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INDEPENDENCE OF CHARACTERIZING PROPERTIES OF $(m+k,m)-$BANDS

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Dedicated to Academician Blagoj Popov on the Occasion of His 85^{th} Birthday

Abstract. An $(m+k,m)-$semigroup which is a direct product of $p-$zero $(m+k,m)-$semigroups is called an $(m+k,m)-$band. The class of $(m+k,m)-$bands is being characterized by the five identities (I), (II), (III), (IV), (V), [3, Proposition 2.2]. Here, identities (III) and (IV) are replaced by a new identity (III′) and independence between these identities is proved, except for the independence between (II) and the rest of the identities.

1. Preliminaries

For a set Q and a positive integer s, Q^s denotes the s-th Cartesian power of Q. We use the notation x_1^s for the elements of Q^s. If $x_1 = x_2 = \cdots = x_s = x$, then x_1^s is denoted by the symbol $\overset{s}{x}$.

Let n, m be positive integers. An $(n,m)-$groupoid is a pair $(Q;[\ ])$ where $Q \neq \emptyset$ and $[\ ]$ is an $(n,m)-$operation i.e. a map from Q^n into Q^m. Every $(n,m)-$operation on Q induces a sequence $[\ ]_1, [\ ]_2, \ldots, [\ ]_m$ of $n-$ary operations on the set Q, such that

$$((\forall i \in \mathbb{N}_m) \quad [x_1^n]_i = y_i) \Leftrightarrow [x_1^n] = y_1^m.$$

Let $m \geq 2, k \geq 1$. An $(m+k,m)-$groupoid $(Q;[\ ])$ is called an $(m+k,m)-$semigroup if for each $i \in \{0,1,2,\ldots,k\}$

$$\left[x_1^i \left[x_{i+1}^{i+m+k}\right] x_{i+m+k+1}^{m+2k}\right] = \left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right].$$

The notions of $p-$zero $(m+k,m)-$semigroups and $(m+k,m)-$bands are introduced in [3]. An $(m+k,m)-$groupoid $(Q;[\ ])$ is said to be a $p-$zero $(m+k,m)-$groupoid, $0 \leq p \leq m$, if $[x_1^{m+k}] = x_1^p x_{p+k+1}^{m+k}$, for any $x_1^{m+k} \in Q^{m+k}$. Any $p-$zero $(m+k,m)-$groupoid is an $(m+k,m)-$semigroup. The $(m+k,m)-$semigroup which is a direct product of $m+1$ $p-$zero $(m+k,m)-$semigroups $(A_m, A_{m-1}, \ldots, A_0)$ is called $(m+k,m)-$band.

The following proposition characterizes $(m+k,m)-$bands.

Proposition 1.1. ([3, *Proposition* 2.2]) *An $(m+k,m)-$semigroup $\mathbf{Q} = (Q,[\ ])$ is an $(m+k,m)-$band if and only if the following conditions are satisfied in $\mathbf{Q}$:*

(I) $\left[x_1^{m+k}\right]_i = \left[y_1^{i-1} x_i y_{i+1}^{i+k-1} x_{i+k} y_{i+k+1}^{m+k}\right]_i$ $i \in \mathbb{N}_m$;

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(II) $\left[\overset{j-1}{a}\left[\overset{i-1}{a}\, x\, \overset{k-1}{a}\, y\, \overset{m-i}{a}\right]_i \overset{k-1}{a}\, z\, \overset{m-j}{a}\right]_j = \left[\overset{i-1}{a}\, x\, \overset{k-1}{a}\left[\overset{j-1}{a}\, y\, \overset{k-1}{a}\, z\, \overset{m-j}{a}\right]_j \overset{m-i}{a}\right]_i$ *, for a fixed element of* Q *and* $j \leq i$;

(III) $\left[\overset{i-1}{a}\left[\overset{j-1}{a}\, x\, \overset{k-1}{a}\, y\, \overset{m-j}{a}\right]_j \overset{k-1}{a}\, z\, \overset{m-i}{a}\right]_i = \left[\overset{i-1}{a}\, x\, \overset{k-1}{a}\, z\, \overset{m-i}{a}\right]_i$ *, for a fixed element of* Q *and* $j \leq i$;

(IV) $\left[\overset{j-1}{a}\, x\, \overset{k-1}{a}\left[\overset{i-1}{a}\, y\, \overset{k-1}{a}\, z\, \overset{m-i}{a}\right]_i \overset{m-j}{a}\right]_j = \left[\overset{j-1}{a}\, x\, \overset{k-1}{a}\, z\, \overset{m-j}{a}\right]_j$ *, for a fixed element of* Q *and* $j \leq i$;

(V) $\left[\overset{m+k}{x}\right] = \overset{m}{x}$.

In section 2 we will prove that the identities (III) and (IV) can be replaced by a new identity, i.e. the class of $(m+k, m)-$bands is being characterized by four identities. The independence between these four identities is proved in section 3, exept for the independence between (II) and the rest of the identities.

2. Replacing identities (III) and (IV) with one new identity

Here, identities (III) and (IV) will be replaced by a new identity

(III') $\left[x_1^{m+rk}\right]_i = \left[x_1^i x_{i+(r-1)k+1}^{m+rk}\right]_i$, $i \in \mathbb{N}_m$, $r \geq 2$.

Proposition 2.1. *Let* $\mathbf{Q} = (Q, [\ ])$ *be* $(m+k, m)-$*semigroup in which* (I), (III) *and* (IV) *are satisfied. Then the identity* (III$'$) *is satisfied in* $\mathbf{Q}$.

Proof. We will consider two cases $k = m+s$, $s \geq 0$ and $k+t = m$, $t \geq 1$.

1. Let $k = m+s$, $s \geq 0$. Then, for $r = 2$ we have:

$\left[x_1^{m+2k}\right]_i = \left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right]_i \stackrel{\mathrm{I}}{=} \left[\overset{i-1}{a}\left[\overset{i-1}{a}\, x_i\, \overset{k-1}{a}\, x_{i+k}\, \overset{m-i}{a}\right]_i \overset{k-1}{a}\, x_{i+2k}\, \overset{m-i}{a}\right]_i$
$\stackrel{\mathrm{III}}{=} \left[\overset{i-1}{a}\, x_i\, \overset{k-1}{a}\, x_{i+2k}\, \overset{m-i}{a}\right]_i \stackrel{\mathrm{I}}{=} \left[x_1^{i-1} x_i x_{i+k+1}^{i+2k-1} x_{i+2k} x_{i+2k+1}^{m+2k}\right]_i = \left[x_1^i x_{i+k+1}^{m+2k}\right]_i.$

Let the statement hold for r. For $r+1$ we have:

$\left[x_1^{m+(r+1)k}\right]_i = \left[\left[x_1^{m+rk}\right] x_{m+rk+1}^{m+(r+1)k}\right]_i \stackrel{\mathrm{I}}{=} \left[\overset{i-1}{a}\left[x_1^{m+rk}\right]_i \overset{k-1}{a}\, x_{m+rk+i+s}\, \overset{m-i}{a}\right]_i.$

Then, by induction

$\left[\overset{i-1}{a}\left[x_1^{m+rk}\right]_i \overset{k-1}{a}\, x_{m+rk+i+s}\, \overset{m-i}{a}\right]_i = \left[\overset{i-1}{a}\left[x_1^i x_{i+(r-1)k+1}^{m+rk}\right]_i \overset{k-1}{a}\, x_{i+(r+1)k}\, \overset{m-i}{a}\right]_i$
$\stackrel{\mathrm{III}}{=} \left[\overset{i-1}{a}\, x_i\, \overset{k-1}{a}\, x_{i+(r+1)k}\, \overset{m-i}{a}\right]_i \stackrel{\mathrm{I}}{=} \left[x_1^{i-1} x_i x_{i+rk+1}^{i+(r+1)k-1} x_{i+(r+1)k} x_{i+(r+1)k+1}^{m+(r+1)k}\right]_i$
$= \left[x_1^i x_{i+rk+1}^{m+(r+1)k}\right]_i.$

2. Let $k+t = m$, $t \geq 1$ and $r = 2$.

a) Let $i \leq t$. Then $i+k \leq t+k = m$.

$\left[x_1^{m+2k}\right]_i = \left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right]_i \stackrel{\mathrm{I}}{=} \left[\overset{i-1}{a}\left[x_1^{m+k}\right]_i \overset{k-1}{a}\left[x_1^{m+k}\right]_{i+k} \overset{m-i}{a}\right]_i$
$\stackrel{\mathrm{I}}{=} \left[\overset{i-1}{a}\left[\overset{i-1}{a}\, x_i\, \overset{k-1}{a}\, x_{i+k}\, \overset{m-i}{a}\right]_i \overset{k-1}{a}\left[\overset{i+k-1}{a}\, x_{i+k}\, \overset{k-1}{a}\, x_{i+2k}\, \overset{m-i-k}{a}\right]_{i+k} \overset{m-i}{a}\right]_i$
$\stackrel{\mathrm{III}}{=} \left[\overset{i-1}{a}\, x_i\, \overset{k-1}{a}\left[\overset{i+k-1}{a}\, x_{i+k}\, \overset{k-1}{a}\, x_{i+2k}\, \overset{m-i-k}{a}\right]_{i+k} \overset{m-i}{a}\right]_i \stackrel{\mathrm{IV}}{=} \left[\overset{i-1}{a}\, x_i\, \overset{k-1}{a}\, x_{i+2k}\, \overset{m-i}{a}\right]_i$
$\stackrel{\mathrm{I}}{=} \left[x_1^{i-1} x_i x_{i+k+1}^{i+2k-1} x_{i+2k} x_{i+2k+1}^{m+2k}\right]_i = \left[x_1^i x_{i+k+1}^{m+2k}\right]_i.$

b) Let $t < i$. Then, $i+k > t+k = m$.

$$\left[x_1^{m+2k}\right]_i = \left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right]_i \stackrel{\mathrm{I}}{=} \left[\overset{i-1}{a} \left[x_1^{m+k}\right]_i \overset{k-1}{a} x_{m+k+i+k-m} \overset{m-i}{a}\right]_i$$
$$\stackrel{\mathrm{III}}{=} \left[\overset{i-1}{a} x_i \overset{k-1}{a} x_{i+2k} \overset{m-i}{a}\right]_i \stackrel{\mathrm{I}}{=} \left[x_1^i x_{i+k+1}^{m+2k}\right]_i.$$

Therefore, the case $r=2$ is true. Let the statement hold for r. Then, for $r+1$ we have:

a)$i \leq t$

$$\left[x_1^{m+(r+1)k}\right]_i = \left[\left[x_1^{m+rk}\right] x_{m+rk+1}^{m+(r+1)k}\right]_i \stackrel{\mathrm{I}}{=} \left[\overset{i-1}{a} \left[x_1^{m+rk}\right]_i \overset{k-1}{a} \left[x_1^{m+rk}\right]_{i+k} \overset{m-i}{a}\right]_i.$$

By induction we obtain

$$\left[\overset{i-1}{a} \left[x_1^{m+rk}\right]_i \overset{k-1}{a} \left[x_1^{m+rk}\right]_{i+k} \overset{m-i}{a}\right]_i$$
$$= \left[\overset{i-1}{a} \left[x_1^i x_{i+(r-1)k+1}^{m+rk}\right]_i \overset{k-1}{a} \left[x_1^{i+k} x_{i+rk+1}^{m+rk}\right]_{i+k} \overset{m-i}{a}\right]_i$$
$$\stackrel{\mathrm{III}}{=} \left[\overset{i-1}{a} x_i \overset{k-1}{a} \left[x_1^{i+k} x_{i+rk+1}^{m+rk}\right]_{i+k} \overset{m-i}{a}\right]_i \stackrel{\mathrm{IV}}{=} \left[\overset{i-1}{a} x_i \overset{k-1}{a} x_{i+(r+1)k} \overset{m-i}{a}\right]_i \stackrel{\mathrm{I}}{=} \left[x_1^i x_{i+rk+1}^{m+(r+1)k}\right]_i.$$

b)$t < i$

$$\left[x_1^{m+(r+1)k}\right]_i = \left[\left[x_1^{m+rk}\right] x_{m+rk+1}^{m+(r+1)k}\right]_i \stackrel{\mathrm{I}}{=} \left[\overset{i-1}{a} \left[x_1^{m+rk}\right]_i \overset{k-1}{a} x_{m+rk+i+k-m} \overset{m-i}{a}\right]_i.$$

Then, by induction

$$\left[\overset{i-1}{a} \left[x_1^{m+rk}\right]_i \overset{k-1}{a} x_{m+rk+i+k-m} \overset{m-i}{a}\right]_i = \left[\overset{i-1}{a} \left[x_1^i x_{i+(r-1)k+1}^{m+rk}\right]_i \overset{k-1}{a} x_{i+(r+1)k} \overset{m-i}{a}\right]_i$$
$$\stackrel{\mathrm{III}}{=} \left[\overset{i-1}{a} x_i \overset{k-1}{a} x_{i+(r+1)k} \overset{m-i}{a}\right]_i \stackrel{\mathrm{I}}{=} \left[x_1^i x_{i+rk+1}^{m+(r+1)k}\right]_i.$$
□

Proposition 2.2. *Let* $\mathbf{Q} = (Q, [\,])$ *be* $(m+k,m)-$*semigroup in which identities* (I) *and* (III′) *are satisfied. Then* (III) *and* (IV) *are satisfied in* $\mathbf{Q}$.

Proof. 1. Let $k = m+s, s \geq 0$, a is a fixed element of Q and $j \leq i$. Then:

$$\left[\overset{i-1}{a} \left[\overset{j-1}{a} x \overset{k-1}{a} y \overset{m-j}{a}\right]_j \overset{k-1}{a} z \overset{m-i}{a}\right]_i \stackrel{\mathrm{I}}{=} \left[\overset{i-j}{a} \left[\overset{j-1}{a} x \overset{k-1}{a} y \overset{m-j}{a}\right] \overset{j+s-1}{a} z \overset{m-i}{a}\right]_i$$
$$= \left[\overset{i-j}{a} \overset{j-1}{a} x \overset{k-1}{a} y \overset{m-j}{a} \overset{j+s-1}{a} z \overset{m-i}{a}\right]_i \stackrel{\mathrm{III}'}{=} \left[\overset{i-1}{a} x \overset{k-1}{a} z \overset{m-i}{a}\right]_i,$$

and therefore (III) holds and

$$\left[\overset{j-1}{a} x \overset{k-1}{a} \left[\overset{i-1}{a} y \overset{k-1}{a} z \overset{m-i}{a}\right]_i \overset{m-j}{a}\right]_j \stackrel{\mathrm{I}}{=} \left[\overset{j-1}{a} x \overset{m-i+s}{a} \left[\overset{i-1}{a} y \overset{k-1}{a} z \overset{m-i}{a}\right] \overset{i-j}{a}\right]_j$$
$$= \left[\overset{j-1}{a} x \overset{m-i+s}{a} \overset{i-1}{a} y \overset{k-1}{a} z \overset{m-i}{a} \overset{i-j}{a}\right]_j \stackrel{\mathrm{III}'}{=} \left[\overset{j-1}{a} x \overset{k-1}{a} z \overset{m-j}{a}\right]_j,$$

so identity (IV) is satisfied.

2. Let $k+t = m, t \geq 1$ and l be the smallest positive integer for which $lk \geq m$. Let $j \leq i$ and a is fixed element of Q. Then:

$$\left[\overset{i-1}{a} \left[\overset{j-1}{a} x \overset{k-1}{a} y \overset{m-j}{a}\right]_j \overset{k-1}{a} z \overset{m-i}{a}\right]_i =$$
$$= \left[\overset{i-1}{a} \left[\overset{j-1}{a} x \overset{k-1}{a} y \overset{m-j}{a}\right]_j \left[\overset{j-1}{a} x \overset{k-1}{a} y \overset{m-j}{a}\right]_{j+1} \ldots \left[\overset{j-1}{a} x \overset{k-1}{a} y \overset{m-j}{a}\right]_m \overset{lk-m+j}{a} \overset{k-1}{a} z \overset{m-i}{a}\right]_i$$
$$\stackrel{\mathrm{I}}{=} \left[\overset{i-j}{a} \left[\overset{j-1}{a} x \overset{k-1}{a} y \overset{m-j}{a}\right] \overset{lk-m+j}{a} \overset{k-1}{a} z \overset{m-i}{a}\right]_i = \left[\overset{i-j}{a} \overset{j-1}{a} x \overset{k-1}{a} y \overset{m-j}{a} \overset{lk-m+j}{a} \overset{k-1}{a} z \overset{m-i}{a}\right]_i$$
$$\stackrel{\mathrm{III}'}{=} \left[\overset{i-1}{a} x \overset{k-1}{a} z \overset{m-i}{a}\right]_i,$$

and therefore (III) is satisfied.

In order to prove that (IV) holds we will consider to separate cases:

a) $i \leq k$. Then:

$$\left[\overset{j-1}{a}\, x \overset{k-1}{a} \left[\overset{i-1}{a}\, y \overset{k-1}{a}\, z \overset{m-i}{a}\right]_i \overset{m-j}{a}\right]_j \overset{\mathrm{I}}{=} \left[\overset{j-1}{a}\, x \overset{k-1}{a} \left[\overset{i-1}{a}\, y \overset{k-1}{a}\, z \overset{m-i}{a}\right] \overset{i-j}{a}\right]_j$$

$$= \left[\overset{j-1}{a}\, x \overset{k-1}{a} \overset{i-1}{a}\, y \overset{k-1}{a}\, z \overset{m-i}{a} \overset{i-j}{a}\right]_j \overset{\mathrm{III}'}{=} \left[\overset{j-1}{a}\, x \overset{k-1}{a}\, z \overset{m-j}{a}\right]_j.$$

b) $k < i$, i.e. $k + k' = i, k' \geq 1$. Then:

$$\left[\overset{j-1}{a}\, x \overset{k-1}{a} \left[\overset{i-1}{a}\, y \overset{k-1}{a}\, z \overset{m-i}{a}\right]_i \overset{m-j}{a}\right]_j \overset{\mathrm{I}}{=}$$

$$= \left[\overset{j-1}{a}\, x \left[\overset{i-1}{a}\, y \overset{k-1}{a}\, z \overset{m-i}{a}\right]_{k'+1} \dots \left[\overset{i-1}{a}\, y \overset{k-1}{a}\, z \overset{m-i}{a}\right]_i \dots \left[\overset{i-1}{a}\, y \overset{k-1}{a}\, z \overset{m-i}{a}\right]_m \overset{i-j}{a}\right]_j$$

$$= \left[\overset{j-1}{a}\, x \overset{lk-k'}{a} \left[\overset{i-1}{a}\, y \overset{k-1}{a}\, z \overset{m-i}{a}\right]_1 \dots \left[\overset{i-1}{a}\, y \overset{k-1}{a}\, z \overset{m-i}{a}\right]_{k'+1} \dots \left[\overset{i-1}{a}\, y \overset{k-1}{a}\, z \overset{m-i}{a}\right]_m \overset{i-j}{a}\right]_j$$

$$= \left[\overset{j-1}{a}\, x \overset{lk-k'}{a} \overset{i-1}{a}\, y \overset{k-1}{a}\, z \overset{m-i}{a} \overset{i-j}{a}\right]_j \overset{\mathrm{III}'}{=} \left[\overset{j-1}{a}\, x \overset{k-1}{a}\, z \overset{m-j}{a}\right]_j. \qquad \square$$

Clearly, according to Proposition 2.1 and Proposition 2.2 we may conclude that an $(m+k,m)-$semigroup $\mathbf{Q} = (Q, [\,])$ is an $(m+k,m)-$band if and only if (I), (II), (III$'$) and (V) are satisfied in $\mathbf{Q}$.

3. Independence of characterizing properties

In this section we will construct three examples of vector valued semigroups that satisfy exactly three if the identities (I), (II), (III$'$) and (V), except for the case when (I), (III$'$) and (V) hold and (II) does not hold. This way we prove that identities (I), (III$'$) and (V) are independent of the other three, while the question of the independence of identity (II) to the rest of the identities is left open.

Example 3.1. (Independence between (V) and the rest of the identities)
Let $(\{0,1,2\}; [\,])$ be $(3,2)-$groupoid, where the $(3,2)-$operation is defined by:

$$[xyz] = \begin{cases} xz, & x \neq 2, z \neq 2 \\ 0z, & x = 2, z \neq 2 \\ x0, & x \neq 2, z = 2 \\ 00, & x = z = 2 \end{cases}$$

Let $x, y, z, a \in \{0,1,2\}$. Since

$$[[xyz]a] = \begin{cases} [xyz]_1\, a, & a \neq 2 \\ [xyz]_1\, 0, & a = 2 \end{cases} = \begin{cases} xa, & x \neq 2, a \neq 2 \\ 0a, & x = 2, a \neq 2 \\ x0, & x \neq 2, a = 2 \\ 00, & x = a = 2 \end{cases}$$

$$[x[yza]] = \begin{cases} x\, [yza]_2\,, & x \neq 2 \\ 0\, [yza]_2\,, & x = 2 \end{cases} = \begin{cases} xy, & x \neq 2, a \neq 2 \\ 0a, & x = 2, a \neq 2 \\ x0, & x \neq 2, a = 2 \\ 00, & x = a = 2 \end{cases}$$

it follows that $(\{0,1,2\}; [\,])$ is a $(3,2)-$semigroup.

Here (I) holds since for each $x, y, z, a \in \{0,1,2\}$

$$[xyz]_1 = \begin{cases} 0, & x = 0,2 \\ 1, & x = 1 \end{cases} = [xya]_1 \text{ and } [xyz]_2 = \begin{cases} 0, & z = 0,2 \\ 1, & z = 1 \end{cases} = [ayz]_2.$$

For $x, y, z, a \in \{0,1,2\}$ we have

$$\left[[xya]_1\, za\right]_1 = [xya]_1 = \begin{cases} 0, & x = 0,2 \\ 1, & x = 1 \end{cases} = \left[x\,[yza]_1\, a\right]_1,$$

$$\left[[axy]_2\, za\right]_1 = [axy]_2 = \begin{cases} 0, & y = 0,2 \\ 1, & y = 1 \end{cases} = [yza]_1 = \left[ax\,[yza]_1\right]_2,$$

$$\left[a\,[axy]_2\, z\right]_2 = \begin{cases} 0, & z = 0,2 \\ 1, & z = 1 \end{cases} = [ayz]_2 = \left[ax\,[ayz]_2\right]_2.$$

So, (II) is satisfied in $(\{0,1,2\};[\,])$.

Let $r \geq 2, x_i \in \{0,1,2\}, i \in \mathbb{N}_{r+2}$. Then:

$$\left[x_1^{r+2}\right]_1 = \left[x_1\left[x_2^{r+2}\right]\right]_1 = \begin{cases} 0, & x_1 = 0,2 \\ 1, & x_1 = 1 \end{cases} = [x_1 x_{r+1} x_{r+2}]_1 \text{ and}$$

$$\left[x_1^{r+2}\right]_2 = \left[\left[x_1^{r+1}\right] x_{r+2}\right]_2 = \begin{cases} 0, & x_{r+2} = 0,2 \\ 1, & x_{r+2} = 1 \end{cases} = [x_1 x_2 x_{r+2}]_2,$$

therefore (III$'$) also holds.

Since $[222] = 00$, (V) is not satisfied in $(\{0,1,2\};[\,])$.

Let assume that $(\{0,1,2\};[\,])$ is a $(3,2)-$band. Then it is isomorphic to a direct product of one $2-$zero $(3,2)-$semigroup $\mathbf{A} = (A;[\,]^A)$, one $1-$zero $(3,2)-$semigroup $\mathbf{S} = (S;[\,]^S)$ and one $0-$zero $(3,2)-$semigroup $\mathbf{B} = (B;[\,]^B)$.
Since $|\{0,1,2\}| = |A \times S \times B| = |A| \cdot |S| \cdot |B|$, we obtain that either $|A| = 3$, $|S| = |B| = 1$ or $|A| = 1, |S| = 3, |B| = 1$ or $|A| = |S| = 1, |B| = 3$. Let $A = \{a,b,c\}, S = \{s\}$ and $B = \{d\}$.
Then $[(x,s,d)(y,s,d)(z,s,d)]^{A\times S\times B} = (x,s,d)(y,s,d)$, and from here $(\{0,1,2\};[\,])$ is a $2-$zero $(3,2)-$semigroup, which is false since $[222] = 00$. Similar, in the second or third case we obtain that $(\{0,1,2\};[\,])$ is a $1-$zero $(3,2)-$semigroup or $0-$zero $(3,2)-$semigroup, which is false ($[222] = 00$).

Therefore, identity (V) is independent of the identities (I), (II) and (III$'$).

The next example shows independence of identity (I) to the identities (II), (III$'$) and (V).

Example 3.2. (Independence between (I) and the rest of the identities)
Let $(\{0,1\};[\,])$ be $(3,2)-$groupoid, where the $(3,2)-$operation is defined by:

$$[xyz] = \begin{cases} 00, & x = 0 \\ 11, & x = 1 \end{cases}.$$

Let $x,y,z,a \in \{0,1\}$. Then

$$[[xyz]\,a] = \begin{cases} [00a]\,, & x = 0 \\ [11a]\,, & x = 1 \end{cases} = \begin{cases} 00, & x = 0 \\ 11, & x = 1 \end{cases} = [x\,[yza]],$$

here for $(\{0,1\};[\,])$ is a $(3,2)-$semigroup.

Let $x,y,z,a \in \{0,1\}$. Using

$$\left[[xya]_1\, za\right]_1 = \begin{cases} 0, & x = 0 \\ 1, & x = 1 \end{cases} = \left[x\,[yza]_1\, a\right]_1,$$

$$\left[[axy]_2\, za\right]_1 = \begin{cases} 0, & a = 0 \\ 1, & a = 1 \end{cases} = \left[ax\,[yza]_1\right]_2,$$

$$\left[a\,[axy]_2\, z\right]_2 = \begin{cases} 0, & a = 0 \\ 1, & a = 1 \end{cases} = \left[ax\,[ayz]_2\right]_2,$$

it follows that (II) holds in $(\{0,1\};[\,])$.

Let $r \geq 2, x_i \in \{0,1\}, i \in \mathbb{N}_{r+2}$. Then

$$\left[x_1^{r+2}\right]_1 = \left[x_1\left[x_2^{r+2}\right]\right]_1 = \begin{cases} 0, & x_1 = 0 \\ 1, & x_1 = 1 \end{cases} = [x_1 x_{r+1} x_{r+2}]_1 \text{ and}$$

$\left[x_1^{r+2}\right]_2 = \left[x_1\left[x_2^{r+2}\right]\right]_2 = \begin{cases} 0, & x_1 = 0 \\ 1, & x_1 = 1 \end{cases} = [x_1x_2x_{r+2}]_2$,
so (III$'$) holds in $(\{0,1\};[\,])$.
Since $[000] = 00$ and $[111] = 11$, (V) is also satisfied in $(\{0,1\};[\,])$.
But identity (I) does not hold in $(\{0,1\};[\,])$, since $[011]_2 = 0 \neq 1 = [111]_2$.

Let assume that $(\{0,1\};[\,])$ is a $(3,2)-$band. Then it is isomorphic to a direct product of one $2-$zero $(3,2)-$semigroup $\mathbf{A} = (A;[\,]^A)$, one $1-$zero $(3,2)-$semigroup $\mathbf{S} = (S;[\,]^S)$ and one $0-$zero $(3,2)-$semigroup $\mathbf{B} = (B;[\,]^B)$.
Since $|\{0,1\}| = |A \times S \times B| = |A| \cdot |S| \cdot |B|$, we obtain that either $|A| = 2$, $|S| = |B| = 1$ or $|A| = 1, |S| = 2, |B| = 1$ or $|A| = |S| = 1, |B| = 2$. Let $A = \{a,b,c\}, S = \{s\}$ and $B = \{d\}$.
Then $[(x,s,d)(y,s,d)(z,s,d)]^{A\times S\times B} = (x,s,d)(y,s,d)$, and from here $(\{0,1\};[\,])$ is a $2-$zero $(3,2)-$semigroup, which is false since $[011] = 00$. Similar, in the second or third case we obtain that $(\{0,1\};[\,])$ is a $1-$zero $(3,2)-$semigroup or $0-$zero $(3,2)-$semigroup, which is false ($[011] = 00$).

Example 3.3. (Independence between (III$'$) and the rest of the identities)
Let $(\{0,1\};[\,])$ be $(3,2)-$groupoid, where the $(3,2)-$operation is defined by:
$[xyz] = \begin{cases} yz, & x = 0 \\ xz, & x = 1 \end{cases}$.

Let $x,y,z,a \in \{0,1\}$. By
$[[xyz]\,a] = \begin{cases} [yza], & x = 0 \\ [xza], & x = 1 \end{cases} = \begin{cases} [yza], & x = 0 \\ xa, & x = 1 \end{cases} = \begin{cases} [yza], & x = 0 \\ x\,[yza]_2, & x = 1 \end{cases}$
$= [x\,[yza]]$,
it follows that $(\{0,1\};[\,])$ be $(3,2)-$semigroup.

Let $x,y,z,a \in \{0,1\}$. The following identities hold in $(\{0,1\};[\,])$:
$-$(I), since $[xyz]_1 = \begin{cases} y, & x = 0 \\ x, & x = 1 \end{cases} = [xya]_1$ and $[xyz]_2 = z = [ayz]_2$;
$-$(II), since $\left[[xyz]_1\,za\right]_1 = \begin{cases} [yza]_1, & x = 0 \\ 1, & x = 1 \end{cases} = \left[x\,[yza]_1\,a\right]_1$,
$\left[[axy]_2\,za\right]_1 = [yza]_1 = \left[ax\,[yza]_1\right]_2$ and $\left[a\,[axy]_2\,z\right]_2 = z = [ayz]_2 = \left[ax\,[ayz]_2\right]_2$;
$-$(V), since $[000] = 00$ and $[111] = 11$.

But identity (III$'$) is not satisfied in $(\{0,1\};[\,])$ since
$[0100]_1 = [100]_1 = 1 \neq 0 = [000]_1$.

Let assume that $(\{0,1\};[\,])$ is a $(3,2)-$band. Then it is isomorphic to a direct product of one $2-$zero $(3,2)-$semigroup $\mathbf{A} = (A;[\,]^A)$, one $1-$zero $(3,2)-$semigroup $\mathbf{S} = (S;[\,]^S)$ and one $0-$zero $(3,2)-$semigroup $\mathbf{B} = (B;[\,]^B)$.
Since $|\{0,1\}| = |A \times S \times B| = |A| \cdot |S| \cdot |B|$, we obtain that either $|A| = 2$, $|S| = |B| = 1$ or $|A| = 1, |S| = 2, |B| = 1$ or $|A| = |S| = 1, |B| = 2$. Let $A = \{a,b,c\}, S = \{s\}$ and $B = \{d\}$.
Then $[(x,s,d)(y,s,d)(z,s,d)]^{A\times S\times B} = (x,s,d)(y,s,d)$, and from here $(\{0,1\};[\,])$ is a $2-$zero $(3,2)-$semigroup, which is false since $[011] = 00$. Similar, in the second or third case we obtain that $(\{0,1\};[\,])$ is a $1-$zero $(3,2)-$semigroup or $0-$zero $(3,2)-$semigroup, which is false ($[010] = 10$ and $[101] = 11$).

Therefore identity (III$'$) is independent of the identities (I), (II) and (V).

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