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ON GENERALIZATION OF THE HERMITE-HADAMARD INEQUALITY III

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Dedicated to Academician Blagoj Popov on the Occasion of His 85^{th} Birthday

Abstract. Generalized form of Hermite-Hadamard inequality for (2n)-convex and (2n-1)-concave or convex Lebesgue integrable functions are obtained through generalization of Taylor's Formula.

1. Introduction and Preliminaries

The classical Hermite-Hadamard inequality gives us an estimate, from below and from above, of the mean value of a convex function $f : [a, b] \to \mathbb{R}$ (see [4], pp. 137.):

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

In [2] S. S. Dragomir and A. Mcandrew gave the following generalization of (1):

Theorem 1. *Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is monotonic and convex on (a, b). Then we have:*

$$\frac{1}{2}\left(\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right) - \frac{1}{b-a}\int_a^b f(x)dx$$
$$\ge \left|\frac{1}{4}[f(b)-f(a)] + \frac{1}{b-a}\int_a^b sgn\left(\frac{a+b}{2}-x\right) f(x)dx\right|.$$

In [3] Sabir Hussain and Matloob Anwar gave the following generalization of Theorem 1:

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Theorem 2. *Assume that $f:[a,b]\longrightarrow\mathbb{R}$ is monotone on $[a,b]$ and convex on (a,b). Then*

$$\frac{1}{2}\left[f(x)+\frac{f(b)(b-x)+f(a)(x-a)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(y)dy$$
$$\geq\left|\frac{1}{b-a}\int_a^b sgn(x-y)f(y)dy-\frac{1}{2(b-a)}[f(x)(a+b-2x)+\right.$$
$$\left.+(a-x)f(a)+(b-x)f(b)]\right|$$

for all $x\in(a,b)$.

In this paper we will give generalizations of Theorem 1 and Theorem 2 for (2n)-convex functions. Let us note that using Taylor's Formula in [1] Matloob Anwar and J. Pečarić proved generalizations of Theorem 2.1 and Theorem 2.2 of [2] and Theorem 1 and Theorem 2 of [3] for (2n)-convex functions i.e they proved the following results:

Theorem 3. *Assume that $f:[a,b]\to\mathbb{R}$ is a continuous and (2n)-convex function. Then*

$$\frac{1}{(b-a)}\int_a^b f(y)dy-(b-a)f(x)-\sum_1^{2n-1}\frac{(b-y)^{k+1}-(a-x)^{k+1}}{(k+1)!(b-a)}f^{(k)}(x)$$
$$\geq\left|\frac{1}{(b-a)}\int_a^b\left|f(y)-f(x)-\sum_1^{2n-2}\frac{(y-x)^k}{k!}f^{(k)}(x)\right|dy-\right.$$
$$\left.-\left|f^{(2n-1)}(x)\right|\frac{(a-x)^{2n}+(b-x)^{2n}}{(2n)!(b-a)}\right|$$

for all $x\in(a,b)$.

Theorem 4. *Assume that $f:[a,b]\to\mathbb{R}$ is a continuous and $(2n)$-convex function. Then*

$$f(x)-\frac{2n}{(b-a)}\int_a^b f(y)dy-\sum_1^{2n-1}\frac{2n-k}{k!(b-a)}[(x-b)^kf^{(k-1)}(b)-(x-a)^kf^{(k-1)}(a)]$$
$$\geq\left|\frac{1}{b-a}\int_a^b\left|f(x)-f(y)-\sum_1^{2n-2}\frac{(x-y)^k}{k!}f^{(k)}(y)\right|dy-\right.$$
$$\left.-\frac{1}{b-a}\int_a^b\left|\frac{(x-y)^{2n-1}}{(2n-1)!}f^{(2n-1)}(y)\right|dy\right|.$$

Let f be a real valued function defined on $[a,b]$. A k-th order divided difference of f at distinct points $x_0,x_1,...,x_n\in[a,b]$ may be defined recursively by(see [1] p-14):

$$[x_i]f=f(x_i)\ \ i=0,1,...,k$$

and

$$[x_0,x_1,....,x_k]f=\frac{[x_1,....,x_k]f-[x_0,x_1,....,x_{k-1}]f}{x_k-x_0}.$$

The value of $[x_0,x_1,....,x_k]f$ is independent of the order of the points $x_0,x_1,....,x_k$.
A function $f:[a,b]\longrightarrow\mathbb{R}$ is said to be (n)-convex, $n\geq 0$ on $[a,b]$ if and only if for all choices of (n+1) distinct points in $[a,b]$,

$$[x_0,x_1,....,x_n]f\geq 0.$$

Letting
$G_k(x)=[c,c,...,c,x]f=$

$$= \begin{cases} (x-c)^{-k}\left(f(x)-f(c)-\sum_{j=1}^{k-1}\frac{f^{(j)}(c)}{j!}(x-c)^j\right), & \text{for } x\neq c; \\ \frac{1}{k!}f^{(k)}(c), & \text{for } x=c. \end{cases} \tag{2}$$

The following result for the function $G_k(x)$ is valid(see [4], p-16).

Theorem 5. *If f is an (n)-convex on $[a,b]$ for $n\geq 2$, then for every $c\in(a,b)$*

(1) *the function G_{n-1} is increasing on $[a,b]$;*
(2) *the function $G_k(n\geq 3, k\in\{1,2,...,n-2\})$ is $(n-k)$-convex on [a,b].*

From this theorem we have the following lemma.

Lemma 1. *Let f be as in Theorem 4 then we have*

$$\left(f(x)-f(c)-\sum_{j=1}^{k-1}\frac{f^{(j)}(c)}{j!}(x-c)^j\right) \begin{array}{ll} \geq 0, & x\geq c \\ \leq 0, & x\leq c. \end{array} \tag{3}$$

Lemma 2. *Let f be integrable on (a,b) and k time differentiable function then the following is valid*

$$\int_a^b (x-y)^k f^{(k)}(y)dy =$$
$$= \sum_{i=0}^{k-1}\frac{k!}{(i+1)!}\left((x-b)^{i+1}f^{(i)}(b)-(x-a)^{i+1}f^{(i)}(a)\right)+k!\int_a^b f(y)dy. \tag{4}$$

2. Main results

We have the following generalization of Theorem 2.

Theorem 6. *Assume that $f:[a,b]\longrightarrow\mathbb{R}$ is a $(2n-1)$-times differentiable and f is $(2n-1)$ and $(2n)$-convex function(or $(2n-1)$-concave and $(2n)$-convex function). Then*

$$f(x)-\frac{2n}{(b-a)}\int_a^b f(y)dy-\sum_{1}^{2n-1}\frac{2n-k}{k!(b-a)}[(x-b)^k f^{(k-1)}(b)-(x-a)^k f^{(k-1)}(a)]$$
$$\geq \frac{1}{b-a}\Bigg|(2x-a-b)f(x)+\sum_{i=0}^{2n-2}\frac{1}{(i+1)!}\left((x-a)^{i+1}f^{(i)}(a)+(x-b)^{i+1}f^{(i)}(b)\right)$$
$$+\sum_{k=1}^{2n-2}\sum_{j=0}^{k-1}\frac{1}{(j+1)!}\left((x-a)^{j+1}f^{(j)}(a)+(x-b)^{j+1}f^{(j)}(b)\right)-$$
$$-(2n-3)\int_a^b sgn(x-y)f(y)dy\Bigg|.$$

Proof. First assume that f is (2n)-convex and (2n-1)-convex.

$$\int_a^b \left| f(x) - f(y) - \sum_1^{2n-2} \frac{(x-y)^k}{k!} f^{(k)}(y) \right| dy =$$

$$= \int_a^x \left| f(x) - f(y) - \sum_1^{2n-2} \frac{(x-y)^k}{k!} f^{(k)}(y) \right| dy +$$

$$+ \int_x^b \left| f(x) - f(y) - \sum_{k=1}^{2n-2} \frac{(x-y)^k}{k!} f^{(k)}(y) \right| dy$$

(using Lemma 1 we have)

$$= \int_a^x \left(f(x) - f(y) - \sum_1^{2n-2} \frac{(x-y)^k}{k!} f^{(k)}(y) \right) dy -$$

$$- \int_x^b \left(f(x) - f(y) - \sum_1^{2n-2} \frac{(x-y)^k}{k!} f^{(k)}(y) \right) dy$$

(by using Lemma 2 we get)

$$= (2x - a - b) f(x) +$$

$$+ \sum_{k=1}^{2n-2} \sum_{j=0}^{k-1} \frac{1}{(j+1)!} \left((x-a)^{j+1} f^{(j)}(a) + (x-b)^{j+1} f^{(j)}(b) \right)$$

$$-(2n-2) \int_a^b sgn(x-y) f(y) dy. \tag{5}$$

$$\int_a^b \left| \frac{(x-y)^{2n-1}}{(2n-1)!} f^{(2n-1)}(y) \right| dy =$$

$$= \int_a^x \left| \frac{(x-y)^{2n-1}}{(2n-1)!} f^{(2n-1)}(y) \right| dy + \int_x^b \left| \frac{(x-y)^{2n-1}}{(2n-1)!} f^{(2n-1)}(y) \right| dy.$$

(f is $(2n-1)$-convex function)

$$= \int_a^x \left(\frac{(x-y)^{2n-1}}{(2n-1)!} f^{(2n-1)}(y) \right) dy - \int_x^b \left(\frac{(x-y)^{2n-1}}{(2n-1)!} f^{(2n-1)}(y) \right) dy.$$

(using Lemma 2 we get)

$$= -\left(\sum_{i=0}^{2n-2} \frac{1}{(i+1)!} \left((x-a)^{i+1} f^{(i)}(a) + (x-b)^{i+1} f^{(i)}(b) \right) \right) +$$

$$+ \int_a^b sgn(x-y) f(y) dy. \tag{6}$$

By substituting (5) and (6) in Theorem 3 we get

$$f(x) - \frac{2n}{(b-a)}\int_a^b f(y)dy - \sum_1^{2n-1}\frac{2n-k}{k!(b-a)}[(x-b)^k f^{(k-1)}(b) - (x-a)^k f^{(k-1)}(a)]$$
$$\geq \frac{1}{b-a}\left|(2x-a-b)f(x) + \sum_{i=0}^{2n-2}\frac{1}{(i+1)!}\Big((x-a)^{i+1}f^{(i)}(a) + (x-b)^{i+1}f^{(i)}(b)\Big)\right.$$
$$+\sum_{k=1}^{2n-2}\sum_{j=0}^{k-1}\frac{1}{(j+1)!}\Big((x-a)^{j+1}f^{(j)}(a) + (x-b)^{j+1}f^{(j)}(b)\Big) -$$
$$\left. -(2n-3)\int_a^b sgn(x-y)f(y)dy\right|.$$

In similar way we can prove for (2n)-convex and (2n-1)-concave functions. ∎

Corollary 1. *Assume that $f : [a,b] \longrightarrow \mathbb{R}$ is a $(2n-1)$-times differentiable and f is $(2n-1)$ and $(2n)$-convex function(or $(2n-1)$-concave and $(2n)$-convex function). Then*

$$f\left(\frac{a+b}{2}\right) - \frac{2n}{(b-a)}\int_a^b f(y)dy +$$
$$+\sum_1^{2n-1}\frac{(2n-k)(b-a)^{k-1}}{2^k k!}[f^{(k-1)}(a) - (-1)^k f^{(k-1)}(b)]$$
$$\geq \left|\sum_{i=0}^{2n-2}\frac{(b-a)^i}{2^{i+1}(i+1)!}\Big(f^{(i)}(a) - (-1)^i f^{(i)}(b)\Big) + \right.$$
$$\sum_{k=1}^{2n-2}\sum_{j=0}^{k-1}\frac{(b-a)^j}{2^{j+1}(j+1)!}\Big(f^{(j)}(a) - (-1)^j f^{(j)}(b)\Big) -$$
$$\left. -\frac{(2n-3)}{b-a}\int_a^b sgn(\frac{a+b}{2} - y)f(y)dy\right|. \quad (7)$$

Proof. Substituting $x = \frac{a+b}{2}$ in Theorem 6 we get (7). ∎

Remark 1. *Corollary 1 is the generalization of Theorem 1 for (2n)-convex functions.*

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