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QUASI-PERIODIC SOLUTIONS TO THE ABEL DIFFERENTIAL EQUATION

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Abstract. As it is known, solving an Abel differential equation is not an easy problem. In this paper, using the idea ([1], [2], [3]) for finding existence conditions of quasi-periodic solutions for linear and nonlinear differential equations of first and second order, we give some conditions of existence quasi-periodic solutions with a constant quasi-period for the Abel differential equation (1) and find them.

1. PRELIMINARY

Let the Abel differential equation

$$y'(x) = f_3(x)y^3(x) + f_2(x)y^2(x) + f_1(x)y(x) + f_0(x), \; f_3(x) \neq 0 \tag{1}$$

be given. We want to find a quasi-periodic solution (QPS) $y = y(x)$ for the DE (1), i.e. to find a solution which satisfies the relation

$$y(x+\omega) = \lambda(x, \omega(x))y(x) = \lambda(x)y(x), \quad x, x+\omega \in D_y, \tag{2}$$

where $\omega = \omega(x)$ is called a quasi-period (QP) and $\lambda = \lambda(x)$ is called a quasi-periodic coefficient (QPC) of the function $y = y(x)$. The following theorem holds.

Theorem 1.1. *If DE (1) has QPS $y = \overline{y}(x)$ with QP $\omega = \omega(x)$ and QPC $\lambda(x)$, then $y = \overline{y}(x)$ is also QPS to an algebraic equation with respect to y*

$$\lambda\left(\frac{1}{t'}f_3(x) - \lambda^2(x)f_3(t)\right)y^3(x) + \lambda(x)\left(\frac{1}{t'}f_2(x) - \lambda(x)f_2(t)\right)y^2(x) + \tag{3}$$
$$+\left(\frac{\lambda'(x)}{t'} + \frac{\lambda(x)}{t'}f_1(x) - \lambda(x)f_1(t)\right)y(x) + \left(\frac{\lambda(x)}{t'}f_0(x) - f_0(t)\right) = 0, t = x + \omega(x)$$

or, to the Riccati DE with respect to y

$$\lambda(x)\left(\frac{1}{t'} - \lambda^2(x)\frac{f_3(t)}{f_3(x)}\right)y'(x) + \left(\frac{1}{t'}\lambda'(x) + \lambda^3(x)\frac{f_3(t)}{f_3(x)}f_1(x) - \lambda(x)f_1(t)\right)y(x) + \tag{4}$$
$$+\lambda(x)\left(\lambda(x)\frac{f_3(t)}{f_3(x)}f_2(x) - f_2(t)\right)y^2(x) + \left(\lambda^3(x)\frac{f_3(t)}{f_3(x)}f_0(x) - f_0(t)\right) = 0, \; t = x + \omega(x)$$

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Proof. Under the conditions of the theorem we form the system:

$$\left.\begin{array}{l} y'(x) = f_3(x)y^3(x) + f_2(x)y^2(x) + f_1(x)y(x) + f_0(x) \\ y'(t) = f_3(t)y^3(t) + f_2(t)y^2(t) + f_1(t)y(t) + f_0(t)_{/t=x+\omega} \\ y(t) = \lambda(x)y(x) \\ \dfrac{d}{dx}y(t) = \lambda'(x)y(x) + \lambda(x)y'(x) \end{array}\right\} \quad (5)$$

from where we get

$$y^3(x) = \frac{1}{f_3(x)}\left(y'(x) - f_2(x)y^2(x) - f_1(x)y(x) - f_0(x)\right) \quad (6)$$

and

$$y'(t) = \frac{1}{t'}\left(\lambda'(x)\,y(x) + \lambda(x)\,y'(x)\right). \quad (7)$$

Substituting expressions (6) and (7) in system (5), after short transformations, we obtain (3) and (4). □

Remark 1.1. In general, solving equations (3) and (4) is not an easy problem. So, in this paper we consider the problem for existence of QPS to the Abel DE(1) with a constant QP $\omega = c$ and a constant QPC $\lambda > 0$.

2. QUASI-PERIODIC SOLUTIONS FOR CONSTANT QP AND CONSTANT QPC

Theorem 2.1. *If DE(1) has QPS $y = \overline{y}(x)$ with a constant QP $\omega = c$ and a constant QPC $\lambda > 0$, then $y = \overline{y}(x)$ is QPS to the algebraic equation*

$$\begin{gathered}\lambda(f_3(x) - \lambda^2 f_3(t))y^3(x) + \lambda(f_2(x) - \lambda f_2(t))y^2(x) + \\ +\lambda(f_1(x) - f_1(t))y(x) + (\lambda f_0(x) - f_0(t)) = 0\end{gathered} \quad (8)$$

or to the Riccati DE

$$\begin{gathered}\lambda(f_3(x) - \lambda^2 f_3(t))y'(x) + \lambda^2(\lambda f_2(x)f_3(t) - f_2(t)f_3(x))y^2(x) + \\ + \lambda(\lambda^2 f_1(x)f_3(t) - f_1(t)f_3(x))y(x) + (\lambda^3 f_0(x)f_3(t) - f_0(t)f_3(x)) = 0\end{gathered} \quad (9)$$

Proof. Substituting $\omega = c$, $\omega' = 0$, $t = x + c$, $t' = 1$ into (3) and (4), we obtain (8) and (9). □

Corrolary 2.1. *Let DE (1) have QPS $y = \overline{y}(x)$ with a constant QP $\omega = c$ and a constant QPC $\lambda > 0$. If the coefficients $f_3(x), f_2(x), f_1(x), f_0(x)$ in DE (1) are QPFs with the same QP $\omega = c$ as $\overline{y}(x)$ and QPC $\lambda_3 = \frac{1}{\lambda^2}, \lambda_2 = \frac{1}{\lambda}, \lambda_1, \lambda_0$ respectively, i.e. the relations*

$$f_3(t) = \frac{1}{\lambda^2}f_3(x),\quad f_2(t) = \frac{1}{\lambda}f_2(x),\quad f_1(t) = \lambda_1 f_1(x), f_0(t) = \lambda_0 f_0(x)$$

are satisfied, then $y = \overline{y}(x)$ is also QPS to the equation

$$\lambda(1 - \lambda_1)f_1(x)y(x) + (\lambda - \lambda_0)f_0(x) = 0. \quad (10)$$

Corrolary 2.2. *Let DE (1) have QPS $y = \overline{y}(x)$ with a constant QP $\omega = c$ and a constant QPC $\lambda > 0$. If the coefficients $f_3(x), f_2(x), f_1(x), f_0(x)$ in DE (1) are QPFs with the same QP $\omega = c$ as $\overline{y}(x)$ and QPC $\lambda_3 = \frac{1}{\lambda^2}, \lambda_2, \lambda_1, \lambda_0$ respectively, i.e. the relations*

$$f_3(t) = \frac{1}{\lambda^2} f_3(x), \quad f_2(t) = \lambda_2 f_2(x), \quad f_1(t) = \lambda_1 f_1(x), f_0(t) = \lambda_0 f_0(x)$$

are satisfied, then $y = \overline{y}(x)$ is also QPS to the equation

$$\lambda(1 - \lambda\lambda_2) f_2(x) y^2(x) + \lambda(1 - \lambda_1) f_1(x) y(x) + (\lambda - \lambda_0) f_0(x) = 0 \qquad (11)$$

Corrolary 2.3. *Let DE (1) have QPS $y = \overline{y}(x)$ with a constant QP $\omega = c$ and a constant QPC $\lambda > 0$. If the coefficients $f_3(x), f_2(x), f_1(x), f_0(x)$ in DE (1) are QPFs with the same QP $\omega = c$ as $\overline{y}(x)$ and QPC $\lambda_3, \lambda_2, \lambda_1, \lambda_0$ respectively, i.e. the relations*

$$f_3(t) = \lambda_3 f_3(x), \quad f_2(t) = \lambda_2 f_2(x), \quad f_1(t) = \lambda_1 f_1(x), f_0(t) = \lambda_0 f_0(x),$$

are satisfied, then $y = \overline{y}(x)$ is also QPS to the equation

$$\lambda(1-\lambda^2\lambda_3) f_3(x) y^3(x) + \lambda(1-\lambda\lambda_2) f_2(x) y^2(x) + \lambda(1-\lambda_1) f_1(x) y(x) + (\lambda-\lambda_0) f_0(x) = 0 \qquad (12)$$

Corrolary 2.4. *Let DE (1) have QPS $y = \overline{y}(x)$ with a constant QP $\omega = c$ and a constant QPC $\lambda > 0$. If the coefficients $f_3(x), f_2(x), f_1(x), f_0(x)$ in DE (1) are QPFs with the same QP $\omega = c$ as $\overline{y}(x)$ and QPC $\lambda_3, \lambda_2 = \lambda\lambda_3, \lambda_1, \lambda_0$ respectively, i.e. the relations*

$$f_3(t) = \lambda_3 f_3(x), \quad f_2(t) = \lambda\lambda_3 f_2(x), \quad f_1(t) = \lambda_1 f_1(x), f_0(t) = \lambda_0 f_0(x)$$

are satisfied, then $y = \overline{y}(x)$ is also QPS to the equation

$$\lambda(1 - \lambda^2\lambda_3) y'(x) + \lambda(\lambda^2\lambda_3 - \lambda_1) f_1(x) y(x) + (\lambda^3\lambda_3 - \lambda_0) f_0(x) = 0 \qquad (13)$$

Corrolary 2.5. *Let DE (1) have QPS $y = \overline{y}(x)$ with a constant QP $\omega = c$ and a constant QPC $\lambda > 0$. If the coefficients $f_3(x), f_2(x), f_1(x), f_0(x)$ in DE (1) are QPFs with the same QP $\omega = c$ as $\overline{y}(x)$ and QPC $\lambda_3 \neq \frac{1}{\lambda^2}, \lambda_2, \lambda_1, \lambda_0$ respectively, i.e. the relations*

$$f_3(t) = \lambda_3 f_3(x), \quad f_2(t) = \lambda_2 f_2(x), \quad f_1(t) = \lambda_1 f_1(x), f_0(t) = \lambda_0 f_0(x),$$

are satisfied, then $y = \overline{y}(x)$ is also QPS to the equation

$$y'(x) + \frac{\lambda^2\lambda_3 - \lambda_2}{1-\lambda^2\lambda_3} f_2(x) y^2(x) + \frac{\lambda^2\lambda_3 - \lambda_1}{1-\lambda^2\lambda_3} f_1(x) y(x) + \frac{\lambda^3\lambda_3 - \lambda_0}{\lambda(1-\lambda^2\lambda_3)} f_0(x) = 0 \qquad (14)$$

Theorem 2.2. *Let DE (1) have QPS $y = \overline{y}(x)$ with a constant QP $\omega = c$ and a constant QPC $\lambda > 0$. If the coefficients $f_3(x), f_2(x), f_1(x), f_0(x)$ in DE (1) are QPFs with the same QP $\omega = c$ as $\overline{y}(x)$ and QPC $\lambda_3 = \frac{1}{\lambda^2}, \lambda_2 = \frac{1}{\lambda}, \lambda_1 = 1, \lambda_0 = \lambda$ respectively, then DE (1) has many QPS with a constant QP $\omega = c$ and a constant QPC $\lambda > 0$.*

Proof. Under the conditions of the theorem, coefficients $f_3(x), f_2(x), f_1(x), f_0(x)$ in DE (1) satisfy the relations

$$f_3(t) = \frac{1}{\lambda^2} f_3(x), \quad f_2(t) = \frac{1}{\lambda} f_2(x), \quad f_1(t) = f_1(x), f_0(t) = \lambda f_0(x),$$

and equations from (10) to (14) can have many QPSs, which are also QPSs to DE (1), since

$$\begin{aligned} &y'(t) - f_3(t)y^3(t) - f_2(t)y^2(t) - f_1(t)y(t) - f_0(t)/_{t=x+c} = \\ &\qquad \lambda(y'(x) - f_3(x)y^3(x) - f_2(x)y^2(x) - f_1(x)y(x) - f_0(x)) = \lambda \cdot 0 = 0. \end{aligned}$$

□

Example 2.1. Let

$$y' = -\frac{e^{-2x}\cos x}{\sin^2 x(1-\sin x)} y^3 + \frac{e^{-x}(\cos x + \sin^2 x - \sin^3 x)}{\sin^2 x(1-\sin x)} y^2 - \sin x \cdot y + e^x \sin x.$$

The coefficients $f_3(x) = -\frac{e^{-2x}\cos x}{\sin^2 x(1-\sin x)}$, $f_2(x) = \frac{e^{-x}(\cos x + \sin^2 x - \sin^3 x)}{\sin^2 x(1-\sin x)}$, $f_1(x) = -\sin x$, $f_0(x) = e^x \sin x$ are QPF with the same QP $\omega = 2\pi$ and QPCs $\lambda_3 = e^{-4\pi} = \frac{1}{\lambda^2}$, $\lambda_2 = e^{-2\pi} = \frac{1}{\lambda}$, $\lambda_1 = 1$, $\lambda_0 = e^{2\pi} = \lambda$, respectively. So, according to the Theorem 2.2 , the given equation can have many particular QPSs. Thus, for instance, $y_1 = e^x \sin x$ and $y_2 = e^x$ are particular solutions which are QPFs with QP $\omega = 2\pi$ and QPC $\lambda = e^{2\pi}$.

Remark 2.1. If DE (1) has four QPSs $y_1(x), y_2(x), y_3(x), y_4(x)$ with a constant QP $\omega = c$ and a constant QPC $\lambda > 0$, then from the system

$$\begin{aligned} f_0(x) + f_1(x)y_1 + f_2(x)y_1^2 + f_3(x)y_1^3 &= y_1'(x) \\ f_0(x) + f_1(x)y_2 + f_2(x)y_2^2 + f_3(x)y_2^3 &= y_2'(x) \\ f_0(x) + f_1(x)y_3 + f_2(x)y_3^2 + f_3(x)y_3^3 &= y_3'(x) \\ f_0(x) + f_1(x)y_4 + f_2(x)y_4^2 + f_3(x)y_4^3 &= y_4'(x) \end{aligned}$$

coefficients $f_3(x), f_2(x), f_1(x), f_0(x)$ are uniquely determined by the relations:

$$f_0(x) = \frac{D_0}{D}, f_1(x) = \frac{D_1}{D}, f_2(x) = \frac{D_2}{D}, f_3(x) = \frac{D_3}{D},$$

where

$$D = \begin{vmatrix} 1 & y_1 & y_1^2 & y_1^3 \\ 1 & y_2 & y_2^2 & y_2^3 \\ 1 & y_3 & y_3^2 & y_3^3 \\ 1 & y_4 & y_4^2 & y_4^3 \end{vmatrix} = (y_1 - y_2)(y_1 - y_3)(y_1 - y_4)(y_2 - y_3)(y_2 - y_4)(y_3 - y_4) \neq 0$$

is Wandermond's determinant and

$$D_0 = \begin{vmatrix} y_1' & y_1 & y_1^2 & y_1^3 \\ y_2' & y_2 & y_2^2 & y_2^3 \\ y_3' & y_3 & y_3^2 & y_3^3 \\ y_4' & y_4 & y_4^2 & y_4^3 \end{vmatrix}, \quad D_1 = \begin{vmatrix} 1 & y_1' & y_1^2 & y_1^3 \\ 1 & y_2' & y_2^2 & y_2^3 \\ 1 & y_3' & y_3^2 & y_3^3 \\ 1 & y_4' & y_4^2 & y_4^3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} 1 & y_1 & y_1' & y_1^3 \\ 1 & y_2 & y_2' & y_2^3 \\ 1 & y_3 & y_3' & y_3^3 \\ 1 & y_4 & y_4' & y_4^3 \end{vmatrix},$$
$$D_3 = \begin{vmatrix} 1 & y_1 & y_1^2 & y_1' \\ 1 & y_2 & y_2^2 & y_2' \\ 1 & y_3 & y_3^2 & y_3' \\ 1 & y_4 & y_4^2 & y_4' \end{vmatrix}.$$

Coefficients $f_3(x), f_2(x), f_1(x), f_0(x)$, determined above, satisfy conditions to the Theorem 2.2.

Remark 2.2. If the DE (1) has four QPSs $y_1(x), y_2(x), y_3(x), y_4(x)$ with a constant QP $\omega = c$ and a constant QPC $\lambda > 0$, then for the system

$$\begin{aligned} f_0(x) + f_1(x)y_1 + f_2(x)y_1^2 + f_3(x)y_1^3 - y_1'(x) = 0 \\ f_0(x) + f_1(x)y_2 + f_2(x)y_2^2 + f_3(x)y_2^3 - y_2'(x) = 0 \\ f_0(x) + f_1(x)y_3 + f_2(x)y_3^2 + f_3(x)y_3^3 - y_3'(x) = 0 \\ f_0(x) + f_1(x)y_4 + f_2(x)y_4^2 + f_3(x)y_4^3 - y_4'(x) = 0 \\ f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3 - y'(x) = 0 \end{aligned} \tag{15}$$

we obtain the equation

$$\begin{vmatrix} 1 & y_1 & y_1^2 & y_1^3 & y_1' \\ 1 & y_2 & y_2^2 & y_2^3 & y_2' \\ 1 & y_3 & y_3^2 & y_3^3 & y_3' \\ 1 & y_4 & y_4^2 & y_4^3 & y_4' \\ 1 & y & y^2 & y^3 & y' \end{vmatrix} = 0 \tag{16}$$

which satisfies conditions of the Theorem 2.2.

Theorem 2.3. *Let the DE (1) have QPS $y = \overline{y}(x)$ with a constant QP $\omega = c$ and a constant QPC $\lambda > 0$ and let the coefficients $f_3(x), f_2(x), f_1(x), f_0(x)$ be QPFs with the same constant QP $\omega = c$ and QPC $\lambda_3, \lambda_2, \lambda_1, \lambda_0 = \lambda\lambda_1$, respectively. Then*

$$\overline{y} = -\frac{f_0(x)}{f_1(x)}, \tag{17}$$

if the relation

$$\left(\frac{f_0(x)}{f_1(x)}\right)' - f_3(x)\left(\frac{f_0(x)}{f_1(x)}\right)^3 + f_2(x)\left(\frac{f_0(x)}{f_1(x)}\right)^2 = 0 \tag{18}$$

is satisfied.

Proof. Using the conditions $f_3(x+c) = \lambda_3 f_3(x)$, $f_2(x+c) = \lambda_2 f_2(x)$, $f_1(x+c) = \lambda_1 f_1(x)$, $f_0(x+c) = \lambda_0 f_0(x)$ relations (8) and (9), and Corollaries 2.1. to 2.5. we have:

a) If $\lambda_3 = \left(\frac{\lambda_1}{\lambda_0}\right)^2, \lambda_2 = \frac{\lambda_1}{\lambda_0}, \lambda_1 \neq 1$, then from the Theorem 2.1 follows that QPS $\overline{y}(x)$ to the DE(1) is also QPS to the equation

$$\lambda\left(\lambda_1 - 1\right) f_1(x) y + \left(\lambda_0 - \lambda\right) f_0(x) = 0.$$

From the last equation we get

$$\bar{y} = -\mu_1 \frac{f_0(x)}{f_1(x)} \tag{19}$$

where $\mu_1 = \frac{\lambda - \lambda_0}{\lambda\left(1 - \lambda_1\right)}$. Solution (19) is QPF with QP $\omega = c$ and QPC $\lambda = \frac{\lambda_0}{\lambda_1}$ for which $\mu_1 = 1$. Thus, from (19) we obtain (17) and (18).

b) If $\lambda_3 = \left(\frac{\lambda_1}{\lambda_0}\right)^2, \lambda_1 \neq 1, \lambda_2 \neq \lambda\lambda_1, i.e. \lambda_1 \neq 1, \lambda_2 \neq \lambda_0, \lambda_3\lambda_0^2 = \lambda_1^2$, then from the Theorem 2.1 follows that QPS $\overline{y}(x)$ to DE(1) is also QPS to the equation

$$\lambda(1 - \lambda\lambda_2) f_2(x) y^2(x) + \lambda(1 - \lambda_1) f_1(x) y(x) + (\lambda - \lambda_0) f_0(x) = 0,$$

whose coefficients are QPF, so its QPS $\overline{y}(x)$ is also QPS to the equation

$$\lambda^2(1 - \lambda_1)(\lambda_2 - \lambda\lambda_1) f_1(x) y(x) + (\lambda - \lambda_0)(\lambda_0 - \lambda^2\lambda_2) f_0(x) = 0.$$

From the last equation, we obtain

$$\overline{y} = -\mu_2 \frac{f_0(x)}{f_1(x)} \tag{20}$$

where $\mu_2 = \frac{(\lambda - \lambda_0)(\lambda_0 - \lambda^2\lambda_2)}{\lambda^2\left(1 - \lambda_1\right)\left(\lambda_2 - \lambda\lambda_1\right)}$. Solution (20) is QPF with QP $\omega = c$ and QPC $\lambda = \frac{\lambda_0}{\lambda_1}$ for which $\mu_2 = 1$. Thus, from (19) we obtain (17) and (18).

c) If $\lambda_1 \neq 1, \lambda_1 \neq \lambda\lambda_2, \lambda_1 \neq \lambda^2\lambda_3$ i.e. $\lambda_1 \neq 1, \lambda_1^3 \neq \lambda_0^2\lambda_3, \lambda_1^2 \neq \lambda_0^2\lambda_2$, then from the Theorem 2.1 follows that QPS $\overline{y}(x)$ to DE(1) is also QPS to the equation

$$\lambda(1 - \lambda^2\lambda_3) f_3(x) y^3(x) + \lambda(1 - \lambda\lambda_2) f_2(x) y^2(x) + \\ + \lambda(1 - \lambda_1) f_1(x) y(x) + (\lambda - \lambda_0) f_0(x) = 0,$$

whose coefficients are QPF, so its QPS $\overline{y}(x)$ is also QPS to the equation

$$\lambda^3(1 - \lambda\lambda_2)(\lambda_2 - \lambda\lambda_3) f_2(x) y^2(x) + \lambda^2(1 - \lambda_1)(\lambda_1 - \lambda^2\lambda_3) f_1(x) y(x) + \\ + (\lambda - \lambda_0)(\lambda_0 - \lambda^3\lambda_3) f_0(x) = 0,$$

i.e. to the equation

$$\lambda^3(1 - \lambda_1)(\lambda_1 - \lambda^2\lambda_3)(\lambda_1 - \lambda\lambda_2) f_1(x) y(x) + (\lambda - \lambda_0)(\lambda_0 - \lambda^3\lambda_3)(\lambda_0 - \lambda^2\lambda_2) f_0(x) = 0.$$

From the last equation we get

$$\overline{y} = -\mu_3 \frac{f_0(x)}{f_1(x)}, \tag{21}$$

where $\mu_3 = \dfrac{(\lambda_0-\lambda)(\lambda_0-\lambda^3\lambda_3)(\lambda_0-\lambda^2\lambda_2)}{\lambda^3(\lambda_1-1)(\lambda_1-\lambda^2\lambda_3)(\lambda_1-\lambda\lambda_2)}$. Solution (21) is QPF with QP $\omega = c$ and QPC $\lambda = \dfrac{\lambda_0}{\lambda_1}$ from where we obtain $\mu_3 = 1$. Thus, from (21) we get (17) and (18).

d) If $\lambda_2 = \lambda\lambda_3,\ \lambda^2\lambda_3 - \lambda_1 \neq 0,\ \lambda_1 \neq 1$ i.e. $\lambda_1 \neq 1,\ \lambda_0^2\lambda_3 \neq \lambda_1^3,\ \lambda_2\lambda_1 = \lambda_0\lambda_3$, then from the Theorem 2.1 follows that QPS $\overline{y}(x)$ to DE (1) is also QPS to the equation

$$\lambda(1-\lambda^2\lambda_3)y'(x) + \lambda(\lambda^2\lambda_3-\lambda_1)f_1(x)y(x) + (\lambda^3\lambda_3-\lambda_0)f_0(x) = 0$$

i.e.[1] to the equation

$$\lambda^2(\lambda_1-1)(\lambda_1-\lambda^2\lambda_3)f_1(x)y(x) + (\lambda_0-\lambda)(\lambda_0-\lambda^3\lambda_3)f_0(x) = 0.$$

From the last equation we find

$$\overline{y} = -\mu_4\frac{f_0(x)}{f_1(x)} \tag{22}$$

where $\mu_4 = \dfrac{(\lambda-\lambda_0)(\lambda_0-\lambda^3\lambda_3)}{\lambda^3(1-\lambda_1)(\lambda_1-\lambda^2\lambda_3)}$. Solution (22) is QPF with QP $\omega = c$ and QPC $\lambda = \dfrac{\lambda_0}{\lambda_1}$ from where we obtain $\mu_4 = 1$. Thus, from (22) we obtain (17) and (18).

e) If $\lambda_3 \neq \left(\dfrac{\lambda_1}{\lambda_0}\right)^2,\ \lambda_1 \neq 1,\ \lambda^2\lambda_3 \neq \lambda_1,\ \lambda_1 \neq \lambda\lambda_2$ i.e. $\lambda_0^2\lambda_3 \neq \lambda_1^3,\ \lambda_1 \neq 1,\ \lambda_1^2 \neq \lambda_0\lambda_2,\ \lambda_0^2\lambda_3 \neq \lambda_1^2$ then QPS to DE (1) is also QPS to the equation

$$y'(x) + \frac{\lambda^2\lambda_3-\lambda_2}{1-\lambda^2\lambda_3}f_2(x)y^2(x) + \frac{\lambda^2\lambda_3-\lambda_1}{1-\lambda^2\lambda_3}f_1(x)y(x) + \frac{\lambda^3\lambda_3-\lambda_0}{\lambda(1-\lambda^2\lambda_3)}f_0(x) = 0,$$

and

$$\frac{\lambda(\lambda^2\lambda_3-\lambda_2)(\lambda\lambda_2-1)}{1-\lambda^2\lambda_3}f_2(x)y^2 + \frac{\lambda(\lambda_1-1)(\lambda^2\lambda_3-\lambda_1)}{1-\lambda^2\lambda_3}f_1(x)y + \\ + \frac{(\lambda_0-\lambda)(\lambda^3\lambda_3-\lambda_0)}{\lambda(1-\lambda^2\lambda_3)}f_0(x) = 0,$$

i.e.

$$\frac{\lambda^2(\lambda_1-1)(\lambda^2\lambda_3-\lambda_1)(\lambda_1-\lambda\lambda_2)}{1-\lambda^2\lambda_3}f_1(x)y + \frac{(\lambda_0-\lambda)(\lambda^3\lambda_3-\lambda_0)(\lambda_0-\lambda^2\lambda_2)}{\lambda(1-\lambda^2\lambda_3)}f_0(x) = 0.$$

From the last equation we get

$$\overline{y} = -\mu_5\frac{f_0(x)}{f_1(x)}, \tag{23}$$

where $\mu_5 = \dfrac{(\lambda_0-\lambda)(\lambda_0-\lambda^2\lambda_2)(\lambda_0-\lambda^3\lambda_3)}{\lambda^3(\lambda_1-1)(\lambda_1-\lambda\lambda_2)(\lambda_1-\lambda^2\lambda_3)}$. Since QPC for $\overline{y}(x)$ is $\lambda = \dfrac{\lambda_0}{\lambda_1}$, then under the given conditions we have $\mu_5 = 1$. Now, from (23) we obtain (17) and (18). □

Example 2.2. The equation

$$y' = 2e^{-5x+3\sin x}\cos x \cdot y^3 + e^{-2x+\sin x}(-2+\cos x)\cdot y^2 + e^{-x+\sin x}\cos x \cdot y - e^x \cos x$$

has coefficients $f_3(x) = 2e^{-5x+3\sin x}\cdot\cos x$, $f_2(x) = e^{-2x+\sin x}\cdot(-2+\cos x)$, $f_1(x) = e^{-x+\sin x}\cos x$, $f_0(x) = -e^x\cdot\cos x$, which are QPF with QP $\omega = 2\pi$ and QPC $\lambda_0 = e^{2\pi}$, $\lambda_1 = e^{-2\pi}$, $\lambda_2 = e^{-4\pi}$, $\lambda_3 = e^{-10\pi}$ respectively. Thus, according to the Theorem 2.3., QPS for the given equation is

$$\bar{y} = -\frac{f_0(x)}{f_1(x)} = e^{2x-\sin x}, \ \left(\omega = 2\pi,\ \lambda = e^{4\pi} = \frac{\lambda_0}{\lambda_1}\right).$$

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