Математички Билтен ISSN 0351-336X
35(LXI)
2011(27-39)
Скопје, Македонија

SOME 2-SUBSPACES OF 2-SPACE

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Abstract. In the paper, some 2-subspaces of 2-vector space are considered and characterized.

The idea of considering a family of subsets of X^n, among which is the class of the 2-subspaces (case $n = 2$), came from the study of 2-norms, 2-seminorms and the skew-symmetric forms.

Let X denote the vector space over the field Φ (Φ is the field of the real numbers or the field of the complex numbers). $M_n(\Phi)$ denotes the set of quadratic matrixes of order n. Δ_2 denotes the set of all ordered pairs $(x,\, y)$, elements of X^2, such that the vectors x, y are linearly dependent. Let S_n denote the set of all permutations of $\{1, 2, \dots, n\}$.

Definition 1. *The function* $||\cdot, \dots, \cdot|| : X^n \to \mathbb{R}$, $n \geq 2$, *which satisfies the conditions:*

a) $||x_1, x_2, ..., x_n|| = 0$ *if and only if the vectors* $x_1, x_2, ..., x_n$ *are linearly dependent;*

b) $||x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}|| = ||x_1, x_2, ..., x_n||$, *for any* $x_1, x_2, ..., x_n \in X$ *and for every* $\pi \in S_n$;

c) $||\alpha x_1, x_2, ..., x_n|| = |\alpha|\, ||x_1, x_2, ..., x_n||$, *for any* $x_1, x_2, ..., x_n \in X$ *and for every scalar* $\alpha \in \Phi$;

d) $||x_1 + x_1', x_2, ..., x_n|| \leq ||x_1, x_2, ..., x_n|| + ||x_1', x_2, ..., x_n||$, *for any* $x_1, x_1', x_2, ..., x_n \in X$,

is called n-norm of the vector space X, *and the ordered pair* $(X, ||\cdot, ..., \cdot||)$ *is called n-normed space.*

As a direct consequence from the definition of 2-norm is the equality

$$||x, y|| = ||x, y + ax|| \tag{1}$$

where $x, y \in X$ and $\alpha \in \Phi$ are arbitrarily chosen.

Using (1) and the definition of 2-norms, it can easily be proved that

$$||a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2|| = |a_{11}a_{22} - a_{12}a_{21}|\, ||x_1, x_2||, \tag{2}$$

for an arbitrary $a_{11}, a_{12}, a_{21}, a_{22} \in \Phi$ and an arbitrary $x_1, x_2 \in X$.

Definition 2. *The function* $p : X^n \to \mathbb{R}$ *which satisfies the conditions:*

a) $p(x_1, x_2, ..., x_n) = 0$ *if* $x_1, x_2, ..., x_n$ *are linearly dependent vectors;*

b) $p(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}) = p(x_1, x_2, ..., x_n)$, *for any* $x_1, x_2, ..., x_n \in X$ *and for every* $\pi \in S_n$*;*

c) $p(\alpha x_1, x_2, ..., x_n) = |\alpha|\, p(x_1, x_2, ..., x_n)$, *for any* $x_1, x_2, ..., x_n \in X$ *and for scalar* $\alpha \in \Phi$*;*

d) $p(x_1 + x_1', x_2, ..., x_n) \leq p(x_1, x_2, ..., x_n) + p(x_1', x_2, ..., x_n)$, *for any* $x_1, x_1', x_2, ..., x_n \in X$,

is called n*-seminorm of the vector space* X*, and the ordered pair* (X, p) *is called* n*-seminormed space.*

As in the case of 2-norm, it is easy to prove that $p(x, y) = p(x, y + \alpha x)$, for any $x, y \in X$ and any $\alpha \in \Phi$.

Even more, the following equality

$$p(a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2) = |a_{11}a_{22} - a_{12}a_{21}|\, p(x_1, x_2) \tag{3}$$

holds for any $a_{11}, a_{12}, a_{21}, a_{22} \in \Phi$ and any $x_1, x_2 \in X$.

In respect to the n-seminorms and n-norms, the most suitable class of functionals for consideration are the skew-symmetric linear forms.

Definition 3. *The function* $\Lambda : X^2 \to \Phi$ *which satisfies the conditions:*

a) $\Lambda(x_1 + x_2, x_3) = \Lambda(x_1, x_3) + \Lambda(x_2, x_3)$, *for any* $x_1, x_2, x_3 \in X$*;*

b) $\Lambda(x_1, x_2) = -\Lambda(x_2, x_1)$, *for any* $x_1, x_2 \in X$*;*

c) $\Lambda(\alpha x_1, x_2) = \alpha\Lambda(x_1, x_2)$, *for any* $x_1, x_2 \in X$ *and* $\alpha \in \Phi$,

is called 2-skew-symmetric linear form.

Immediate consequence of this definition is the following equality

$$\Lambda(a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2) = (a_{11}a_{22} - a_{12}a_{21})\, \Lambda(x_1, x_2). \tag{4}$$

Having in mind the equalities (2), (3) and (4), we introduce the denotation $(a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2) = A(x_1, x_2)^T$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2(\Phi)$, and the multiplication on the right hand side of the last equality is the same as the multiplication of a matrix with a vector. For simplification, we can use the denotation $A(x_1, x_2)^T = A(x_1, x_2)$.

On X^2, the following operations are defined:

i) $(x_1, x_3) + (x_2, x_3) = (x_1 + x_2, x_3)$;

ii) $(x_3, x_1) + (x_3, x_2) = (x_3, x_1 + x_2)$;

iii) $A(x_1, x_2) = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2)$,

for an arbitrary $x_1, x_2, x_3 \in X$ and an arbitrary $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2(\Phi)$. X^2, with the defined operations, is called 2-vector space over the field Φ or simply 2-space.

Now, the definitions of 2-norm, 2-seminorm and 2-skew-symmetric linear form get simpler form.

Definition 4. *The function* $||\cdot, \cdot|| : X^2 \to \mathbb{R}$ *which satisfies the conditions:*

a) $||x_1, x_2|| = 0$ *if and only if* x_1, x_2 *are linearly dependent vectors;*

b) $||A(x_1, x_2)|| = |\det A|\, ||x_1, x_2||$, *for any* $x_1, x_2 \in X$ *and for every* $A \in M_2(\Phi)$*;*

c) $||x_1 + x_2, x_3|| \leq ||x_1, x_3|| + ||x_2, x_3||$, *for any* $x_1, x_2, x_3 \in X$,

is called 2-norm of the vector space X, *and the ordered pair* $(X, ||\cdot, \cdot||)$ *is called 2-normed space.*

Definition 5. *The function* $p : X^2 \to \mathbb{R}$, *which satisfies the conditions:*

a) $p(A(x_1, x_2)) = |\det A|\, p(x_1, x_2)$, *for any* $x_1, x_2 \in X$ *and for every* $A \in M_2(\Phi)$;
b) $p(x_1 + x_2, x_3) \leq p(x_1, x_3) + p(x_2, x_3)$, *for any* $x_1, x_2, x_3 \in X$,

is called 2-seminorm of the vector space X, *and the ordered pair* (X, p) *is called 2-seminormed space.*

Definition 6. *The function* $\Lambda : X^2 \to \Phi$ *which satisfies the conditions:*

a) $\Lambda(x_1 + x_2, x_3) = \Lambda(x_1, x_3) + \Lambda(x_2, x_3)$, *for any* $x_1, x_2, x_3 \in X$;
b) $\Lambda(A(x_1, x_2)) = (\det A)\Lambda(x_1, x_2)$, *for any* $x_1, x_2 \in X$ *and for every* $A \in M_2(\Phi)$

is called 2-skew-symmetric linear form.

It is easy to prove that the definitions 1 and 4, 2 and 5, and 3 and 6, in the case $n = 2$, are equivalent.

Definition 7. *The set* $W \subseteq X^2$, *where* X *is a vector space over the field* Φ, *is 2-invariant if* $AW \subseteq W$ *for every* $A \in M_2(\Phi)$ *for which* $\det A = 1$.

Let $||\cdot, \cdot||, p : X^2 \to \mathbb{R}$ and $\Lambda : X^2 \to \Phi$ be 2-norm, 2-seminorm and 2-skew-symmetric linear form, respectively. We consider the following sets

$N_{||\cdot,\cdot||} = \{(x_1, x_2) \,|\, x_1, x_2 \in X,\ ||x_1, x_2|| = 0\}$
$N_p = \{(x_1, x_2) \,|\, x_1, x_2 \in X,\ p(x_1, x_2) = 0\}$
$N_\Lambda = \{(x_1, x_2) \,|\, x_1, x_2 \in X,\ \Lambda(x_1, x_2) = 0\}$

Using the definitions of 2-norm, 2-seminorm and 2-skew-symmetric linear form, we get:

$(x_1, x_3) + (x_2, x_3) \in S$;
$(x_3, x_1) + (x_3, x_2) \in S$;
$A(x_1, x_2) \in S$,

for any $(x_1, x_3), (x_2, x_3), (x_3, x_1), (x_3, x_2) \in S$ and $A \in M_2(\Phi)$, when S is any of the sets $N_{||\cdot,\cdot||}$, N_p or N_Λ.

The following definition comes naturally.

Definition 8. *Let* X *be a vector space over the field* Φ. *The subset* $S \subseteq X^2$ *which satisfies the conditions:*

$(x_1, x_3) + (x_2, x_3) \in S$;
$(x_3, x_1) + (x_3, x_2) \in S$;
$A(x_1, x_2) \in S$,

for any $(x_1, x_3), (x_2, x_3), (x_3, x_1), (x_3, x_2) \in S$ *and for every* $A \in M_2(\Phi)$ *is called 2-subspace of the 2-space* X^2.

In [2] we proved the following theorems.

Theorem 1. *The intersection of an arbitrary family of 2-subspaces of* X^2 *is a 2-subspace.*

For the set B, $B \subseteq X^2$, with S_B, we denote the family of all 2-subspaces which contain the set B.

If B is an arbitrary subset of X^2, the smallest 2-subspace which contains the set B will be denoted with P_B.

Theorem 2. *If B, $B \subseteq X^2$ is an arbitrary set, then $P_B = \bigcap\limits_{S \in S_B} S$.*

Note. If the 2-subspace contains the pair (x, y) then it, also, contains the pair (y, x) and, therefore when (x, y) is a generator element of the 2-subspace, we do not consider the pair (y, x) as a generator element of 2-subspace.

Theorem 3. *Every 2-subspace of the 2-vector space X^2 is 2-invariant.*

In the subsequent part, we will consider special cases for B. The elements for each considered pair of B are linearly independent vectors.

In the subsequent part, we will consider the sum of two ordered pairs with respect to first coordinate (the second coordinates of the ordered pairs are the same). The considerations in the other case are analogues because of Theorem 1, Theorem 2, Theorem 3 and the note after Theorem 2.

$L(u, v)$ denotes a vector subspace of X generated with the vectors u and v.

Case 1. $B = \{(x_1, x_2)\}$.

In this case, it is easy to prove that $P_B = \{(a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2) \mid a_{11}, a_{12}, a_{21}, a_{22} \in \Phi\}$. It is clear that, for every $S \in S_B$, $\{(a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2) \mid a_{11}, a_{12}, a_{21}, a_{22} \in \Phi\} \subseteq S$, and $\{(a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2) \mid a_{11}, a_{12}, a_{21}, a_{22} \in \Phi\}$ is a 2-subspace which contains $B = \{(x_1, x_2)\}$ as its subset.
We note that in this case we have $P_B = L(x_1, x_2) \times L(x_1, x_2)$.

Case 2. $B = \{(x_1, x_2), (x_3, x_4)\}$. Let L be the subspace of X which is generated with the vectors x_1, x_2, x_3, x_4, i.e. $L = L(x_1, x_2, x_3, x_4)$. There are 3 possible sub cases.

Sub case 1. $\dim L = 4$.

Since x_1, x_2 and x_3, x_4 are linearly independent, we get that $L(x_1, x_2) \cap L(x_3, x_4) = \{0\}$. Because of case 1, we have
$P_{B_1} = L(x_1, x_2) \times L(x_1, x_2)$, $P_{B_2} = L(x_3, x_4) \times L(x_3, x_4)$, for $B_1 = \{(x_1, x_2)\}$ and $B_2 = \{(x_3, x_4)\}$. Then $P_{B_1} \cap P_{B_2} = \{(0, 0)\}$ and therefore $P_B = P_{B_1} \cup P_{B_2} \cup P_{\{\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 \mid \alpha, \beta, \gamma, \delta \in \Phi\}}$.

Sub case 2. $\dim L = 2$.

If $\dim L(x_1, x_2) = 2$ and $\dim L(x_3, x_4) = 2$ we have
$P_{B_1} = \{(a_{11}x_1 + a_{12}x_2, a_{12}x_1 + a_{22}x_2) \mid a_{11}, a_{12}, a_{12}, a_{22} \in \Phi\}$,
$P_{B_2} = \{(b_{11}x_3 + b_{12}x_4, b_{12}x_3 + b_{22}x_4) \mid b_{11}, b_{12}, b_{12}, b_{22} \in \Phi\}$,
and therefore $P_{B_1} = P_{B_2} = P_B$.
The other possibilities of this sub case 2 are trivial and we do not consider them.

Sub case 3. $\dim L = 3$.

Without loss of generality, we may assume that x_1, x_2, x_3 are linearly independent vectors. Then $x_4 = \alpha x_1 + \beta x_2 + \gamma x_3$, and the 2-subspace generated with $B_1 = \{(x_1, x_2)\}$ is the same with the 2-subspace generated with $B_2 = \{(\alpha x_1 + \beta x_2, x_2)\}$, i.e. $P_{B_1} = P_{B_2}$. On the other hand, the 2-subspace generated with $B_3 = \{(x_3, x_4)\}$ is the same with the 2-subspace generated with $B_4 = \{(x_3, \alpha x_1 + \beta x_2)\}$, i.e. $P_{B_3} = P_{B_4}$, and the 2-subspace generated with $B = \{(x_1, x_2), (x_3, x_4)\}$ is the same with the 2-subspace generated with $\{(\alpha x_1 + \beta x_2, x_2), (x_3, \alpha x_1 + \beta x_2)\}$.

Let $y_1 = x_1, y_2 = \alpha x_1 + \beta x_2, y_3 = x_3$. Then this 2-subspace is completely described in [2], and it is equal to $P_B = \bigcup\limits_{\sigma\in\pi(y_1,y_3;y_2)} \sigma \times \sigma$, where σ is two dimensional subspace defined with the vectors $\alpha y_1+\beta y_3$ and y_2, i.e. $\sigma = L(\alpha y_1+\beta y_3, y_2)$.

Therefore $P_B = \bigcup\limits_{\alpha,\beta\in\Phi} L(\alpha y_1 + \beta y_3, y_2) \times L(\alpha y_1 + \beta y_3, y_2)$.

Note. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2(\Phi)$ such that $\det A \neq 0$, then the vectors $a_{11}y_1 + a_{12}y_3, a_{21}y_1 + a_{22}y_3$ are linearly independent. Let us suppose that there exist $\alpha, \beta \in \Phi$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in M_2(\Phi)$, such that

$A(y_1, y_3) = B(\alpha y_1 + \beta y_3, y_2)$,

$(a_{11}y_1 + a_{12}y_3, a_{21}y_1 + a_{22}y_3) = (b_{11}(\alpha y_1 + \beta y_3) + b_{12}y_2, b_{21}(\alpha y_1 + \beta y_3) + b_{22}y_2)$.

The last equality is possible if $b_{12} = b_{22} = 0$ and $a_{11}y_1 + a_{12}y_3 = b_{11}(\alpha y_1 + \beta y_3)$, $a_{21}y_1 + a_{22}y_3 = b_{21}(\alpha y_1 + \beta y_3)$.

The last equalities are not possible because the vectors $b_{11}(\alpha y_1+\beta y_3)$ and $b_{21}(\alpha y_1+\beta y_3)$ are linearly dependent.

Therefore, $P_B \neq L \times L$, where $L = L(y_1, y_2, y_3)$.

Note. Before we go to the next sub case, we will consider one 2-subspace generated with 3 elements, i.e. generated with the set $B = \{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}$, where x_1, x_2, x_3 are linearly independent vectors. We will show that, in this case, the 2-subspace generated with the set B is $L \times L$, where $L = L(x_1, x_2, x_3)$.

Indeed, if $x = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$ and $y = b_{11}x_1 + b_{12}x_2 + b_{13}x_3$, then

$$
\begin{aligned}
(x, y) &= (a_{11}x_1 + a_{12}x_2 + a_{13}x_3, b_{11}x_1 + b_{12}x_2 + b_{13}x_3) \\
&= \left(\begin{bmatrix} a_{11} & a_{12} \\ b_{11} & 0 \end{bmatrix} (x_1, x_2) + \begin{bmatrix} 0 & a_{13} \\ b_{11} & 0 \end{bmatrix} (x_1, x_3) \right) + \\
&\quad + \left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & b_{12} \end{bmatrix} (x_1, x_2) + \begin{bmatrix} 0 & a_{13} \\ b_{12} & 0 \end{bmatrix} (x_2, x_3) \right) \\
&\quad + \left(\begin{bmatrix} a_{11} & 0 \\ 0 & b_{13} \end{bmatrix} (x_1, x_3) + \begin{bmatrix} a_{12} & a_{13} \\ 0 & b_{13} \end{bmatrix} (x_2, x_3) \right).
\end{aligned}
$$

This example gives motivation to define one characteristic type of 2-subspace of 2-vector space X^2.

Definition 9. *Let X be a vector space over the field Φ and let $L \subseteq X$ be its vector subspace. The set $L \times L$, which is 2-subspace of X^2, will be called **kernel 2-subspace**.*

We note that the dimension of L might be an arbitrary one.

Considering the case 2, it becomes clear that consideration of 2-subspaces generated with the subsets of X^2 which have more than two generator elements, would be very complicated in this way. Therefore, we will continue the considerations in a different way.

Case 3. Let L be the subspace of X generated with the linearly independent vectors x_1, x_2, x_3, x_4, i.e. $L = L(x_1, x_2, x_3, x_4)$. If $Q = \{x_1, x_2, x_3, x_4\}$ then the set $(Q \times Q) \setminus \Delta_2$, has 6 elements, i.e. $U = (Q \times Q) \setminus \Delta_2 = \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4), (x_3, x_4)\}$.

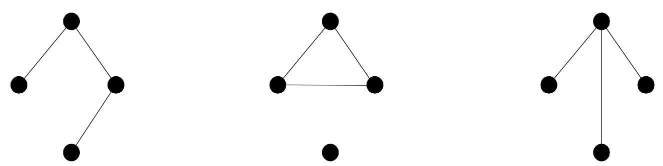
The subsets of U with one or two elements are of no interest since they generate 2-subspaces which are, already, described in the previous cases.

There are 20 subsets of U with 3 elements, 15 subsets of U with 4 elements and 6 subsets of U with 5 elements. So, there are 41 different 2-subspaces generated with 3, 4 or 5 elements that have to be described.

For an easier characterization of the 2-subspaces in this case, we will give a schematic description for each of them. The vectors will be represented with points in the plane and the ordered pair, which is a generator element, will be represented with a segment that connects the elements of the ordered pair. Considering all the possible schemes for a generator set with 3 elements that generates 2-subspace different than the previously described, we notice that there are two different types. They will be described in the sub cases 1 and 2. Considering all the possible schemes for the 2-subspaces generated with 4 elements, we notice that there are two different types of 2-subspaces and that they will be described in the sub cases 3 and 4.

Schematic description of the types of generator sets (with 3 and 4 generator elements) which are subsets of $\{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4), (x_3, x_4)\}$, where x_1, x_2, x_3, x_4 are linearly independent vectors:

Types of generator sets with 3 elements:



type 1 type 2 type 3

Types of generator sets with 4 elements:

type 4 type 5

Sub case 1. $B = \{(x_1, x_2), (x_1, x_3), (x_1, x_4)\}$ (type 3)

The 2-subspace generated with the elements of the subset $B_1 = \{(x_1, x_2), (x_1, x_3)\}$ is described in the sub case 3 of the case 2. This 2-subspace is of the form

$$\bigcup_{\alpha,\beta\in\Phi} L(\alpha x_2 + \beta x_3, x_1) \times L(\alpha x_2 + \beta x_3, x_1),$$

and, at the same time, it is 2-subspace of P_B.

If $y \in L(x_2, x_3, x_4)$ is an arbitrary chosen element, i.e. $y = \alpha x_2 + \beta x_3 + \gamma x_4$, then

$$(y, x_1) = (\alpha x_2 + \beta x_3 + \gamma x_4, x_1) = (\alpha x_2 + \beta x_3, x_1) + (\gamma x_4, x_1),$$

and since $(\alpha x_2+\beta x_3, x_1), (\gamma x_4, x_1) = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} (x_4, x_1) \in P_B$, we get that $(y, x_1) \in P_B$.

It is not hard to prove that $\bigcup_{\alpha,\beta,\gamma\in\Phi} L(\alpha x_2+\beta x_3+\gamma x_4, x_1)\times L(\alpha x_2+\beta x_3+\gamma x_4, x_1)$ is 2-subspace, which is 2-subspace of every 2-subspace S, $S \in S_B$. Therefore

$$P_B = \bigcup_{\alpha,\beta,\gamma\in\Phi} L(\alpha x_2 + \beta x_3 + \gamma x_4, x_1) \times L(\alpha x_2 + \beta x_3 + \gamma x_4, x_1)$$
$$= \bigcup_{u\in L(x_2,x_3,x_4)} L(u, x_1) \times L(u, x_1)$$

Sub case 2. $B = \{(x_1, x_2), (x_2, x_3), (x_3, x_4)\}$ (type 1).

For the sets $B_1 = \{(x_1, x_2), (x_2, x_3)\}$ and $B_2 = \{(x_2, x_3), (x_3, x_4)\}$, the 2-subspaces P_{B_1} and P_{B_2} are 2-subspaces of every 2-subspace S, $S \in S_B$. Let $(a_{11}(\alpha x_1+\beta x_3)+a_{12}x_2, a_{21}(\alpha x_1+\beta x_3)+a_{22}x_2)$ and $(b_{11}(\gamma x_2+\delta x_4)+b_{12}x_3, b_{21}(\gamma x_2+\delta x_4) + b_{22}x_3)$ be an arbitrary elements of P_{B_1} and P_{B_2}, respectively. Adding two elements is possible if and only if

$$a_{21}\alpha x_1 + a_{21}\beta x_3 + a_{22}x_2 = b_{21}\gamma x_2 + b_{21}\delta x_4 + b_{22}x_3.$$

Because of the assumed linear independence, the last equality is possible if and only if $a_{21}\alpha = 0$, $b_{21}\delta = 0$, $a_{21}\beta = b_{22}$ and $b_{21}\gamma = a_{22}$.

i) $a_{21} = 0, b_{21} = 0$.

Then $a_{22} = 0$, $b_{22} = 0$ and the result of the adding is an element of Δ_2, i.e.

$$(a_{11}(\alpha x_1 + \beta x_3) + a_{12}x_2, 0) + (b_{11}(\gamma x_2 + \delta x_4) + b_{12}x_3, 0) =$$
$$= (a_{11}\alpha x_1 + (a_{11}\beta + b_{12})x_3 + (a_{12} + b_{11}\gamma)x_2 + b_{11}\delta x_4, 0) \in \Delta_2.$$

ii) $\alpha = 0$, $b_{21} = 0$

In this case, $a_{22} = 0$ and the adding is possible if and only if $a_{21}\beta = b_{22} = s$, and the result of the adding is

$$(a_{11}\beta x_3 + a_{12}x_2, sx_3) + (b_{11}(\gamma x_2 + \delta x_4) + b_{12}x_3, sx_3) =$$
$$= ((a_{11}\beta + b_{12})x_3 + (a_{12} + b_{11}\gamma)x_2 + b_{11}\delta x_4, sx_3).$$

Because of the 2-invariance of P_B, we get that the ordered pair on the right hand side of the last equality is an element of P_{B_2}.

iii) $a_{21} = 0, \delta = 0$.

In this case, $b_{22} = 0$ and the adding is possible if and only if $b_{21}\gamma = a_{22} = t$, and the result of the adding is $(a_{11}(\alpha x_1 + \beta x_3) + a_{12}x_2, tx_2) + (b_{11}\gamma x_2 + b_{12}x_3, tx_2) = (a_{11}\alpha x_1 + (a_{11}\beta + b_{12})x_3 + (a_{12} + b_{11}\gamma)x_2, tx_2)$

Again, because of the 2-invariance of P_B, we get that the ordered pair on the right hand side of the last equality is an element of P_{B_1}.

iv) $\alpha = 0, \delta = 0$.

In this case, the adding is possible if and only $a_{21}\beta = b_{22} = s$ and $b_{21}\gamma = a_{22} = t$, and the result of the adding is

$(a_{11}\beta x_3 + a_{12}x_2, sx_3 + tx_2) + (b_{11}\gamma x_2 + b_{12}x_3, tx_2 + sx_3) =$
$= ((a_{12} + b_{11}\gamma)x_2 + (a_{11}\beta + b_{12})x_3, tx_2 + sx_3)$.

The element from the right hand side of the last equality belongs to P_{B_1} and to P_{B_2}.

The other possibilities of adding elements of P_B are trivial.

From the above discussion, in this case we get that $P_B = P_{B_1} \cup P_{B_2}$.

From the 2-subspaces generated with 4 elements, it is enough to consider two types.

Sub case 3. $B = \{(x_1, x_2), (x_2, x_3), (x_3, x_1), (x_1, x_4)\}$ (type 5) .

In the sub case 3 of the case 2, we saw that the 2-subspace generated with the set $B_1 = \{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}$ is $P_{B_1} = L(x_1, x_2, x_3) \times L(x_1, x_2, x_3)$. Let $u \in L(x_1, x_2, x_3)$ be an arbitrary element. If S is any 2-subspace which contains B, then (u, x_1) and (x_4, x_1) belong to S, therefore S contains

$\bigcup\limits_{\alpha,\delta\in\Phi} L(\alpha u + \delta x_4, x_1) \times L(\alpha u + \beta x_4, x_1)$, as its 2-subspace.

On the other hand, since α and β are arbitrary and having in mind the form of u, it is not hard to see that

$$\begin{aligned}\bigcup_{u\in L(x_1,x_2,x_3)} \bigcup_{\eta,\delta\in\Phi} L(\eta u + \delta x_4, x_1) \times L(\eta u + \beta x_4, x_1) =\\ = \bigcup_{\alpha,\beta,\gamma,\delta\in\Phi} L(\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4, x_1) \times L(\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4, x_1) =\\ = \bigcup_{v\in L(x_1,x_2,x_3,x_4)} L(v, x_1) \times L(v, x_1)\end{aligned}$$

is 2-subspace of S, $S \in S_B$. Therefore $P_B = \bigcup\limits_{v\in L(x_1,x_2,x_3,x_4)} L(v, x_1) \times L(v, x_1)$.

Sub case 4. $B = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1)\}$ (type 4) For the sets $B_1 = \{(x_1, x_2), (x_2, x_3)\}$, $B_2 = \{(x_2, x_3), (x_3, x_4)\}$, $B_3 = \{(x_3, x_4), (x_4, x_1)\}$ and $B_4 = \{(x_4, x_1), (x_1, x_2)\}$, the 2-subspaces

$$P_{B_1} = \bigcup_{\alpha,\beta\in\Phi} L(\alpha x_1 + \beta x_3, x_2) \times L(\alpha x_1 + \beta x_3, x_2)$$

$$P_{B_2} = \bigcup_{\gamma,\delta\in\Phi} L(\gamma x_2 + \delta x_4, x_3) \times L(\gamma x_2 + \delta x_4, x_3)$$

$$P_{B_3} = \bigcup_{\mu,\nu\in\Phi} L(\mu x_1 + \nu x_3, x_4) \times L(\mu x_1 + \nu x_3, x_4)$$

and

$$P_{B_4} = \bigcup_{\eta,\theta\in\Phi} L(\eta x_2 + \theta x_4, x_1) \times L(\eta x_2 + \theta x_4, x_1)$$

are 2-subspaces which are 2-subspaces of every 2-subspace S, $S \in S_B$.

On the other hand, $P_\circ = \bigcup\limits_{\alpha,\beta,\gamma,\delta} L(\alpha x_1 + \beta x_3, \gamma x_2 + \delta x_4) \times L(\alpha x_1 + \beta x_3, \gamma x_2 + \delta x_4)$ is 2-subspace which is 2-subspace of every 2-subspace S, $S \in S_B$.

Using the results of the sub case 2 of this case, it is not hard to prove that $P = P_{B_1} \cup P_{B_2} \cup P_{B_3} \cup P_{B_4} \cup P_\circ$ is 2-subspace which is 2-subspace of every 2-subspace S, $S \in S_B$.

Therefore $P_B = P_{B_1} \cup P_{B_2} \cup P_{B_3} \cup P_{B_4} \cup P_\circ$.

There exists only one type of 2-subspaces generated with 5 elements.

Sub case 5. $B = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1), (x_1, x_3)\}$.

We note that, in this case the 2-subspaces of P_B are the kernel 2-subspaces generated with $B_1 = \{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}$ and $B_2 = \{(x_1, x_4), (x_4, x_3), (x_3, x_1)\}$. Their form is

$P_{B_1} = L(x_1, x_2, x_3) \times L(x_1, x_2, x_3)$ and $P_{B_2} = L(x_1, x_3, x_4) \times L(x_1, x_3, x_4)$.

For two elements $(a_{11}x_1 + a_{12}x_3 + a_{13}x_4, a_{21}x_1 + a_{22}x_3 + a_{23}x_4)$ and $(b_{11}x_1 + b_{12}x_2 + b_{13}x_3, b_{21}x_1 + b_{22}x_2 + b_{23}x_3)$ of P_{B_2} and P_{B_1}, respectively, the adding is possible if and only if $a_{23} = b_{22} = 0$, and, because of the linear independence, we have that $a_{21} = b_{21} = s$ and $a_{22} = b_{23} = t$. In this case, we have

$(a_{11}x_1 + a_{12}x_3 + a_{13}x_4, sx_1 + tx_3) + (b_{11}x_1 + b_{12}x_2 + b_{13}x_3, sx_1 + tx_3) = ((a_{11} + b_{11})x_1 + b_{12}x_2 + (a_{12} + b_{13})x_3 + a_{13}x_4, sx_1 + tx_3)$

Since $a_{11}, b_{11}, b_{12}, a_{12}, b_{13}, a_{13} \in \Phi$ are arbitrarily chosen, we get that

$$T = \bigcup_{\substack{u \in L(x_1, x_2, x_3, x_4) \\ v \in L(x_1, x_3)}} L(u, v) \times L(u, v)$$

is 2-subspace of every 2-subspace S, $S \in S_B$. Therefore $T \subseteq P_B$.

We notice that the sum of two elements in P_{B_1} is an element in P_{B_1}, the sum of two elements in P_{B_2} is an element in P_{B_2}, the sum of two elements in T is an element in T, and the sum of an element in T and an element in P_{B_1} or P_{B_2} is an element in T.

Therefore $P_{B_1} \cup P_{B_2} \cup T$ is 2-subspace of every 2-subspace S, $S \in S_B$. Finally, $P_B = P_{B_1} \cup P_{B_2} \cup T$.

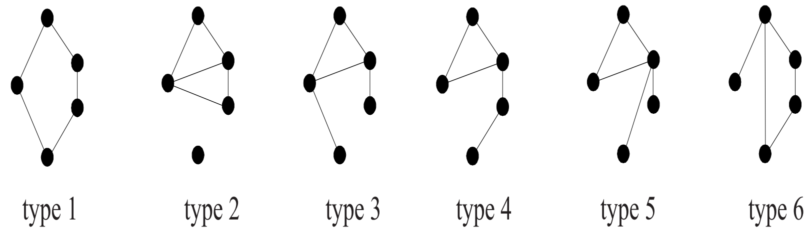
The further on approach in this way, like in the case 2 and in the case 3, as well as the previous approaches in the description of the 2-subspaces of 2-vector space X^2 is very complicated and technically almost impossible. We have to consider enormous number of cases, i.e. types of 2-subspaces in respect to the number of generator elements.

For example, if we consider 5 vectors x_1, x_2, x_3, x_4, x_5 which are linearly independent, then the set $(R \times R) \setminus \Delta_2$, where $R = \{x_1, x_2, x_3, x_4, x_5\}$ has 10 elements, i.e.

$W = (R \times R) \setminus \Delta_2 = \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_1, x_5), (x_2, x_3), (x_2, x_4),$
$(x_2, x_5), (x_3, x_4), (x_3, x_5), (x_4, x_5)\}$.

Now, the number of different generator subsets of W is $2^{10} - 1 = 1023$. There are $252, 210, 120$, 45 and 9 generator sets with $5, 6, 7, 8$ and 9 elements, respectively, and the number of different 2-subspaces is the same, i.e. 636. So, in this case, for complete description, should, also, be considered some other 2-subspaces generated with 4 elements.

Schematic description of the types of generator subsets (with 5 elements) of the set W:



If we consider 6 vectors $x_1, x_2, x_3, x_4, x_5, x_6$ which are linearly independent and then repeat the same procedure as before, we get that the number of 2-subspaces generated with the elements of the set $(T \times T) \backslash \Delta_2$, where $T = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, is $2^{15} - 1 = 22787$. The number of different types of 2-subspaces generated with the elements is smaller than 22787, but still it is a big number.

What we can see from the schematic description is that some of the 2-subspaces have a specific structure. So, the 2-subspace from the sub case 4 is generated with elements of the form $\{(a, b), (b, c), (c, d), (d, a)\}$, where the elements a, b, c, d are linearly independent vectors. We will call these 2-subspaces **cycle 2-subspaces** and give a general definition of this type of 2-subspaces.

Definition 10. *Let X be a vector space over a field Φ, and let $x_1, x_2, ..., x_n,\ n > 3$ be linearly independent vectors. The 2-subspace generated with the elements of the set $\{(x_1, x_2), (x_2, x_3), ..., (x_{n-1}, x_n), (x_n, x_1)\}$ will be called **cycle 2-subspace**.*

From the description of the sub case 2 of the case 2, we notice another type of specific 2-subspace of the 2-vector space X^2. So, if a, b, c, d are linearly independent vectors then the 2-subspace generated with the set $\{(a, b), (b, c), c, d)\}$ will be called **branch 2-subspace**. These types of 2-subspaces may be considered in a general case (with more than 3 generator elements).

Definition 11. *Let X be a vector space over a field Φ, and let $x_1, x_2, ..., x_n, ...$ be linearly independent vectors. The 2-subspace generated with the elements of the set $\{(x_1, x_2), (x_2, x_3), ..., (x_{n-1}, x_n), ...\}$ will be called **branch 2-subspace**.*

Note. The generator set in the previous definition might be finite one.

The 2-subspaces in the case 3 give motivation for the definition, in a general sense, of another type of 2-subspaces of 2-vector space.

Definition 12. *Let $x_1, x_2, x_3, ..., x_n, ...$ be linearly independent vectors of the vector space X. The 2-subspace generated with the set $\{(x_1, x_i) \mid i = 2, 3, 4, ...\}$ will be called **loop 2-subspace**.*

Note. The generator set in the previous definition might be finite one.

The 2-subspaces which do not have cycle 2-subspases and kernel 2-subspace as their own subspaces will be called **tree 2-subspaces**.

The discussion in the sub case 2 of the case 3 is a motivation for complete description of the branch 2-subspaces.

Theorem 4. *If M is a branch 2-subspace generated with the set $\{(x_1, x_2), (x_2, x_3), ..., (x_{m-1}, x_m), ...\}$, where $\{x_1, x_2, x_3, ..., x_m, ...\}$ is linearly independent set, then*

$$M = \bigcup_{i\in\mathbb{N}\setminus\{1\}} \bigcup_{a_{i-1},a_{i+1}\in\Phi} L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i).$$

Proof. Let $B_{i-1} = \{(x_{i-2}, x_{i-1}), (x_{i-1}, x_i)\}$ and $B_{i+1} = \{(x_i, x_{i+1}), (x_{i+1}, x_{i+2})\}$. Then the 2-subspaces generated with B_{i-1} and B_{i+1} are

$P_{B_{i-1}} = \bigcup_{\alpha,\beta\in\Phi} L(\alpha x_{i-2} + \beta x_i, x_{i-1}) \times L(\alpha x_{i-2} + \beta x_i, x_{i-1})$

$P_{B_{i+1}} = \bigcup_{\gamma,\delta\in\Phi} L(\gamma x_i + \delta x_{i+2}, x_{i+1}) \times L(\gamma x_i + \delta x_{i+2}, x_{i+1})$

Arbitrary two elements from $P_{B_{i-1}}$ and $P_{B_{i+1}}$,
$(a_{11}(\alpha x_{i-2} + \beta x_i) + a_{12}x_{i-1}, a_{21}(\alpha x_{i-2} + \beta x_i) + a_{22}x_{i-1})$ and
$(b_{11}(\gamma x_i + \delta x_{i+2}) + b_{12}x_{i+1}, b_{21}(\gamma x_i + \delta x_{i+2}) + b_{22}x_{i+1})$
may be added if and only if $a_{22} = b_{22} = 0$, $a_{21}\alpha = 0$, $b_{21}\delta = 0$ and $a_{21}\beta = b_{21}\gamma = s$.

If $\alpha = \delta = 0$, then
$(a_{11}\beta x_i + a_{12}x_{i-1}, sx_i) + (b_{11}\gamma x_i + b_{12}x_{i+1}, sx_i) = (a_{12}x_{i-1} + (a_{11}\beta + b_{11}\gamma)x_i + b_{12}x_{i+1}, sx_i) \in P_{B_i}$ where $B_i = \{(x_{i-1}, x_i), (x_i, x_{i+1})\}$.

The rest of the cases $a_{21} = 0, \delta = 0$; $b_{21} = 0, \alpha = 0$ or $a_{21} = 0, b_{21} = 0$ reduce to adding of elements of Δ_2.

Therefore, for two elements whose sum is defined, the sum always belongs to some of the 2-subspaces P_{B_k}, where $B_k = \{(x_{k-1}, x_k), (x_k, x_{k+1})\}$.

Repeating the procedure from the sub case 2 of the case 3 for the subspaces $P_{B_{i-1}}$ and P_{B_i}, for $i = 2, 3, 4, ...$ and using the previous part of the proof of the theorem, we conclude the proof. □

Note. If the generator set in the previous theorem is finite, $\{(x_1, x_2), (x_2, x_3), ..., (x_{n-1}, x_n)\}$, then

$$M = \bigcup_{i=2}^{n-1} \bigcup_{a_{i-1},a_{i+1}\in\Phi} L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i) \times L(a_{i+1}x_{i+1} + a_{i-1}x_{i-1}, x_i).$$

Because of the description of the 2-subspace in the sub case 1 of the case 3, and using the principle of mathematical induction, we can easily prove the following theorem.

Theorem 5. *If M is loop 2-subspace generated with the set $\{(x_1, x_i) \mid i = 2, 3, 4, ...\}$, then*

$$M = \bigcup_{u\in L(x_2,x_3,...)} L(u, x_1) \times L(u, x_1),$$

where $L = L(x_2, x_3, ...)$ is the subspace of X generated with $\{x_2, x_3, ...\}$.

Proof. The proof follows from the sub case 1 of the case 3, using the principle of mathematical induction. □

Theorem 6. *The cycle 2-subspace generated with the elements of the set $\{(x_1, x_2), (x_2, x_3), ..., (x_{n-1}, x_n), (x_n, x_1)\}$ for $n \geq 5$ is*

$$M = \bigcup_{\alpha,\beta\in\Phi} [L(\alpha x_1 + \beta x_3, x_2) \times L(\alpha x_1 + \beta x_3, x_2)] \cup$$
$$\cup \bigcup_{\gamma,\delta\in\Phi} [L(\gamma x_2 + \delta x_4, x_3) \times L(\gamma x_2 + \delta x_4, x_3)] \cup$$
$$\cup \ldots \cup \bigcup_{\theta,\eta\in\Phi} [L(\theta x_{n-1} + \eta x_1, x_n) \times L(\theta x_{n-1} + \eta x_1, x_n)].$$

Proof. The proof is a direct consequence from theorem 4 and the sub case 2 of the case 3. □

Note. A cycle 2-subspace for $n = 4$ is described in the sub case 4 of the case 3. A cycle 2-subspace for $n = 3$ is a kernel 2-subspace described in the note after the sub case 3 of the case 2.

The following problem concerning the 2-subspaces of 2-vector space X^2 arises from the consideration of the previous examples of the 2-subspaces.

Problem. Whether every 2-subspace S of a given 2-vector space has at least one minimal generator set $\{(x_\alpha, x_\beta) \,|\, \alpha, \beta \in A\}$, i.e. a set which satisfies

$(x_\gamma, x_\delta) \notin P_B$, where $B = \{(x_\alpha, x_\beta) \,|\, \alpha, \beta \in A\} \backslash \{(x_\gamma, x_\delta)\}$, for every $(x_\gamma, x_\delta) \in \{(x_\alpha, x_\beta) \,|\, \alpha, \beta \in A\}$ and $S = P_B$?

Such minimal generator set of 2-subspace S, if it exists, will be called **2-base** of S.

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НЕКОИ 2-ПОДПРОСТОРИ НА 2-ПРОСТОР

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Р е з и м е

Во трудот ќе бидат разгледувани некои 2-подпростори од 2-векторски простор и ќе биде дадена нивна карактеризација.

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