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ABOUT ONE RELATION CONCERNING TWO CIRCLES, WHERE ONE IS INSIDE OF THE OTHER

MIRKO RADIĆ AND ZORAN KALIMAN

Abstract. The following theorem and some of its corollaries will be proved.

THEOREM 1. Let C_1 and C_2 be any given two circles such that C_1 is inside of the C_2 and let A_1, A_2, A_3 be any given three different points on C_2 such that there are points T_1 and T_2 on C_1 with property

$$|A_1A_2| = t_1 + t_2, \quad |A_2A_3| = t_2 + t_3, \tag{1a}$$

where

$$t_1 = |A_1T_1|, \quad t_2 = |T_1A_2|, \quad t_3 = |T_2A_3|. \tag{1b}$$

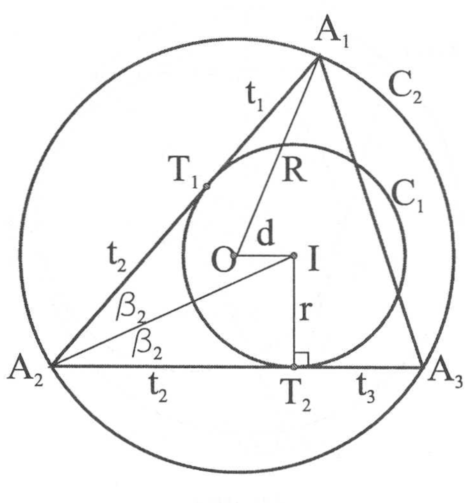

Figure 1

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Then

$$|A_1A_3| = k(t_1 + t_3), \tag{2a}$$

where

$$k = \frac{2rR}{R^2 - d^2}, \tag{2b}$$

r = radius of C_1, R = radius of C_2, $d = |IO|$, I is center of C_1 and O is center of C_2. (See Figure 1.)

Proof. First we prove how t_2 and t_3 can be expressed it t_1 is given. (See Figure 2.) For this purpose we prove the following lemma.

Lemma 1. If t_1 is given then t_2 can be calculated using the expression

$$(t_2)_{1,2} = \frac{t_1(R^2 - d^2) \pm \sqrt{D_1}}{r^2 + t_1^2} \tag{3a}$$

where

$$D_1 = t_1^2(R^2 - d^2)^2 + (r^2 + t_1^2)\left[4R^2d^2 - r^2t_1^2 - (R^2 + d^2 - r^2)^2\right]. \tag{3b}$$

Proof. From rectangular triangles A_1IT_1 and A_2IT_1 it follows

$$t_1^2 + r^2 = (x_1 - d)^2 + y_1^2 = R^2 + d^2 - 2dx_1, \quad t_2^2 + r^2 = R^2 + d^2 - 2dx_2 \tag{4}$$

or

$$x_1 = \frac{-t_1^2 + R^2 - r^2 + d^2}{2d}, \quad x_2 = \frac{-t_2^2 + R^2 - r^2 + d^2}{2d}. \tag{5}$$

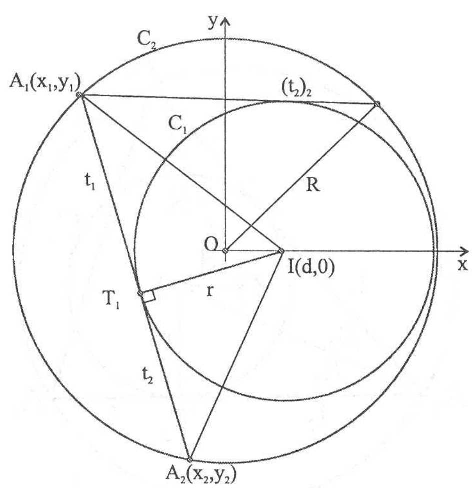

Figure 2

Since for area of triangle A_1A_2I it holds

$$(t_1 + t_2)^2r^2 = \left[x_1(y_2 - 0) + x_2(0 - y_1) + d(y_1 - y_2)\right]^2, \tag{6}$$

we can write

$$(t_1 + t_2)^2 r^2 = [y_1(d - x_2) - y_2(d - x_1)]^2, \tag{7a}$$

$$4[y_1 y_2 (d - x_1)(d - x_2)]^2 = [y_1^2(d - x_2)^2 + y_2^2(d - x_1)^2 - (t_1 + t_2)^2 r^2]^2. \tag{7b}$$

The above equation using the expressions $y_1^2 = R^2 - x_1^2$, $y_2^2 = R^2 - x_2^2$, and (5) can be written as

$$F \cdot [(r^2 + t_1^2)t_2^2 - 2t_1 t_2 (R^2 - d^2) - 4R^2 d^2 + r^2 t_1^2 + (R^2 + d^2 - r^2)^2] = 0, \tag{8a}$$

where

$$\begin{aligned} F =&(t_1 + t_2)^2 \\ &\Big(4d^4r^2 + 8d^2r^4 + 4r^6 - 8d^2r^2R^2 - 8r^4R^2 + 4r^2R^4 + d^4t_1^2 + 2d^2r^2t_1^2 \\ &+ 5r^4t_1^2 - 2d^2R^2t_1^2 - 6r^2R^2t_1^2 + R^4t_1^2 + r^2t_1^4 - 2d^4t_1t_2 - 12d^2r^2t_1t_2 \\ &- 2r^4t_1t_2 + 4d^2R^2t_1t_2 + 4r^2R^2t_1t_2 - 2R^4t_1t_2 - 2d^2t_1^3t_2 - 2r^2t_1^3t_2 \\ &+ 2R^2t_1^3t_2 + d^4{t_2}^2 + 2d^2r^2{t_2}^2 + 5r^4{t_2}^2 - 2d^2R^2{t_2}^2 - 6r^2R^2{t_2}^2 \\ &+ R^4{t_2}^2 + 4d^2t_1^2{t_2}^2 + 6r^2t_1^2{t_2}^2 - 4R^2t_1^2{t_2}^2 + t_1^4{t_2}^2 - 2d^2t_1{t_2}^3 \\ &- 2r^2t_1{t_2}^3 + 2R^2t_1{t_2}^3 - 2t_1^3{t_2}^3 + r^2{t_2}^4 + t_1^2{t_2}^4\Big). \end{aligned} \tag{8b}$$

It is not difficult to see that the factor F has no geometrical meaning important for our theorem, i.e. we get the following equation for t_2

$$(r^2 + t_1^2)t_2^2 - 2t_1 t_2 (R^2 - d^2) - 4R^2 d^2 + r^2 t_1^2 + (R^2 + d^2 - r^2)^2 = 0. \tag{9}$$

Thus, we have

$$(t_2)_{1,2} = \frac{t_1(R^2 - d^2) \pm \sqrt{D_1}}{r^2 + t_1^2} \tag{10a}$$

where

$$D_1 = t_1^2(R^2 - d^2)^2 + (r^2 + t_1^2)\left[4R^2d^2 - r^2t_1^2 - (R^2 + d^2 - r^2)^2\right]. \tag{10b}$$

(The length $(t_2)_1$ in Figure 2 is denoted by t_2.) □

First from Figure 2 we see that

$$|A_1A_3|^2 = (t_1 + t_2)^2 + (t_2 + t_3)^2 - 2(t_1 + t_2)(t_2 + t_3)\frac{t_2^2 - r^2}{t_2^2 + r^2}, \tag{11a}$$

since

$$\cos 2\beta_2 = \frac{1 - \tan^2 \beta_2}{1 + \tan^2 \beta_2} = \frac{1 - \left(\frac{r}{t_2}\right)^2}{1 + \left(\frac{r}{t_2}\right)^2} = \frac{t_2^2 - r^2}{t_2^2 + r^2}. \tag{11b}$$

The tangent length $t_2 = (t_2)_1$ is given by (10) and tangent length t_3 can be written as

$$t_3 = \frac{t_2(R^2-d^2)+\sqrt{D_2}}{r^2+t_2^2}, \tag{12a}$$

where

$$D_2 = t_2^2(R^2-d^2)^2 + (r^2+t_2^2)\left[4R^2d^2 - r^2t_2^2 - (R^2+d^2-r^2)^2\right] \tag{12b}$$

First, we form the equation

$$k^2 - \frac{|A_1A_3|^2}{(t_1+t_3)^2} = 0. \tag{13}$$

In this equation we have to eliminate square roots. We eliminate $\sqrt{D_2}$ by solving the equation (13) for $\sqrt{D_2}$. Square of the solution we equate with the expression for the D_2, Eq. (12b). New equation is

$$\frac{a_1\sqrt{D_1}+a_0}{n} = 0, \tag{14}$$

where a_0 and a_1 are function of (R, r, d, t_1). Terms a_0 and a_1 have common factor $d^4k^2 - 2d^2k^2R^2 - 4r^2R^2 + k^2r^2$ while the rest is still function of all variables (R, r, d, t_1). Evidently, the equation (14) can be valid (for all t_1) only if common factor vanish.

Using computer, it can be found that

$$k(t_1+t_3) - |A_1A_3| = 0 \Leftrightarrow d^4k^2 - 2d^2k^2R^2 - 4r^2R^2 + k^2R^4 = 0. \tag{15}$$

But, $d^4k^2 - 2d^2k^2R^2 - 4r^2R^2 + k^2R^4 = 0$ if $k = \frac{2rR}{R^2-d^2}$. This proves Theorem 1. □

Before we state some of its corollaries, let us remark that a polygon which is both tangential and chordal, for short called bicentric polygon.

***Corollary** 1.1.* Let $A_1 \dots A_n$ be a bicentric polygon. Then

$$\frac{|A_iA_{i+2}|}{t_i+t_{i+2}} = \frac{2rR}{R^2-d^2}, \quad i = 1,\dots,n. \tag{16}$$

***Corollary** 1.2.* Let $A_1 \dots A_n$ be a tangential polygon with property that there is $k > 0$ such that

$$\frac{|A_iA_{i+2}|}{t_i+t_{i+2}} = k \quad \text{for each } i = 1,\dots,n, \tag{17}$$

where indices are calculated modulo n. Then this polygon is also a chordal one, that is a bicentric n-gon.

Proof. Let A_1, A_2, A_3, A_4 be four consecutive vertices of $A_1 \dots A_n$ and let C_2 be circumcircle of the triangle $A_1A_2A_3$, R radius of C_2 and r radius of C_1. (Figure 3). We have to prove that A_4 lies on C_2, that is $\varphi_2 = \varphi_1$. Thus, we have to prove that the situation is like this shown in Figure 3. The proof is as follows.

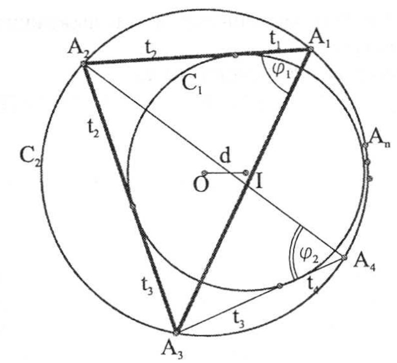

Figure 3.

Supposing that

$$|A_1A_3| = k(t_1 + t_3), \quad |A_2A_4| = k(t_2 + t_4), \tag{18}$$

we have the following two equations

$$k^2(t_1 + t_3)^2 = (t_1 + t_2)^2 + (t_2 + t_3)^2 - 2(t_1 + t_2)(t_2 + t_3)\frac{t_2^2 - r^2}{t_2^2 + r^2} \tag{19a}$$

$$k^2(t_2 + t_4)^2 = (t_2 + t_3)^2 + (t_3 + t_4)^2 - 2(t_2 + t_3)(t_3 + t_4)\frac{t_3^2 - r^2}{t_3^2 + r^2}. \tag{19b}$$

From the first we can calculated t_1 and from the second t_4, so we have

$$t_1 = \frac{a_1 \pm 2t_2\sqrt{D}}{n_1}, \quad t_4 = \frac{a_4 \pm 2t_3\sqrt{D}}{n_4} \tag{20a}$$

where

$$\begin{aligned} D &= k(r^2 + t_2^2)(r^2 + t_3^2) - r^2(t_2 + t_3)^2, \\ a_1 &= 2r^2t_2 + r^2t_3 - k^2r^2t_3 - t_2^2t_3 - k^2t_2^2t_3 \\ a_4 &= 2r^2t_3 + r^2t_2 - k^2r^2t_2 - t_2t_3^2 - k^2t_2t_3^2 \\ n_1 &= (r^2 + t_2^2)(k^2 - 1) \\ n_4 &= (r^2 + t_3^2)(k^2 - 1) \end{aligned} \tag{20b}$$

The values $\cos\varphi_1$ and $\cos\varphi_2$ can be expressed as

$$\begin{aligned} \cos\varphi_1 &= \frac{-(t_2 + t_3)^2 + (t_1 + t_2)^2 + |A_1A_3|^2}{2(t_1 + t_2)\,|A_1A_3|}, \\ \cos\varphi_2 &= \frac{-(t_2 + t_3)^2 + (t_3 + t_4)^2 + |A_2A_4|^2}{2(t_3 + t_4)\,|A_2A_4|}. \end{aligned} \tag{21}$$

Using expressions for t_1 and t_4 given by (20a), t_1 can be eliminated from (19a) and t_4 from (19b), so that each of $\cos\varphi_1$ and $\cos\varphi_2$ can be expressed only by t_2 and t_3. Together with Eq. (18) complex but straightforward calculation shows that $\cos\varphi_1 = \cos\varphi_2$, that is $\varphi_1 = \varphi_2$. □

University of Rijeka, Faculty of Philosophy, Department of Mathematics, 51000 Rijeka, Omladinska 14, Croatia
E-mail address: mradic@ffri.hr, kaliman@ffri.hr