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SEVERAL PROPERTIES OF CONVERGENT AND CAUCHY SEQUENCES IN A QUASI 2-NORMED SPACE

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Abstract. In [7] C. Park has generalized the term quasi-normed space, i.e. has given the term quasi 2-normed space. Further, C. Park has proven few properties of quasi 2-norm, in [3] M. Kir and M. Acikgoz have given the procedure for completing the quasi 2-normed space and in [1], [4] and [6] are proven few inequalities about quasi 2-normed spaces. In this paper are proven several properties of Cauchy and convergent sequences in a quasi 2-normed space. Some of them are analogy to the properties of such sequences in n-normed spaces ([5], [8]).

1. INTRODUCTION

S. Gähler (1965) gave the terms of 2-normed spaces. Parallelepiped inequality, which is one of the basic in the theory of 2-normed spaces, is one of the axioms of 2-norm. Precisely this inequality, analogously as in the normed spaces, C. Park has replaced by new condition. Thus, he actually derived the following definition for quasi 2-normed space.

Definition 1 ([7]). Let L be a real vector space and $\dim L \geq 2$. *Quasi 2-norm* is a real function $\|\cdot,\cdot\|: L\times L \to [0,\infty)$ such that:

i) $\|x,y\|\geq 0$, for all $x,y\in L$ and $\|x,y\|=0$ if and only if the set $\{x,y\}$ is linearly dependent;

ii) $\|x,y\|=\|y,x\|$, for all $x,y\in L$;

iii) $\|\alpha x,y\|=|\alpha|\cdot\|x,y\|$, for all $x,y\in L$ and for each $\alpha\in\mathbf{R}$, and

iv) It exists a constant $C\geq 1$ such that $\|x+y,z\|\leq C(\|x,z\|+\|y,z\|)$, for all $x,y,z\in L$.

The ordered pair $(L,\|\cdot,\cdot\|)$ is *quasi 2-normed space* if $\|\cdot,\cdot\|$ is quasi 2-norm. The smallest number K such that it satisfies *d*) is *modulus of concavity* of the quasi 2-norm $\|\cdot,\cdot\|$

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Definition 2. Let L be a real vector space and $\dim L \geq 2$. Quasi 2-norms $\|\cdot,\cdot\|_1$ and $\|\cdot,\cdot\|_2$ defined on L are equivalent if it exists $m, M > 0$ such that

$$m\|x,y\|_1 \leq \|x,y\|_2 \leq M\|x,y\|_1, \text{ for all } x,y \in L.$$

Further, in [3] M. Kir and M. Acikgoz gave few examples of trivial quasi 2-normed spaces and considered the question about completing the quasi 2-normed space. C. Park in [7] gave a characterization of quasi 2-normed space (Theorem 1), and [4] (Lemma 1) is proven an inequality which is characteristic for quasi 2-normed spaces.

Theorem 1 ([7]). Let $(L,\|\cdot,\cdot\|)$ be a quasi 2-normed space. It exists $p,\ 0 < p \leq 1$ and an equivalent quasi 2-norm $|||\cdot,\cdot|||$ on L such that

$$|||x+y,z|||^p \leq |||x,z|||^p + |||y,z|||^p, \tag{1}$$

for all $x,y,z \in L$. ■

Lemma 1 ([4]). If L is a quasi 2-normed space with modulus of concavity $C \geq 1$, then for each n>1 and for all $z, x_1, x_2, ..., x_n \in L$ it holds true that

$$\|\sum_{i=1}^{n} x_i, z\| \leq C^{1+[\log_2(n-1)]} \sum_{i=1}^{n} \|x_i, z\|. \ ■$$

Definition 3 ([7]). Quasi 2-norm $|||\cdot,\cdot|||$ given in Theorem 1 is called $(2,p)-$*norm*, and quasi 2-normed space L is called $(2,p)-$*normed space*.

2. CONVERGENT SEQUENCES IN QUASI 2-NORMED SPACE

In [7] C. Park has defined a convergent sequence in quasi 2-normed space, i.e. has given the following definition.

Definition 4 ([7]). Let L be a quasi 2-normed space. The sequence $\{x_n\}_{n=1}^{\infty}$ on L is called convergent sequence if it exists $x \in L$ so that

$$\lim_{n\to\infty} \|x_m - x, z\| = 0, \text{ for each } z \in L.$$

The vector $x \in L$ is called *bound for the sequence* $\{x_n\}_{n=1}^{\infty}$.

By the following statements will be proven several basic properties of convergent sequences in quasi 2-normed spaces.

Lemma 2. Let $\|\cdot,\cdot\|_1$ and $\|\cdot,\cdot\|_2$ be equivalent quasi 2-norms on the real vector space L. The sequence $\{x_n\}_{n=1}^{\infty}$ converges on the quasi 2-normed space $(L,\|\cdot,\cdot\|_1)$ if and only if it converges on the quasi normed space $(L,\|\cdot,\cdot\|_2)$,

Proof. The proof is directly implied by definitions 2 and 4. ■

Theorem 2. Let $(L,\|\cdot,\cdot\|)$ be a quasi 2-normed space with modulus of concavity C.

a) If $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} \alpha_n = \alpha$ and $\lim_{n\to\infty} \beta_n = \beta$, then

$$\lim_{n\to\infty} (\alpha_n x_n + \beta_n y_n) = \alpha x + \beta y. \tag{2}$$

b) If $\dim L \geq 2$, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$, then $x = y$.

Proof. a) Since Lemma 1, it implies that for each $z \in L$

$$\|\alpha_n x_n + \beta_n y_n - (\alpha x + \beta y), z\| = \|\alpha_n (x_n - x) + (\alpha_n - \alpha)x + \beta_n (y_n - y) + (\beta_n - \beta) y, z\|$$
$$\leq C^2 (|\alpha_n| \cdot \|x_n - x, z\| + |\alpha_n - \alpha| \cdot \|x, z\| + |\beta_n| \cdot \|y_n - y, z\| + |\beta_n - \beta| \cdot \|y, z\|).$$

holds true.

By applying that $n \to \infty$ for the latter, and also using the fact that each convergent sequence of real numbers is bounded, we derive the equality (2).

b) Since definition 1, it implies that for each $z \in L$

$$\|x - y, z\| \leq C(\|x - x_n, z\| + \|x_n - y, z\|) \text{ holds true.}$$

By applying that $n \to \infty$ for the latter, we get that $\|x - y, z\| = 0$, for each $z \in L$. So, $x - y$ and z are linearly dependent for each $z \in L$ and since $\dim L \geq 2$ we get that $x = y$. ■

Theorem 3. Let L be a quasi 2-normed space with modulus of concavity $C \geq 1$, $\{x_n\}_{n=1}^{\infty}$ be a sequence on L and $y \in L$ be so that $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$. Then, for each $x \in L$ the real sequence $\{\|x_n - x, y\|\}_{n=1}^{\infty}$ include a convergent subsequence.

Proof. According to theorem 1, there exist p, $0 < p \leq 1$ and an equivalent quasi 2-norm $|||\cdot,\cdot|||$ on L so that for any $x, y, z \in L$ the inequality (1) holds true. Thus, there exist $m, M \in \mathbf{R}$ such that

$$m\|x, y\| \leq |||x, y||| \leq M\|x, y\| \tag{3}$$

holds true for all $x, y \in L$.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence on L and $y \in L$ be so that $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ is satisfied. The inequalities (3) imply that $\lim_{m,n\to\infty} |||x_n - x_m, y||| = 0$. Further, by applying the inequality (1) we get that

$$|||x_n - x, y|||^p = |||x_n - x_m + x_m - x, y|||^p \leq |||x_n - x_m, y|||^p + |||x_m - x, y|||^p$$

holds true, So,

$$|||x_n - x, y|||^p - |||x_m - x, y|||^p \leq |||x_n - x_m, y|||^p .$$

Analogously,

$$|||x_m - x, y|||^p - |||x_n - x, y|||^p \leq |||x_n - x_m, y|||^p .$$

The last two inequalities imply that

$$\big|\, |||x_m - x, y|||^p - |||x_n - x, y|||^p \,\big| \leq |||x_n - x_m, y|||^p ,$$

and since $\lim\limits_{m,n\to\infty} |||x_n - x_m, y||| = 0$, we get that the real sequence $\{|||x_n - x, y|||^p\}_{n=1}^{\infty}$ is Cauchy, that is convergent sequence. Thus, the sequence $\{|||x_n - x, y|||\}_{n=1}^{\infty}$ is convergent, that is bounded sequence. Then, the inequalities (3) imply that the real sequence $\{||x_n - x, y||\}_{n=1}^{\infty}$ is bounded, therefore it consists a convergent subsequence $\{||x_{n_k} - x, y||\}_{k=1}^{\infty}$. ■

Theorem 4. . Let L be a quasi 2-normed space with modulus of concavity $C \geq 1$, $\{x_n\}_{n=1}^{\infty}$ be a sequence on L and $x, y \in L$ be so that $\lim\limits_{n\to\infty} ||x_n - x, y|| = 0$. Then, the real sequence $\{||x_n, y||\}_{n=1}^{\infty}$ include a convergent subsequence.

Proof. According to theorem 1, there exist p, $0 < p \leq 1$ and an equivalent quasi 2-norm $|||\cdot,\cdot|||$ on L so that for any $x, y, z \in L$ the inequality (1) holds true. Thus, there exist $m, M \in \mathbf{R}$ such that for all $x, y \in L$ the inequality (3) holds true.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence on L and $x, y \in L$ be so that $\lim\limits_{n\to\infty} ||x_n - x, y|| = 0$. Now, the inequalities (3) imply that $\lim\limits_{n\to\infty} |||x_n - x, y||| = 0$. Analogously to the proof of theorem 3, we conclude that

$$\big|\, |||x_n, y|||^p - |||x, y|||^p \,\big| \leq |||x_n - x, y|||^p , \tag{4}$$

and since $\lim\limits_{n\to\infty} |||x_n - x, y||| = 0$, by the inequality (4) we get that the real sequence $\{|||x_n, y|||\}_{n=1}^{\infty}$ is convergent, that is the latter is bounded. Now, the inequality (3) implies that the real sequence $\{||x_n, y||\}_{n=1}^{\infty}$ is bounded. Therefore, it includes a convergent subsequence $\{||x_{n_k}, y||\}_{k=1}^{\infty}$. ■

3. CAUCHY SEQUENCES IN QUASI 2-NORMED SPACE

Definition 3 ([7]). Let L be a quasi 2-normed space. The sequence $\{x_n\}_{n=1}^{\infty}$ on L is called a Cauchy sequence if

$$\lim_{n\to\infty} \| x_m - x_n, z \| = 0 \text{, for each } z \in L .$$

The quasi 2-normed space L is called *a complete* if each Cauchy sequence converges on L .

Theorem 5. Let $(L, \|\cdot,\cdot\|)$ be a quasi 2-normed space with modulus of concavity $C \geq 1$. If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence on L, then for each $z \in L$ the real sequence $\{\| x_n, z \|\}_{n=1}^{\infty}$ includes a convergent subsequence.

Proof. According to the theorem 1, there exist p, $0 < p \leq 1$ and an equivalent quasi 2-norm $|||\cdot,\cdot|||$ on L such that for all $x, y, z \in L$ the inequality (1) holds true. Thus, there exist $m, M \in \mathbf{R}$ such that for all $x, y \in L$, the inequality (3) holds true.

Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence on $(L, \|\cdot,\cdot\|)$ and $z \in L$. Since the inequality (3) we get that the sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence on $(L, |||\cdot,\cdot|||)$. Further, the inequality (1) implies that

$$||| x_n, z |||^p = ||| (x_n - x_m) + x_m, z |||^p \leq ||| x_n - x_m, z |||^p + ||| x_m, z |||^p ,$$

that is

$$||| x_n, z |||^p - ||| x_m, z |||^p \leq ||| x_n - x_m, z |||^p .$$

Analogously

$$||| x_m, z |||^p - ||| x_n, z |||^p \leq ||| x_n - x_m, z |||^p .$$

The last two inequalities imply

$$\left| ||| x_m, z |||^p - ||| x_n, z |||^p \right| \leq ||| x_n - x_m, z |||^p \to 0$$

for $m, n \to \infty$. Thus, the sequence $\{||| x_n, z |||^p\}_{n=1}^{\infty}$ is Cauchy sequence. The latter means that also the sequence $\{||| x_n, z |||\}_{n=1}^{\infty}$ is Cauchy sequence. So, it is bounded sequence. Finally the inequalities (3) imply that the real sequence $\{\| x_n, z \|\}_{n=1}^{\infty}$ is bounded, that is it consists a convergent subsequence. ■

Theorem 6. Let L be a quasi 2-normed space with modulus of concavity $C \geq 1$. If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences on L and $\{\alpha_n\}_{n=1}^{\infty}$ is a real Cauchy sequence, then $\{x_n + y_n\}_{n=1}^{\infty}$ and $\{\alpha_n x_n\}_{n=1}^{\infty}$ are Cauchy sequences on X .

Proof.

$$\| x_n + y_n - (x_m + y_m), z \| = \| (x_n - x_m) + (y_n - y_m), z \|$$
$$\leq C(\| x_n - x_m, z \| + \| y_n - y_m, z \|)$$

holds true. Thus, $\| x_n + y_n - (x_m + y_m), z \| \to 0$ for $m,n \to \infty$. So, $\{x_n + y_n\}_{n=1}^{\infty}$ is Cauchy sequence on L.

The real sequence $\{\alpha_n\}_{n=1}^{\infty}$ is Cauchy, so it is bounded. Further, the proof of theorem 5 implies that the real sequence $\{\| x_n, z \|\}_{n=1}^{\infty}$ is bounded. Since

$$\| \alpha_n x_n - \alpha_m x_m, z \| = \| (\alpha_n x_n - \alpha_n x_m) + (\alpha_n x_m - \alpha_m x_m), z \|$$
$$\leq C(\| \alpha_n x_n - \alpha_n x_m, z \| + \| \alpha_n x_m - \alpha_m x_m, z \|)$$
$$= C(| \alpha_n | \cdot \| x_n - x_m, z \| + | \alpha_n - \alpha_m | \cdot \| x_m, z \|$$

it follows that $\| \alpha_n x_n - \alpha_m x_m, z \| \to 0$ for $m,n \to \infty$. Thus, the sequence $\{\alpha_n x_n\}_{n=1}^{\infty}$ in Cauchy sequence on L. ■

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