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REFLEXIVE ALGEBRAIC OPERATORS

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Abstract. In this note, a characterization of reflexive algebraic operators is given. Also, it is proven that the reflexivity of algebraic operators is a quasisimilarity invariant.

1. Introduction and preliminaries

For a bounded linear operator A on a complex Hilbert space $\mathcal{H}$, denote by $a(A)$ the weakly closed subalgebra of $B(\mathcal{H})$ generated by A and the identity I.

Other algebras can be associated with A as follows. If $\mathcal{U}$ is a collection of (bounded, linear) operators on $\mathcal{H}$, then $\operatorname{Lat}\mathcal{U}$ denotes the family of all subspaces of $\mathcal{H}$ that are invariant under all the operators in $\mathcal{U}$, i.e. $\mathcal{M} \in \operatorname{Lat}\mathcal{U}$ if $A\mathcal{M} \subseteq \mathcal{M}$ for every $A \in \mathcal{U}$.

If $\mathcal{F}$ is a family of subspaces of $\mathcal{H}$, than $\operatorname{Alg}\mathcal{F}$ denotes the algebra of all operators on $\mathcal{H}$ which leave invariant every member of $\mathcal{F}$. $\operatorname{Alg}\mathcal{F}$ is always a weakly closed sub algebra of $B(\mathcal{H})$. We can thus form for every algebra $\mathcal{U}$ the larger algebra $\operatorname{Alg}\mathcal{U}$.

An algebra $\mathcal{U}$ is said to be reflexive if $\mathcal{U} = \operatorname{Alg}\operatorname{Lat}\mathcal{U}$. An operator is said to be reflexive if $a(A)$ is reflexive. We clearly have that A is reflexive if $\operatorname{Lat} A \subset \operatorname{Lat} B \Longrightarrow B \in a(A)$ i.e. $\operatorname{Alg}\operatorname{Lat} A = a(A)$.

There exists in the literature a number of results about reflexivity. Sarason in 1966 proved that every normal operator is reflexive and that every Teoplitz operator is reflexive. In particular the unilateral shift is reflexive. The word 'reflexive' was suggested by Halmos. Deddens in 1971 proved that every isometry V on a Hilbert space is reflexive. Deddens and Fillmore gave a complete characterization of reflexive operators on a finite dimensional space. But, this question is very difficult for arbitrary spaces.

Any general characterization of reflexive algebras seems to be very difficult. Is $A \oplus A$ reflexive operator for every $A \in B(\mathcal{H})$? A stronger statement conjectured by P. Rosenthal would be: $\operatorname{Lat} A \subset \operatorname{Lat} B$ and $AB = BA$ implies that B is in the weakly closed algebra generated by A and I.

In this note we show that this statement holds for an algebraic operator.

Recall some basic facts about the structure of algebraic operators. If A is algebraic, there is a unique monic polynomial p of minimal degree such that $p(A) = 0$; p is the minimal polynomial of A, and its existence follows from the observation

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that the set of all polynomials q such that $q(A)$ is an ideal. Consider the factorization of $p(z)$ into linear factors: $p(z) = \prod_{j=1}^{k}(z-\lambda_j)^{n_j}$ with the $\{\lambda_j\}$ distinct. Then $\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}$. Let $\mathcal{M}_j = \operatorname{Ker}(A - \lambda_j)^{n_j}$, $j = 1, \ldots, n$, then $\mathcal{M}_j \in \operatorname{Lat} A$. It can be shown that $\mathcal{H} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_k$. Let $A_j = A \mid \mathcal{M}_j$, then $A = A_1 \oplus A_2 \oplus \ldots \oplus A_k$. We call this the Riesz decomposition of A. For each j, the integer n_j is the algebraic multiplicity of λ_j. Every operator on a finite-dimensional space is algebraic, of course, but operators on infinite-dimensional spaces are rarely algebraic.

2. Main results

Theorem 1. *Let A be an algebraic operator and $B \in \{A\}'$. Then* $\operatorname{Lat} A \subset \operatorname{Lat} B$ *if and only if $B = q(A)$ for some polynomial q.*

Proof. Let $p(z) = \prod_{j=1}^{k}(z - \lambda_j)^{n_j}$ be the minimal polynomial for the operator A, $\mathcal{M}_j = \operatorname{Ker}(A - \lambda_j)^{n_j}$, $j = 1, \ldots, k$, $A_j = A \mid \mathcal{M}_j$. Then $A = \bigoplus_{j=1}^{k} A_j$. Let $B \in \{A\}'$ and $\operatorname{Lat} A \subset \operatorname{Lat} B$. It is easy to verify that $B = \bigoplus_{j=1}^{k} B_j$, $\mathcal{M}_j \in \operatorname{Lat} A_j \subset \operatorname{Lat} B_j, B_j \in \{A_j\}'$. For fixed $j \in \{1, \ldots, k\}$, let $x \in \mathcal{M}_j$, $\mathcal{M}_x = \vee\{A^n x\} \in \operatorname{Lat} A \subset \operatorname{Lat} B$. Then there exists a polynomial p_x, $\deg p_x \leq \deg p$, such that $B_j x = p_x(A_j)x$. Since B_j commutes with A_j, for arbitrary $x' \in \mathcal{M}_x$ there exists a polynomial $r_{x'}$ such that $x' = r_{x'}(A_j)x$. So, $B_j x' = B_j r_{x'}(A_j)x = r_{x'}(A_j)B_j x = r_{x'}(A_j)p_x(A_j)x = p_x(A_j)r_{x'}(A_j)x = p_x(A_j)x'$, which means that $B_j = p_x(A_j)$ on $\mathcal{M}_x$.

Let $y \neq x$. In like manner, for $\mathcal{M}_y$ there exists a polynomial p_y such that $B_j = p_y(A_j)$ on $\mathcal{M}_y$. Consider now the vector $z = x + y$. Then $\mathcal{M}_z \subset \mathcal{M}_j$, p_z is a polynomial such that $B_j = p_z(A_j)$ on $\mathcal{M}_z$. We than have $B_j x = p_z(A_j)x$ and $B_j y = p_z(A_j)y$, and therefore $p_z = p_x + k_x m_x = p_y + k_y m_y$ for suitable polynomials k_x and k_y; m_x and m_y are the minimum polynomials of $A_j \mid \mathcal{M}_x$ and $A_j \mid \mathcal{M}_y$, respectively. Since the operators $A_j \mid \mathcal{M}_x$ and $A_j \mid \mathcal{M}_y$ are p-primes, we can assume that $m_x \mid m_y$ and we conclude that $p_x(A) = p_y(A)$ on $\mathcal{M}_y$. Hence $B_j = p_j(A_j)$ on $\mathcal{M}_j$ and $B = \bigoplus_{j=1}^{k} p_j(A_j)$. The other implication is trivial. □

There are few immediate consequences of Theorem 1 for the reflexivity of algebraic operators.

Corollary 1. *An algebraic operator A is reflexive if and only if* $\operatorname{Alg}\operatorname{Lat} A \subseteq \{A\}'$.

Proof. The inclusion $\operatorname{Alg}\operatorname{Lat} A \subseteq \{A\}'$ is clearly true if A is reflexive. Conversely, if A is algebraic, then by Theorem 1 we have the inclusion $\operatorname{Alg}\operatorname{Lat} A = \operatorname{Alg}\operatorname{Lat} A \cap \{A\}' \subseteq a(A)$. The inclusion $a(A) \subseteq \operatorname{Alg}\operatorname{Lat} A \cap \{A\}'$ is true for an arbitrary operator. □

Corollary 2. *If A_1 and A_2 are reflexive algebraic operators with relatively prime minimal polynomials p_1 and p_2, then:*

(1) *The operator $A_1 \oplus A_2$ is reflexive.*

(2) *$a(A_1 \oplus A_2)$ splits.*

Proof. (1) Let $B \in \operatorname{Alg Lat}(A_1 \oplus A_2)$ and $\mathcal{M} \in \operatorname{Lat}(A_1 \oplus A_2) \subset \operatorname{Lat} B$. Since $(p_1, p_2) = 1$, we have that the operator $A_1 \oplus A_2$ splits. So, $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, $\mathcal{M}_j \in \operatorname{Lat} A_j$, $j = 1,2$. It is easy to verify that $B = B_1 \oplus B_2$, $\operatorname{Lat} A_j \subset \operatorname{Lat} B_j$ and $A_j B_j = B_j A_j, j = 1,2$. Finally we have: $(A_1 \oplus A_2)B = B(A_1 \oplus A_2)$ and $\operatorname{Lat}(A_1 \oplus A_2) \subset \{A_1 \oplus A_2\}'$, so $A_1 \oplus A_2$ is reflexive.

(2) $\mathfrak{a}(A_1 \oplus A_2) = \operatorname{Alg Lat}(A_1 \oplus A_2) = \operatorname{Alg Lat} A_1 \oplus \operatorname{Alg Lat} A_2 \supseteq \mathfrak{a}(A_1) \oplus \mathfrak{a}(A_2)$. Since $\mathfrak{a}(A_1) \oplus \mathfrak{a}(A_2) \subseteq \mathfrak{a}(A_1 \oplus A_2)$ holds for arbitrary operators, the corollary follows. □

Corollary 3. *Let A be an algebraic operator, $\mathcal{M}_x = \vee\{A^n x\}, x \in \mathcal{H}$. If $A \mid \mathcal{M}_x$ is reflexive for every x, then the operator A is reflexive.*

Proof. Let $B \in \operatorname{Alg Lat} A$, then $\mathcal{M}_x \in \operatorname{Lat} B$ for every $x \in \mathcal{H}$ and $B \mid \mathcal{M}_x \in \operatorname{Alg Lat}(A \mid \mathcal{M}_x)$. If $A \mid \mathcal{M}_x$ is reflexive, then $B \mid \mathcal{M}_x \in \{A \mid \mathcal{M}_x\}'$, so we have $\operatorname{Ker}(BA - AB) \supseteq \vee_x\{\mathcal{M}_x\} = \mathcal{H}$, i.e. $B \in \{A\}'$. The reflexivity of A follows from corollary 1. □

Next we will prove that the reflexivity of algebraic operators is a quasisimilarity invariant. We will then be able to characterize reflexive operators in terms of their Jordan models (see [5]). Recall, an operator $A \in B(\mathcal{H})$ is a *quasi-affine transform* of the operator $B \in B(\mathcal{H})$ if there exists a quasi-affinity (an injection with dense range) operator X such that $XA = BX$. A and B are said to be quasi-similar if there are quasi-affine transforms of each other.

Corollary 4. *Let A and A' be two quasi-similar algebraic operators such that $XY \in \operatorname{Alg Lat} A$, for the quasi-affinities X and Y. Then A is reflexive if and only if A' is reflexive.*

Proof. Let X and Y be quasi-affine transforms such that $AX = XA'$ and $YA = A'Y$. Assume that A is reflexive, $B' \in \operatorname{Alg Lat} A'$ and $\mathcal{M} \in \operatorname{Lat} A$. Then we have $\overline{Y\mathcal{M}} \in \operatorname{Lat} A' \subseteq \operatorname{Lat} B'$, so that $XB'Y\mathcal{M} \subset X\overline{Y\mathcal{M}} \subseteq \overline{XY\mathcal{M}} = \overline{q(A)\mathcal{M}} \subseteq \mathcal{M}$. Note that $q(A) = XY$ from Theorem 1, since $XY \in \operatorname{Alg Lat} A$ and A is algebraic. We conclude that $XB'Y \in \operatorname{Alg Lat} A$. Now A is reflexive, so $XB'Y \in \{A\}'$ i.e. $AXB'Y = XB'YA$. Hence, $XA'B'Y = XB'A'Y$. Because X and Y are quasi-affinities, the last equality shows that B' commutes with A'. The operator B' was arbitrary in $\operatorname{Alg Lat} A'$, and therefore A' is reflexive from corollary 1.

Now, remark that $X(YX - q(A')) = (XY - q(A))X = 0$, and hence $YX = q(A')$ because X is quasi-affinity. The corollary follows obviously by reasons of symmetry. □

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