l_{k,i}$ is different to 0;

(*iii*) $l = 0$, if (6.1) is without left distortion and all of numbers p and $l_{k,i}$ are equal to 0, [26].

By *right characteristic* of an identity (6.1) we mean the left characteristic of the identity $\overline{u} = \overline{v}$.

By $\pi\mathcal{R}$ ($\mathcal{G}$, $\mathcal{N}$, $\mathcal{CS}$, $\mathcal{UG}$) we denote the class of π-regular semigroups (groups, nil-semigroups, completely simple semigroups, union of groups).

Theorem 30. [26] *The following conditions for an identity* (6.1) *are equivalent:*

(i) (6.1) *is a* $\pi\mathcal{R} \triangleright (\mathcal{G} \circ \mathcal{N}) \circ \mathcal{B}$*-identity;*

(ii) (6.1) *is not satisfied in semigroups* $B_2,\ L_{3,1},\ R_{3,1},\ LZ(d)$ *and* $RZ(d)$ *for* $d \in \mathbf{Z}^+,\ d \ge 2$*;*

(iii) (6.1) *is a* $\pi\mathcal{R} \triangleright (\mathcal{CS} \circledast \mathcal{N}) \circ \mathcal{S}$*-identity of the left and the right characteristic equal to* 1*.*

Using existing classifications of bands (see, for example, [51]), descriptions of identities which induce decompositions of π-regular semigroups into some special types of bands of π-groups can be obtained.

Theorem 31. [26] *The following conditions for an identity* (6.1) *are equivalent:*

(i) (6.1) *is a* $\mathcal{UG} \triangleright \mathcal{G} \circ \mathcal{B}$*-identity;*

(ii) (6.1) *is not satisfied in semigroups* $LZ(d)$ *and* $RZ(d)$ *for* $d \in \mathbf{Z}^+,\ d \geq 2$*;*

(iii) (6.1) *is an identity of left and right characteristic equal to* 1.

The description of band of t-Archimedeans identities in general case is an open problem. Identities of this type are considered only by M. S. Putcha [55], where M. S. Putcha proved that the identity $(xy)^2 = x^2y^2$ is a band of t-Archimedean identity.

The identities of the form $(xy)^n = x^n y^n,\ n \in \mathbf{Z}^+,\ n \geq 2$, are considered many a times. By Theorem 30 we obtain that this identity is a $\pi\mathcal{R} \triangleright (\mathcal{G} \circ \mathcal{N}) \circ \mathcal{B}$- identity if and only if $n = 2$.

For connections with semigroup varieties we refer to V. V. Rasin [61] and L. N. Shevrin i E. V. Suhanov [70].

7. Bands of η-simple semigroups

In this section, on an arbitrary semigroup, are defined a few different types of relations and its congruence extensions. Also, here are described the structure of semigroups in which these relations are band congruences. The components of such obtained band decompositions are in some sense simple semigroups.

First we give the following lemma which is the helpful result for the further work.

Lemma 8. [21] *Let* ξ *be a congruence relation on a semigroup* S*. Then* $R(\xi) = \xi$ *if and only if* ξ *is a band congruence on* S*.*

7.1. **The η relations.** On a semigroup S we define the following relations:

$$(a, b) \in \eta \ \Leftrightarrow\ (\exists i, j \in \mathbf{Z}^+)\ a^i = b^j,$$

$$(a, b) \in \eta^\flat \ \Leftrightarrow\ (\forall x, y \in S^1)\ (xay, xby) \in \eta.$$

It is easy to verify that η is an equivalence relation on a semigroup S.

A semigroup S is η-*simple* if

$$(\forall a, b \in S)\ (a, b) \in \eta.$$

These semigroups are well-known in the literature as power joined semigroups.

The important result is the following lemma.

Lemma 9. [21] *If* ξ *is a band congruence on a semigroup* S*, then* $\xi \subseteq \eta$ *if and only if every* ξ*-class of* S *is an* η*-simple semigroup.*

By the following theorem we describe the structure of semigroups in which the relation η is a congruence relation. These semigroups in a different way have been treated by S. Bogdanović in [1].

Theorem 32. [21] *The following conditions on a semigroup* S *are equivalent:*

(i) S *is a band of* η*-simple semigroups;*

(ii) η *is a* (*band*) *congruence on* S*;*

(iii) $\eta^\flat$ *is a band congruence on* S*;*

(iv) $(\forall a \in S)(\forall x, y \in S^1)\ xay\,\eta\,xa^2y$*;*

(v) $R(\eta^\flat) = \eta^\flat$.

Let $m, n \in \mathbf{Z}^+$. On a semigroup S we define a relation $\overline{\eta}_{(m,n)}$ by

$$(a,b) \in \overline{\eta}_{(m,n)} \;\Leftrightarrow\; (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \eta.$$

If instead of η we assume the equality relation, then we obtain the relation which discussed by S. J. L. Kopamu in [40] and [41]. The main characteristic of previous defined relation gives the following theorem.

Theorem 33. [21] *Let S be a semigroup and let $m, n \in \mathbf{Z}^+$. Then $\overline{\eta}_{(m,n)}$ is a congruence relation on S.*

Remark 7.1: Let μ be an equivalence relation on a semigroup S and let $m, n \in \mathbf{Z}^+$. Then a relation $\overline{\mu}_{(m,n)}$ defined on S by

$$(a,b) \in \overline{\mu}_{(m,n)} \;\Leftrightarrow\; (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \mu$$

is a congruence relation on S. But, there exists a relation μ which is not equivalence, for example $\mu = \longrightarrow$, for which the relation $\overline{\mu}_{(m,n)}$ is a congruence on S.

The complete description of $\overline{\mu}_{(m,n)}$ congruence, for $\mu = \longrightarrow$, was given by S. Bogdanović, Ž. Popović and M. Ćirić in [22].

Theorem 34. [21] *Let $m, n \in \mathbf{Z}^+$. The following conditions on a semigroup S are equivalent:*

(i) *$\overline{\eta}_{(m,n)}$ is a band congruence on S;*
(ii) *$(\forall x \in S^m)(\forall y \in S^n)(\forall a \in S)\ xay\,\eta\,xa^2y$;*
(iii) *$\eta \subseteq \overline{\eta}_{(m,n)}$;*
(iv) *$R(\overline{\eta}_{(m,n)}) = \overline{\eta}_{(m,n)}$.*

Proposition 7.1. *Let $m, n \in \mathbf{Z}^+$. If $\overline{\eta}_{(m,n)}$ is a band congruence on a semigroup S, then S is a band of $\overline{\eta}_{(m,n)}$-simple semigroups.*

7.2. **The η_k relations.** Let $k \in \mathbf{Z}^+$ be a fixed integer. On a semigroup S we define the following relations by

$$(a,b) \in \eta_k \;\Leftrightarrow\; a^k = b^k;$$

$$(a,b) \in \eta_k^\flat \;\Leftrightarrow\; (\forall x, y \in S^1)\ (xay, xby) \in \eta_k.$$

It is easy to verify that η_k is an equivalence relation on a semigroup S.

A semigroup S is η_k-*simple* if

$$(\forall a, b \in S)\ (a,b) \in \eta_k.$$

These semigroups are periodic.

Lemma 10. [21] *Let $k \in \mathbf{Z}^+$. If ξ is a band congruence on a semigroup S, then $\xi \subseteq \eta_k$ if and only if every ξ-class of S is an η_k-simple semigroup.*

By the following theorem we give structural characterization of bands of η_k-simple semigroups.

Theorem 35. [21] *Let $k \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is a band of η_k-simple semigroups;*

(ii) $(\forall a, b \in S)\ (\,(ab)^k = (a^k b^k)^k \ \wedge\ a^k = a^{2k}\,)$;

(iii) *η_k is a band congruence on S;*

(iv) *$\eta_k^\flat$ is a band congruence on S;*

(v) $(\forall a \in S)(\forall x, y \in S^1)\ xay\,\eta_k\,xa^2y$;

(vi) *$R(\eta_k) = \eta_k$ and η_k is a congruence on S;*

(vii) $R(\eta_k^\flat) = \eta_k^\flat$.

Let $k, m, n \in \mathbf{Z}^+$. On a semigroup S we define a relation $\overline{\eta}_{(k;m,n)}$ by

$$(a, b) \in \overline{\eta}_{(k;m,n)} \ \Leftrightarrow\ (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \eta_k.$$

The following lemma holds.

Lemma 11. [21] *Let S be a semigroup and let $k, m, n \in \mathbf{Z}^+$, then $\overline{\eta}_{(k;m,n)}$ is a congruence relation on S.*

Theorem 36. [21] *Let $k, m, n \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *$\overline{\eta}_{(k;m,n)}$ is a band congruence on S;*

(ii) $(\forall x \in S^m)(\forall y \in S^n)(\forall a \in S)\ xay\,\eta_k\,xa^2y$;

(iii) $R(\overline{\eta}_{(k;m,n)}) = \overline{\eta}_{(k;m,n)}$.

Proposition 7.2. [21] *Let $k, m, n \in \mathbf{Z}^+$. If $\overline{\eta}_{(k;m,n)}$ is a band congruence on a semigroup S, then $\eta_k \subseteq \overline{\eta}_{(k;m,n)}$.*

7.3. **The τ relations.** Further, by previous defined relations on a semigroup S we define the following relations:

$$(a, b) \in \tau \ \Leftrightarrow\ (\exists k \in \mathbf{Z}^+)\ (a, b) \in \eta_k;$$

$$(a, b) \in \tau^\flat \ \Leftrightarrow\ (\forall x, y \in S^1)\ (xay, xby) \in \tau.$$

It is easy to verify that the relation τ is an equivalence on a semigroup S.

A semigroup S is *τ-simple* if

$$(\forall a, b \in S)\ (a, b) \in \tau.$$

By the following theorem we describe the structure of bands of τ-simple semigroups. S. Bogdanović in [2] gave some other characterizations of these semigroups.

Theorem 37. [21] *The following conditions on a semigroup S are equivalent:*

(i) *S is a band of τ-simple semigroups;*

(ii) *τ is a band congruence on S;*

(iii) *$\tau^\flat$ is a band congruence on S;*

(iv) $(\forall a \in S)(\forall x, y \in S^1)\ xay\,\tau\,xa^2y$;

(v) *$R(\tau) = \tau$ and τ is a congruence on S;*

(vi) $R(\tau^\flat) = \tau^\flat$.

Let $m,n \in \mathbf{Z}^+$. On a semigroup S we define a relation $\overline{\tau}_{(m,n)}$ by

$$(a,b) \in \overline{\tau}_{(m,n)} \Leftrightarrow (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \tau.$$

The following theorem holds.

Theorem 38. [21] *Let S be a semigroup and let $m,n \in \mathbf{Z}^+$. Then $\overline{\tau}_{(m,n)}$ is a congruence relation on S.*

Theorem 39. [21] *Let $m,n \in \mathbf{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *$\overline{\tau}_{(m,n)}$ is a band congruence on S;*
(ii) *$(\forall x \in S^m)(\forall y \in S^n)(\forall a \in S)\ xay\,\tau\,xa^2y$;*
(iii) *$R(\overline{\tau}_{(m,n)}) = \overline{\tau}_{(m,n)}$.*

Proposition 7.3. [21] *Let $m,n \in \mathbf{Z}^+$. If $\overline{\tau}_{(m,n)}$ is a band congruence on a semigroup S, then $\tau \subseteq \overline{\tau}_{(m,n)}$.*

8. Some open problems

In this section we give some open problems on bands decompositions of semigroups.

Problem 1: How we can effective to construct the band congruence on an arbitrary semigroup?

Problem 2: Describe the structure of bands of λ-simple semigroups?

Problem 3: Describe the structure of left zero bands of left zero bands of power joined semigroups?

Problem 4: Describe the structure of right zero bands of left zero bands of power joined semigroups?

Problem 5: Describe the structure of bands (semilattices) of semigroups forom the class was given in the Problem 3?

Problem 6: Describe the structure of bands (semilattices) of semigroups forom the class was given in the Problem 4?

Problem 7: By $\mathcal{M}$ we denote the class of all matrices (rectangular bands). Let

$$\Lambda \circ \mathcal{M}^{k+1} = \left(\Lambda \circ \mathcal{M}^k\right) \circ \mathcal{M}, \quad k \in \mathbf{Z}^+.$$

Describe the structure of semigroups from the following classes

$$\Lambda \circ \mathcal{M}^{k+1}, \quad \left(\Lambda \circ \mathcal{M}^{k+1}\right) \circ \mathcal{B}, \quad \left(\Lambda \circ \mathcal{M}^{k+1}\right) \circ \mathcal{S}.$$

The previous problem can be formulated in the same way if instead the class Λ we take the class of all power-joined semigroups or the class of all λ_n-simple semigroups.

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