Math. Maced.
Vol. 8 (2010)
69-77

RELATIONSHIP BETWEEN VECTOR VALUED SEMIGROUP AUTOMATA AND RANDOM PROCESSES

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Dedicated to Academician Ǵorǵi Čupona

Abstract. The aim of the talk is to examine a vector valued semigroup automata as a discrete random process. There will be established a relationship between a transition function from one to other state of vector valued semigroup automata and an appropriate matrix probabilities for transition from one to other state. Then, using this matrix probabilities the properties of vector valued semigroup automata will be examined. Also it will be discussed the ergodicity on the set of states over the matrix probability with their properties and the probability of all states after n-transitions which forms a matrix of final probabilities.

1. Introduction

The notion of semigroup automaton is introduced in [1] and [2]. Here we recall the necessary definitions and known results on vector valued semigroup automata.

From now on, let n, k be positive integers, such that $1 \leq k \leq n$.

Let B be a nonempty set and let $\{\ \} : B^{n+k} \to B^n$ be a mapping (B^i denotes the i-th Cartesian power of B). Then we say that $(B, \{\ \})$ is $(n+k, n)$-**groupoid** or **vector valued groupoid**. If $\{\ \}((x_1, ..., x_{n+k})) = (y_1, ..., y_n)$, then we denote $\{x_1^{n+k}\} = (y_1^n)$; the symbol z_i^j will denote the sequence $z_i z_{i+1} ... z_j$ when $i \leq j$, and empty sequence when $i > j$.

An $(n+k, n)$-groupoid $(B, \{\ \})$ is called $(n+k, n)$-**semigroup** or **vector valued semigroup** iff $\{\{x_1^{n+k}\}x_{n+k+1}^{n+2k}\} = \{x_1^j\{x_{j+1}^{j+n+k}\}x_{j+n+k+1}^{n+2k}\}$ for every $1 \leq j \leq k$.

Key words and phrases. vector valued semigroup automata, random process, matrix of probability.

Example 1.1. *Let* $B = \{a, b\}$, $n = 2$, $k = 1$ *and let* $\{\ \} : B^3 \to B^2$ *be given by the Table 1.1. Than* $(B, \{\})$ *is a* $(3,2)$*-semigroup.*

This example of $(3,2)$*-semigroup is generated by an appropriate computer program.*

{ }	
a a a	(b,a)
a a b	(a,a)
a b a	(a,a)
a b b	(b,a)
b a a	(a,a)
b a b	(b,a)
b b a	(b,a)
b b b	(a,a)

Table 1.1

A **semigroup automaton** is a triple $(S, (B, \bullet), f)$, where S is a set, $(B, \bullet)$ is a semigroup, and $f : S \times B \to S$ is a map satisfying

$$f(f(s, x), y) = f(s, x \bullet y) \text{ for every } s \in S,\ x, y \in B.$$

The set S is called the set of states of $(S, (B, \bullet), f)$ and f is called the **transition function** of $(S, (B, \bullet), f)$.

An $(n+k, n)$**-semigroup automaton** or **vector valued semigroup automaton** (VVSA) is a triple $(S, (B, \{\ \}), f)$, where S is a set, $(B, \{\ \})$ is $(n+k, n)$-semigroup, and $f : S \times B^n \to S \times B^{n-k}$ is a map satisfying $f(f(s, x_1^n), y_1^k) = f(s, \{x_1^n y_1^k\})$ for every $s \in S,\ x_1, ..., x_n, y_1, ..., y_k \in B$.
The set S is called set of **states** of $(S, (B, \{\ \}), f)$ and f is called **transition function** of $(S, (B, \{\ \}), f)$.
Example 1.1'. Let $(B, \{\ \ \})$ be $(3,2)$-semigroup from Example 1.1 and $S = \{s_0, s_1, s_2\}$. The $(3,2)$-semigroup automaton $(S, (B, \{\ \}), f)$is a set S together with a mapping $f : S \times B^2 \to S \times B$ satisfying $f(f(s, x_1^2), x_3) = f(s, \{x_1^3\})$ for every $s \in S,\ x_1, x_2, x_3 \in B$.

This $(3,2)$-semigroup automaton is given by the Table 1.1' and the Figure 1.

f	(a,a)	(a,b)	(b,a)	(b,b)
s_0	(s_1, b)	(s_2, b)	(s_2, b)	(s_1, b)
s_1	(s_1, b)	(s_0, a)	(s_2, b)	(s_1, b)
s_2	(s_2, b)	(s_0, b)	(s_1, b)	(s_2, b)

Table 1.1′

This example of $(3,2)$-semigroup automaton is generated by computer.

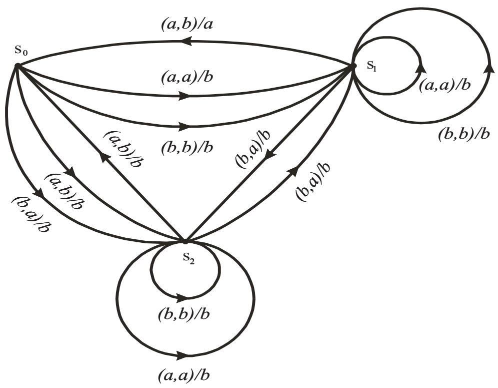

FIGURE 1

Let $\{X_t,\ t \in T\}$ b a random process with the argument $t \in T$.

In the theory of random processes terms such as probability vector and stochastic matrix are used, so we will define them first.

The vector $p = (p_1, p_2, ..., p_r)$ is called **probability vector** if its components are nonegative numbers and their sum is 1, i..

$$\sum_{i=1}^{r} p_i = 1.$$

Square matrix $P = [p_{ij}]$ is called **stochastic matrix** if every row is a probability vector, i.e. if every element of the matrix is nonegative number and the elements' sum in every row is 1.

The stochastic matrix P is a **regular matrix** if all elements on a degree $P^n,\ n \in N$ are positive numbers.

The basic properties of regular matrix are explained in the following theorem.

Theorem 1.1. *Let P be regular stochastic matrix. Then:*

a) for the matrix P there is only one fixed vector t which components are positive numbers and it is true that $tP = t$;

b) the array of degrees $P,\ P^2,\ P^3, ...$ on the matrix P converges to the matrix T which rows are equal to the fixed vector t on the matrix P;

c) if p is a probability vector then the vectors array $pP,\ pP^2,\ pP^3, ...$ converges to the fixed vector t.

2. VECTOR VALUED SEMIGROUP AUTOMATA AND RANDOM PROCESS

In this section interpretation of vector valued semigroup automata will be given through the random processes. Namely, vector valued semigroup automata can be explained as a mappings set of states set S in itself. On each transition from one to other state an appropriate probability can be followed. In addition, the probability of transition which refers to the following vector valued semigroup automata transitions, if the present state is fully known, won't be changed if additional information are known about the past of vector valued semigroup automata. That's why vector valued semigroup automata can be discussed directly as Markov's discrete processes, which are called Markov's chains.

Let $(S,(B,\{\ \}),f)$ be $(n+k,n)$-semigroup automaton. The alphabet B and the set of states S are finite sets, so let $|B|=m$ and $|S|=l$. We define a mapping $p: S\times B^{n-k}\to[0,1]$ so $p((s_i,y_1,y_2,\dots,y_{n-k}))=\frac{h}{m^n}$, where as h is a number of the pairs $((x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_{n-k}))$ satisfying $f(s_i,x_1,\ x_2,\ ...,x_n)=(s_j,y_1,\ y_2,\ ...,y_{n-k})$, for each $(s_j,y_1,y_2,\dots,y_{n-k})\in S\times B^{n-k}$.

Let's prove that p is a well defined mapping. If $f(s_i,x_1,x_2,...,x_n)=(s_j,y_1,y_2,...,y_{n-k})$ and $f(s_i,x_1,x_2,...,x_n)=(s_l,y_1,y_2,...,y_{n-k})$, we should prove that $p((s_j,y_1,y_2,...,y_{n-k}))=p((s_l,y_1,y_2,...,y_{n-k}))$. But, as f is a transition function on $(n+k,n)$-semigroup automaton $(S,(B,\{\ \}),f)$, it is uniquely determined, so it implicates that $(s_j,y_1,y_2,...,y_{n-k})=(s_l,y_1,y_2,...,y_{n-k})$, i.. $p((s_j,y_1,y_2,...,y_{n-k}))=p((s_l,y_1,y_2,...,y_{n-k}))$.

The mapping p is called **transition probability** on $(n+k,n)$-semigroup automaton $(S,(B,\{\ \}),f)$.

If the transition probability p_{ij} are known for each pair of states (s_i,s_j), then they are written in square matrix. The matrix with transition probabilities on $(n+k,n)$-semigroup automaton $(S,(B,\{\ \}),f)$ is called **a matrix on transition in one step on $(n+k,n)$-semigroup automaton** $(S,(B,\{\ \}),f)$:

$$P=\begin{bmatrix} p_{11}\ p_{12}\ \cdots\ p_{1l}\\ p_{21}\ p_{22}\ \cdots\ p_{2l}\\ \cdots\ \cdots\ \cdots\ \cdots\\ p_{l1}\ p_{l2}\ \cdots\ p_{ll}\end{bmatrix}=\begin{matrix} \\ s_1\\ s_2\\ \cdots\\ s_l\end{matrix}\begin{matrix} s_1\ \ s_2\ \cdots\ s_l\\ \begin{bmatrix} p_{11}\ p_{12}\ \cdots\ p_{1l}\\ p_{21}\ p_{22}\ \cdots\ p_{2l}\\ \cdots\ \cdots\ \cdots\ \cdots\\ p_{l1}\ p_{l2}\ \cdots\ p_{ll}\end{bmatrix}\end{matrix}.$$

On every state s_i $(i=1,\ 2,\ ...,\ l)$ corresponds the probabilities $p_{i1},\ p_{i2},\ ...,\ p_{il}$, for which $\sum_{j=1}^{l}p_{ij}=1$, and they consist the i-th row of the matrix P. If $(n+k,n)$-semigroup automaton is in state s_i, then this row represents the probabilities of all possible results in the following step. From here the following theorem results.

Theorem 2.1. *The probability matrix on transition P on $(n+k,n)$-semigroup automaton $(S,(B,\{\ \}),f)$ is a stochastic matrix.*

Example 2.1. *Let's determine the transition probability of $(3,2)$-semigroup automaton $(S,(B,\{\ \}),f)$ from the Example 1.1'. As the functioning of $(3,2)$-semigroup automaton is based on transition of one state to other, the representation of $(3,2)$-semigroup automaton given in Table 1.1' we transform it into Table 2.1. Here the transition function $f(s_i,x,y)=(s_j,z)$ appropriates to the representation $(x,y)/z$, which signs a transition on the word (x,y) from state s_i to the word z in state s.*

f	s_0	s_1	s_2
s_0	$\emptyset$	$\{(a,a)/b,(b,b)/b\}$	$\{(a,b)/b,(b,a)/b\}$
s_1	$\{(a,b)/a\}$	$\{(a,a)/a,(b,b)/b\}$	$\{(b,a)/b\}$
s_2	$\{(a,b)/b\}$	$\{(b,a)/b\}$	$\{(a,a)/b,(b,b)/b\}$

Table 2.1

The probability matrix on transition P on $(3,2)$-semigroup automaton $(S,(B,\{\ \}),f)$ is

$$P = \begin{array}{c} \\ s_0 \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{ccc} s_0 & s_1 & s_2 \end{array} \\ \left[\begin{array}{ccc} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{array}\right] \end{array}.$$

We note $\sum\limits_{j=1}^{3} p_{ij} = 1$ for $i = 1,2,3$, so P is stochastic matrix.

The state from which vector valued semigroup automata starts can be known or can be determined after stochastic rule. For some vector valued semigroup automata **an initial vector with probabilities** $p(0) = (p_1^0, p_2^0, ..., p_l^0)$ is given, in which p_i^0 is a probability that s_i is an initial state of vector valued semigroup automata.

The transition probability from state s_i to state s_j for q steps is called **transition probability in q steps** and it is signed as $p_{ij}^{(q)}$.

The square matrix formed from the all probabilities $p_{ij}^{(q)}$ is called **transition probability mtrix in q steps**.

Theorem 2.2. *If $P = P(1)$ is a transition probability matrix in one step for finite $(n+k,n)$-semigroup automaton $(S,(B,\{\ \}),f)$, then the transition probability matrix in q steps $P(q)$ is*
$P(q) = P^q$.

Proof. It is clear that $p_{ij}^{(1)} = p_{ij}$, where p_{ij} are elements of the matrix $P = P(1)$. The probability $p_{ij}(2)$ is transition probability from state s_i to state s_j in two steps (Figure 2.). These transitions are $s_i \to s_1 \to s_j,\ s_i \to s_2 \to s_j, \ldots, s_i \to s_t \to s_j$ in dependence of $(x_1, x_2, ..., x_n) \in B^n$. The probabilities of this transitions are appropriately, $p_{i1} \cdot p_{1j},\ p_{i2} \cdot p_{2j}, \ldots, p_{il} \cdot p_{lj}$. So, the transition probability of $(n+k,n)$-semigroup automaton from state s_i to state s_j in two steps is sum from these probabilities, i.e.

$$p_{ij}^{(2)} = \sum_{r=1}^{l} p_{ir} \cdot p_{rj}.$$

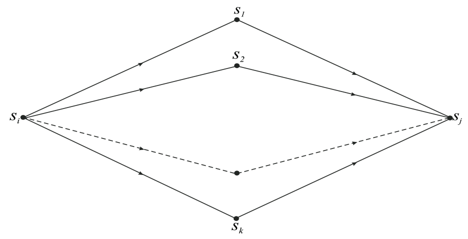

FIGURE 2

According to the rule for multipling matixes, the right side of the last formula is an element of the matrix P^2, which is found in the section of i-th row and j-th column:

$$P^2 = P \cdot P = \begin{bmatrix} p_{11} & p_{12} \cdots p_{1j} \cdots p_{1l} \\ \cdots & \cdots \\ p_{i1} & p_{i2} \cdots p_{ij} \cdots p_{il} \\ \cdots & \cdots \\ p_{l1} & p_{l2} \cdots p_{lj} \cdots p_{ll} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \cdots p_{1j} \cdots p_{1l} \\ \cdots & \cdots \\ p_{i1} & p_{i2} \cdots p_{ij} \cdots p_{il} \\ \cdots & \cdots \\ p_{l1} & p_{l2} \cdots p_{lj} \cdots p_{ll} \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{r=1}^{l} p_{1r}p_{r1} & \sum_{r=1}^{l} p_{1r}p_{r2} \cdots & \sum_{r=1}^{l} p_{1r}p_{rj} \cdots & \sum_{r=1}^{l} p_{1r}p_{rl} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{r=1}^{l} p_{ir}p_{r1} & \sum_{r=1}^{l} p_{ir}p_{r2} \cdots & \sum_{r=1}^{l} p_{ir}p_{rj} \cdots & \sum_{r=1}^{l} p_{ir}p_{rl} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{r=1}^{l} p_{lr}p_{r1} & \sum_{r=1}^{l} p_{lr}p_{r2} \cdots & \sum_{r=1}^{l} p_{lr}p_{rj} \cdots & \sum_{r=1}^{l} p_{lr}p_{rl} \end{bmatrix}$$

We can note that the matrix P^2 is equal with the matrix $P(2) = \left[p_{ij}^{(2)}\right]$, i.e. $P(2) = P^2$.

Continuing this treatment we come to

$$p_{ij}(q+1) = \sum_{r=1}^{l} p_{ir} p_{rj}^{(q)},$$

where according to the mathematical induction criterion we can prove that the Theorem 2.2 is true for every q. □

In general case, if p_r^0, for $r = 1, 2, \ldots, l$ $\left(\sum_{r=1}^{l} p_r^0 = 1\right)$, are probabilities of the initial states of $(n+k, n)$-semigroup automaton $(S, (B, \{\ \}), f)$ and if $p_j(w)$ are probabilities after w steps $(n+k, n)$-semigroup automaton $(S, (B, \{\ \}), f)$ will be found in state s_j, than

$$p_j(w) = \sum_{r=1}^{l} p_r^0 p_{rj}(w).$$

The probability $p_j(w)$ is j-th element in the vector-row, which is gotten with multiplying the probability vector in the initial state $p(0) = (p_1^0, p_2^0, \ldots, p_t^0)$ and the matrix P^w. With this the following theorem is proved:

Theorem 2.3. *If $p(w)$ is a probability vector with (present) states of $(n+k, n)$-semigroup automaton $(S, (B, \{\ \}), f)$ after w steps, then*

$$p(w) = p(0) \cdot P^w,$$

where $p(0)$ is vector with probabilities of the initial states.

Theorem 2.4. *Let P be a transition probability matrix of $(n+k, n)$-semigroup automaton $(S, (B, \{\ \}), f)$. If $p = (p_1, p_2, \ldots, p_l)$ is a vector with probabilities of the initial states, then pP is a vector with probabilities of the following step and pP^w is a vector with probabilities of states in the w-th step. In special case,*

$$p(1) = p(0)P,\ p(2) = p(1)P = p(0)P^2, \ldots, p(w) = p(0)P^w$$

3. Ergodicity of vector valued semigroup automata

Let S be a set of states, so that $S = S' \cup S''$ and $S' \cup S'' = \emptyset$. If every state of the set S' can be approached to every state of the set S', and in the set S' can't make a transition nor from one state of the set S'', then the set S' is called **rgodic set of states**. Once $(n+k, n)$-semigroup automaton has made a transition into an ergodic set it can no longer leave the set and the further transitions will be made inside the set.

If the ergodic set is contained from one state, than that state is called **absorbing state**. Once $(n+k, n)$-semigroup automaton has transited in that state, it stays in it.

It is proved that a finite $(n+k, n)$-semigroup automaton must have at least one ergodic set of states.

Theorem 3.1. *If P is a regular transition matrix with dimensions $l \times l$, then:*

a) P^w converges to the stochastic matrix T, when $w \to \infty$;

b) every row of the matrix T is a vector $(t_1, t_2, \ldots, t_l)$, where $t_1, t_2, \ldots, t_l$ are probabilities and $\sum_{i=1}^{l} t_i = 1$.

The matrix T is called ***matrix of final probabilities*** *of $(n+k, n)$-semigroup automaton $(S, (B, \{\ \}), f)$.*

Example 3.1. *Let's determine the matrix of final probabilities of the $(3, 2)$-semigroup automaton $(S, (B, \{\ \}), f)$ from Example 2.1. According to Theorem 1.1 for the matrix P there is single fixed vector t, whose components are positive numbers and* *for which it is true that $tP = t$. From this condition, taking $t = (x, y, 1-x-y)$ and with solving the equation*

$$(x, y, 1-x-y)\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} = (x, y, 1-x-y),$$

we have the system

$$\begin{aligned} &\frac{y}{4} + \frac{1-x-y}{4} = x \\ &\frac{x}{2} + \frac{y}{2} + \frac{1-x-y}{4} = y \\ &\frac{x}{2} + \frac{y}{4} + \frac{1-x-y}{2} = 1-x-y \end{aligned} ,$$

and with solving it we get $x = \frac{1}{5}$ and $y = \frac{2}{5}$. That means the fixed vector is $t = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$. On the other hand,

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix};\ P^2 = \begin{bmatrix} 1/4 & 3/8 & 3/8 \\ 3/16 & 7/16 & 3/8 \\ 3/16 & 3/8 & 7/16 \end{bmatrix};$$

$$P^3 = \begin{bmatrix} 3/16 & 13/32 & 13/32 \\ 13/64 & 13/32 & 25/64 \\ 13/64 & 25/64 & 13/32 \end{bmatrix};\ P^4 = \begin{bmatrix} 13/64 & 51/128 & 51/128 \\ 51/256 & 103/256 & 51/128 \\ 51/256 & 51/128 & 103/256 \end{bmatrix};$$

$$P^5 = \begin{bmatrix} 102/512 & 205/512 & 205/512 \\ 205/1024 & 205/512 & 409/1024 \\ 205/1024 & 409/1024 & 205/512 \end{bmatrix}.$$

It is proved that the convergation velocity on P^w to T is very high and it is equal to the convergation velocity of the geometry progression with quotient smaller than 1. In many examples it can be appropriate, i.e. $P^5 \approx T$.

Theorem 3.2. *If P is a regular transition probability matrix, T is a matrix with final probabilities on $(n+k, n)$-semigroup automaton $(S, (B, \{\ \}), f)$ characterised with the matrix P and $(t_1, t_2, \ldots, t_l)$ is a vector with final probabilities (limit distribution of probabilities on the states on $(n+k, n)$-semigroup automaton $(S, (B, \{\ \}), f)$), then*

$$p(0)P^w \to (t_1, t_2, \ldots, t_l), \quad \text{when } w \to \infty,$$

where $p(0)$ is a vector with initial probabilities.

Proof.

$$\lim_{w\to\infty} p(0)P^w = p(0)\lim_{w\to\infty} P^w = p(0)T = (t_1, t_2, \ldots, t_l).$$

□

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