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NEW CHARACTERIZATION OF 2-PRE-HILBERT SPACE

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Abstract. The problem of finding necessary and sufficient conditions a 2-normed space to be treated as 2-pre-Hilbert space is the focus of interest of many mathematicians. Few characterizations of 2-inner product are given in [1], [3], [5], [6], [8] and [9]. In this paper a new necessary and sufficient condition for existence of 2-inner product into 2-normed space is given.

1. INTRODUCTION

Let L be a real vector space with dimension greater than 1 and $\|\cdot,\cdot\|$ be a real function on $L\times L$ such that:

a) $\|x,y\|\geq 0$, for all $x,y\in L$ and $\|x,y\|=0$ if and only if the set $\{x,y\}$ is linearly dependent,

b) $\|x,y\|=\|y,x\|$, for all $x,y\in L$,

c) $\|\alpha x,y\|=|\alpha|\cdot\|x,y\|$, for all $x,y\in L$ and for each $\alpha\in\mathbf{R}$, and

d) $\|x+y,z\|\leq\|x,z\|+\|y,z\|$, for all $x,y,z\in L$.

The function $\|\cdot,\cdot\|$ is said to be 2-*norm of L*, and $(L,\|\cdot,\cdot\|)$ is said to be *vector* 2-*normed space* ([7]). The inequality in the axiom *d*) is said to be parallelepiped inequality.

Let $n>1$ be a positive integer, L be a real vector space, $\dim L\geq n$ and $(\cdot,\cdot|\cdot)$ be a real function over $L\times L\times L$ such that:

i) $(x,x|y)\geq 0$, for all $x,y\in L$ and $(x,x|y)=0$ if and only if x and y are linearly dependent,

ii) $(x,y|z)=(y,x|z)$, for all $x,y,z\in L$,

iii) $(x,x|y)=(y,y|x)$, for all $x,y\in L$,

iv) $(\alpha x,y|z)=\alpha(x,y|z)$, for all $x,y,z\in L.$ and for each $\alpha\in\mathbf{R}$, and

v) $(x+x_1,y|z)=(x,y|z)+(x_1,y|z)$, for all $x_1,x,y,z\in L$.

The function $(\cdot,\cdot|\cdot)$ is said to be 2-*inner product,* and $(L,(\cdot,\cdot|\cdot))$ is said to be 2-*pre-Hilbert space* ([3]).

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The concepts of 2-norm and 2-inner product are two dimensional analogies of the concepts of norm and inner product. R. Ehret proved ([7]) that if $(L,(\cdot,\cdot\,|\,\cdot))$ is a 2-pre-Hilbert space, then

$$\| x,y \| = (x,x\,|\,y)^{1/2}, \tag{1}$$

for all $x,y\in L$ defines 2-norm. So, we get vector 2-normed space $(L,\|\cdot,\cdot\|)$ and moreover, for all $x,y,z\in L$ the following equalities are satisfied:

$$(x,y\,|\,z) = \frac{\|x+y,z\|^2-\|x-y,z\|^2}{4}, \tag{2}$$

$$\| x+y,z \|^2 + \| x-y,z \|^2 = 2(\| x,z \|^2 + \| y,z \|^2), \tag{3}$$

The equality (3) is actually analogous to the parallelogram equality and it is called parallelepiped equality. Further, 2-normed space L is 2-pre-Hilbert space if and only if for all $x,y,z\in L$ the equality (3) holds true.

2. CHARACTERIZATION OF 2-PRE-HILBERT SPACE

The problem of characterization of 2-pre-Hilbert spaces, i.e. finding the necessary and sufficient conditions the 2-normed spaces to be treated as 2-pre-Hilbert space is of particular interest while studying the 2-normed spaces. Thus, in [1] is given characterization of 2-pre-Hilbert space using the equality of Euler-Lagrange type, in [8] is given characterization using the strictly convex norm with modulus c, and in [9] are given characterizations using the Mercer inequality for 2-normed space and its equivalent inequality. In the following theorem are given some of the already known characterizations of 2-pre-Hilbert spaces, which are necessary for our further considerations.

Theorem 1 ([3]). Let $(L,\|\cdot,\cdot\|)$ be 2-normed space. L is 2-pre-Hilbert space if and only if for each $z\in L\setminus\{0\}$ one of the following conditions is satisfied:

II_1. For all $x,y,z\in L$ such that $\|x,z\|=\|y,z\|$ and for all $m,n\in\mathbf{R}$ it holds true that

$$\| mx+ny,z \| = \| nx+my,z \|.$$

II_2. $\|x+y,z\|=\|x-y,z\|$, $x,y,z\in L$ implies that

$$\| x+y,z \|^2 = \| x,z \|^2 + \| y,z \|^2$$

II_3. It exists a real number $\alpha\neq 0,\pm 1$ such that $\|x,z\|=\|y,z\|$, $x,y,z\in L$ implies that $\|x+\alpha y,z\|=\|\alpha x+y,z\|$.

II_4. It exists a real number $\alpha\neq 0,\pm 1$ such that $\|x+y,z\|=\|x-y,z\|$, $x,y,z\in L$ implies that $\|x+\alpha y,z\|=\|x-\alpha y,z\|$.

II_5. $\|x,z\|=\|y,z\|$, $x,y,z\in L$ implies that for each real number $\alpha>0$ it holds true that

$$\| \alpha x + \alpha^{-1} y, z \| \geq \| x + y, z \|.$$

II_6. For all $x_1, x_2, x_3, z \in L$ such that $\sum_{i=1}^{3} x_i = 0$ and $\| x_1, z \| = \| x_2, z \|$ it holds true that

$$\| x_1 - x_3, z \| = \| x_2 - x_3, z \|.$$

II_7. For all $x_1, x_2, x_3, x_4, z \in L$ such that $\sum_{i=1}^{4} x_i = 0$ and $\| x_1, z \| = \| x_2, z \|$ and $\| x_3, z \| = \| x_4, z \|$ it holds true that

$$\| x_1 - x_3, z \| = \| x_2 - x_4, z \| \text{ and } \| x_2 - x_3, z \| = \| x_1 - x_4, z \|.$$

II_8. For all $x_1, x_2, x_3, z \in L$ the value of the expression

$$F(x_1, x_2, x_3, z) = \| x_1 + x_2 + x_3, z \|^2 + \| x_1 + x_2 - x_3, z \|^2 - \\ - \| x_1 - x_2 - x_3, z \|^2 - \| x_1 - x_2 + x_3, z \|^2$$

does not depend on x_3.

II_9. For all $x_1, ..., x_n, z \in L$, $n \geq 3$ such that $\sum_{i=1}^{n} x_i = 0$ it holds true that

$$\sum_{i \neq k} \| x_i - x_k, z \|^2 = 2n \sum_{i=1}^{n} \| x_i, z \|^2 . \ \blacksquare$$

In the following theorem a new characterization of 2-pre-Hilbert space will be given.

Theorem 2. Let $(L, \| \cdot, \cdot \|)$ be a real 2-normed space. Then L is 2-pre-Hilbert space if and only if the following condition is satisfied

II_{10}. If $n \geq 3$, $x_1, x_2, ..., x_n, z \in L$ and $\alpha_1, \alpha_2, ..., \alpha_n$ are real numbers such that $\sum_{i=1}^{n} \alpha_i = 0$, then

$$\| \sum_{i=1}^{n} \alpha_i x_i, z \|^2 = - \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \| x_i - x_j, z \|^2 .$$

Proof. Let the condition II_{10} be satisfied. If $x_1, x_2, z \in L$, then for $x_3 = 0$ and the condition II_{10} applied to the vectors $x_1, x_2, x_3, z \in L$ and the real numbers $\alpha_1 = \alpha_2 = 1, \alpha_3 = -2$ follow the following equalities

$$\begin{aligned} \| x_1 + x_2, z \|^2 &= \| x_1 + x_2 + (-2) \cdot 0, z \|^2 \\ &= -1 \cdot (-2) \| x_1 - 0, z \|^2 - 1 \cdot (-2) \| x_2 - 0, z \|^2 - 1 \cdot 1 \| x_1 - x_2, z \|^2 \\ &= 2 \| x_1, z \|^2 + 2 \| x_2, z \|^2 - \| x_1 - x_2, z \|^2, \end{aligned}$$

The latter implies the parallelepiped equality, which actually means that L is 2-pre-Hilbert space.

Let L be 2-pre-Hilbert space. Applying the principle of mathematical induction we will prove that the condition II_{10} is satisfied. Let $n=3$, $\alpha_1,\alpha_2,\alpha_3$ be real numbers such that $\alpha_1+\alpha_2+\alpha_3=0$ and $x_1,x_2,x_3,z\in L$. Then by the properties of 2-inner product and since $\alpha_1+\alpha_2=-\alpha_3$ we get that

$$\begin{aligned}
\|\alpha_1x_1+\alpha_2x_2+\alpha_3x_3,z\|^2 &= \|\alpha_1(x_1-x_3)+\alpha_2(x_2-x_3),z\|^2\\
&=\alpha_1^2\|x_1-x_3,z\|^2+\alpha_1\alpha_2(x_1-x_3,x_2-x_3\,|\,z)\\
&\quad+\alpha_1\alpha_2(x_2-x_3,x_1-x_3\,|\,z)+\alpha_2^2\|x_2-x_3,z\|^2\\
&=\alpha_1^2\|x_1-x_3,z\|^2+\alpha_1\alpha_2(x_1-x_3,x_1-x_3+x_2-x_1\,|\,z)\\
&\quad+\alpha_1\alpha_2(x_2-x_3,x_2-x_3+x_1-x_2\,|\,z)+\alpha_2^2\|x_2-x_3,z\|^2\\
&=\alpha_1^2\|x_1-x_3,z\|^2+\alpha_1\alpha_2\|x_1-x_3,z\|^2+\alpha_1\alpha_2(x_1-x_3,x_2-x_1\,|\,z)\\
&\quad+\alpha_1\alpha_2\|x_2-x_3,z\|^2+\alpha_1\alpha_2(x_2-x_3,x_1-x_2\,|\,z)+\alpha_2^2\|x_2-x_3,z\|^2\\
&=\alpha_1(\alpha_1+\alpha_2)\|x_1-x_3,z\|^2+\alpha_2(\alpha_1+\alpha_2)\|x_2-x_3,z\|^2\\
&\quad-\alpha_1\alpha_2[(x_1-x_3,x_1-x_2\,|\,z)+(x_3-x_2,x_1-x_2\,|\,z)]\\
&=-\alpha_1\alpha_3\|x_1-x_3,z\|^2-\alpha_2\alpha_3\|x_2-x_3,z\|^2-\alpha_1\alpha_2\|x_1-x_2,z\|^2,
\end{aligned}$$

which means that the condition II_{10} holds true.

Let in the 2-pre-Hilbert space L the condition II_{10} be satisfied for some positive integer $n\geq 3$. Let $x_1,x_2,...,x_n,x_{n+1},z\in L$ and $\alpha_1,\alpha_2,...,\alpha_n,\alpha_{n+1}$ be real numbers such that $\sum_{i=1}^{n+1}\alpha_i=0$ and let

$$\beta=\alpha_1+\alpha_2+...+\alpha_{n-1}=-(\alpha_n+\alpha_{n+1})\,.$$

Then, since

$$\frac{\beta}{-\alpha_{n+1}}+\frac{\alpha_n}{-\alpha_{n+1}}-1=0 \text{ and } 1+\sum_{i=1}^{n+1}\frac{\alpha_i}{\beta}=0$$

the inductive assumption implies that

$$\begin{aligned}
\|\sum_{i=1}^{n+1}\alpha_ix_i,z\|^2 &= \alpha_{n+1}^2\|\sum_{i=1}^{n-1}\frac{\alpha_i}{-\alpha_{n+1}}x_i+\frac{\alpha_n}{-\alpha_{n+1}}x_n+(-1)x_{n+1},z\|^2\\
&=\alpha_{n+1}^2\|\frac{\beta}{-\alpha_{n+1}}(\sum_{i=1}^{n-1}\frac{\alpha_i}{\beta}x_i)+\frac{\alpha_n}{-\alpha_{n+1}}x_n+(-1)x_{n+1},z\|^2\\
&=\alpha_{n+1}^2[-\frac{\beta\alpha_n}{\alpha_{n+1}^2}\|\sum_{i=1}^{n-1}\frac{\alpha_i}{\beta}x_i-x_n,z\|^2-\frac{\beta}{\alpha_{n+1}}\|\sum_{i=1}^{n-1}\frac{\alpha_i}{\beta}x_i-x_{n+1},z\|^2-\frac{\alpha_n}{\alpha_{n+1}}\|x_{n+1}-x_n,z\|^2\\
&=-\beta\alpha_n\|\sum_{i=1}^{n-1}\frac{\alpha_i}{\beta}x_i-x_n,z\|^2-\beta\alpha_{n+1}\|\sum_{i=1}^{n-1}\frac{\alpha_i}{\beta}x_i-x_{n+1},z\|^2-\alpha_n\alpha_{n+1}\|x_{n+1}-x_n,z\|^2\\
&=-\beta\alpha_n[-\sum_{1\leq i<j\leq n-1}\frac{\alpha_i\alpha_j}{\beta^2}\|x_i-x_j,z\|^2+\sum_{i=1}^{n}\frac{\alpha_i}{\beta}\|x_i-x_n,z\|^2]
\end{aligned}$$

$$-\beta\alpha_{n+1}[-\sum_{1\le i<j\le n-1}\frac{\alpha_i\alpha_j}{\beta^2}\|x_i-x_j,z\|^2+\sum_{i=1}^{n}\frac{\alpha_i}{\beta}\|x_i-x_{n+1},z\|^2]$$
$$-\alpha_n\alpha_{n+1}\|x_{n+1}-x_n,z\|^2$$
$$=\frac{\alpha_n+\alpha_{n+1}}{\beta}\sum_{1\le i<j\le n-1}\alpha_i\alpha_j\|x_i-x_j,z\|^2-\sum_{i=1}^{n}\alpha_i\alpha_n\|x_i-x_n,z\|^2$$
$$-\sum_{i=1}^{n}\alpha_i\alpha_{n+1}\|x_i-x_{n+1},z\|^2-\alpha_n\alpha_{n+1}\|x_{n+1}-x_n,z\|^2$$
$$=-\sum_{1\le i<j\le n+1}\alpha_i\alpha_j\|x_i-x_j,z\|^2.$$

The latter means that the condition II_{10} also holds true for $n+1$. So, the principle of mathematical induction implies that II_{10} holds true for each positive integer. ■

The theorems 1 and 2 imply that in 2-normed space the conditions II_1-II_{10} are equivalent to each other. In the further considerations we will prove that the condition II_{10} directly implies some of the conditions II_1 - II_9.

Lemma 1. Let L be 2-normed space. Then the condition II_{10} implies the condition II_9.

Proof. Let $x_1,...,x_n,z\in L$, $n\ge 3$ be such that $\sum_{i=1}^{n}x_i=0$. Then the condition II_{10} implies the following

$$0=\|\sum_{i=1}^{n}x_i,z\|^2=\|x_1+x_2+...+x_n-n\cdot 0,z\|^2=n\sum_{i=1}^{n}\|x_i-0,z\|^2-\sum_{1\le i<k\le n}\|x_i-x_k,z\|^2,$$

which implies that

$$\sum_{i\ne k}\|x_i-x_k,z\|^2=\sum_{1\le i<k\le n}\|x_i-x_k,z\|^2+-\sum_{1\le k<i\le n}\|x_k-x_i,z\|^2=2n\sum_{i=1}^{n}\|x_i,z\|^2,$$

i.e. the condition II_9 is satisfied. ■

Lemma 2. Let L be 2-normed space. Then the condition II_{10} implies the condition II_5.

Proof. Let $\|x,z\|=\|y,z\|$, $x,y,z\in L$ and $\alpha>0$ be real number. Then the condition II_{10} implies the following

$$\|\alpha x+\alpha^{-1}y,z\|^2=\|\alpha x+(-\alpha^{-1})(-y)+(\alpha^{-1}-\alpha)0,z\|^2$$
$$=-\alpha(\alpha^{-1}-\alpha)\|x,z\|^2+\alpha^{-1}(\alpha^{-1}-\alpha)\|y,z\|^2+\|x+y,z\|^2$$
$$=(-1+\alpha^2+\frac{1}{\alpha^2}-1)\|x,z\|^2+\|x+y,z\|^2$$
$$=(\alpha+\frac{1}{\alpha})^2\|x,z\|^2+\|x+y,z\|^2\ge\|x+y,z\|^2,$$

thus $\| \alpha x+\alpha^{-1}y,z \| \geq \| x+y,z \|$, i.e. the condition II_5 is satisfied. ■

Lemma 3. Let L be 2-normed space. Then the condition II_{10} implies the condition II_1.

Proof. Let $x,y,z\in L$ be such that $\| x,z \| = \| y,z \|$ $m,n\in \mathbf{R}$. Then the condition II_{10} implies that

$$\begin{aligned}\| mx+ny,z \|^2 &= \| mx+ny+(-m-n)0,z \|^2 \\ &= m(m+n)\| x,z \|^2 + n(m+n)\| y,z \|^2 - mn\| x-y,z \|^2\end{aligned}$$

and

$$\begin{aligned}\| nx+my,z \|^2 &= \| nx+my+(-m-n)0,z \|^2 \\ &= n(m+n)\| x,z \|^2 + m(m+n)\| y,z \|^2 - mn\| x-y,z \|^2 .\end{aligned}$$

Further, since $\| x,z \| = \| y,z \|$, the last two equalities imply that

$$\| nx+my,z \|^2 = \| mx+ny,z \|^2 \text{, i.e. } \| mx+ny,z \| = \| nx+my,z \| ,$$

The latter means that the condition II_1 is satisfied. ■

Lemma 4. Let L be 2-normed space. Then the condition II_{10} implies the condition II_3.

Proof. Let $\| x,z \| = \| y,z \|$, $x,y,z\in L$ an let α be a real number such that $\alpha\neq 0,\pm 1$. Then the condition II_{10} implies that

$$\begin{aligned}\| x-y,z \|^2 &= \| \tfrac{1}{\alpha}(\alpha x)+(-y)+(-1-\tfrac{1}{\alpha})0,z \|^2 \\ &= \tfrac{\alpha+1}{\alpha^2}\| \alpha x,z \|^2 + \tfrac{\alpha+1}{\alpha}\| -y,z \|^2 - \tfrac{1}{\alpha}\| \alpha x-(-y),z \|^2 \\ &= (\alpha+1)\| x,z \|^2 + \tfrac{\alpha+1}{\alpha}\| y,z \|^2 - \tfrac{1}{\alpha}\| \alpha x+y,z \|^2\end{aligned}$$

and

$$\begin{aligned}\| x-y,z \|^2 &= \| x+\tfrac{1}{\alpha}(-\alpha y)+(-1-\tfrac{1}{\alpha})0,z \|^2 \\ &= \tfrac{\alpha+1}{\alpha}\| x,z \|^2 + \tfrac{\alpha+1}{\alpha^2}\| -\alpha y,z \|^2 - \tfrac{1}{\alpha}\| x-(-\alpha y),z \|^2 \\ &= \tfrac{\alpha+1}{\alpha}\| x,z \|^2 + (\alpha+1)\| y,z \|^2 - \tfrac{1}{\alpha}\| x+\alpha y,z \|^2 .\end{aligned}$$

Further, since $\| x,z \| = \| y,z \|$ holds true, the last two equalities imply that

$$\| \alpha x+y,z \|^2 = \| x+\alpha y,z \|^2 \text{, т.е. } \| \alpha x+y,z \| = \| x+\alpha y,z \| ,$$

The latter means that the condition II_3 is satisfied. ■

Lemma 5. Let L be 2-normed space. Then the condition II_{10} implies the condition II_6.

Proof. Let $x_1,x_2,x_3,z\in L$ be such that $\sum_{i=1}^{3} x_i = 0$ and $\| x_1,z \| = \| x_2,z \|$. For $\alpha = 2$, the Lemma 4 implies that

$$\| 2x_1 + x_2, z \| = \| x_1 + 2x_2, z \|$$

holds true. Further, since $\sum_{i=1}^{3} x_i = 0$ we get that $x_3 = -x_1 - x_2$, thus

$$\| x_1 - x_3, z \| = \| x_1 - (-x_1 - x_2 (, z \| = \| 2x_1 + x_2, z \| = \| x_1 + 2x_2, z \|$$
$$= \| x_2 - (-x_1 - x_2), z \| = \| x_2 - x_3, z \|,$$

The latter means that the condition II_6 is satisfied. ■

Lemma 6. Let L be 2-normed space. Then the condition II_{10} implies the condition II_7.

Proof. Let $x_1, x_2, x_3, x_4, z \in L$ be such that $\sum_{i=1}^{4} x_i = 0$ and $\| x_1, z \| = \| x_2, z \|$ and $\| x_3, z \| = \| x_4, z \|$. Further, since $x_1 + x_2 + (x_3 + x_4) = 0$ and $\| x_1, z \| = \| x_2, z \|$ holds true, the Lemma 5 implies that $\| x_1 - x_3 - x_4, z \| = \| x_2 - x_3 - x_4, z \|$. Further, the condition II_{10} implies that

$$\| x_1 - x_3 - x_4, z \|^2 = \| x_1 - x_3 - x_4 + 0, z \|^2$$
$$= -\| x_1, z \|^2 + \| x_3, z \|^2 + \| x_4, z \|^2 + \| x_1 - x_3, z \|^2 + \| x_1 - x_4, z \|^2 - \| x_3 - x_4, z \|^2$$

and

$$\| x_2 - x_3 - x_4, z \|^2 = \| x_2 - x_3 - x_4 + 0, z \|^2$$
$$= -\| x_2, z \|^2 + \| x_3, z \|^2 + \| x_4, z \|^2 + \| x_2 - x_3, z \|^2 + \| x_2 - x_4, z \|^2 - \| x_3 - x_4, z \|^2$$

and since $\| x_1 - x_3 - x_4, z \| = \| x_2 - x_3 - x_4, z \|$ and $\| x_1, z \| = \| x_2, z \|$ we get that

$$\| x_1 - x_3, z \|^2 + \| x_1 - x_4, z \|^2 = \| x_2 - x_3, z \|^2 + \| x_2 - x_4, z \|^2 . \qquad (1)$$

Analogously can be proven the following

$$\| x_3 - x_1, z \|^2 + \| x_3 - x_2, z \|^2 = \| x_4 - x_1, z \|^2 + \| x_4 - x_2, z \|^2 . \qquad (2)$$

Finally, (1) and (2) imply that $\| x_1 - x_3, z \| = \| x_2 - x_4, z \|$ and $\| x_2 - x_3, z \| = \| x_1 - x_4, z \|$. The latter means that the condition II_7 is satisfied. ■

Lemma 7. Let L be 2-normed space. Then the condition II_{10} implies the condition II_8.

Proof. Let $x_1, x_2, x_3, z \in L$. Then the condition II_{10} implies

$$\| 2x_1 + 2x_2, z \|^2 = \| x_1 + (x_2 + x_3) - (x_3 - x_2) - (-x_1), z \|^2$$
$$= -\| x_1 - (x_2 + x_3), z \|^2 + \| x_1 - (x_3 - x_2), z \|^2 + \| x_1 - (-x_1), z \|^2$$
$$+ \| x_2 + x_3 - (x_3 - x_2), z \|^2 + \| x_2 + x_3 - (-x_1), z \|^2 - \| x_3 - x_2 - (-x_1), z \|^2$$
$$= -\| x_1 - x_2 - x_3), z \|^2 + \| x_1 - x_3 + x_2, z \|^2 + \| 2x_1, z \|^2$$
$$+ \| 2x_2, z \|^2 + \| x_2 + x_3 + x_1, z \|^2 - \| x_3 - x_2 + x_1, z \|^2,$$

thus

$$F(x_1,x_2,x_3,z) = \| x_1 + x_2 + x_3, z \|^2 + \| x_1 + x_2 - x_3, z \|^2$$
$$- \| x_1 - x_2 - x_3, z \|^2 - \| x_1 - x_2 + x_3, z \|^2$$
$$= \| 2x_1 + 2x_2, z \|^2 - \| 2x_1, z \|^2 - \| 2x_2, z \|^2,$$

The later means that the condition II_8 is satisfied. ■

Lemma 8. Let L be 2-normed space. Then the condition II_{10} implies the condition II_4.

Proof. Let $\| x+y, z \| = \| x-y, z \|$, $x,y,z \in L$ and let α be a real number such that $\alpha \neq 0, \pm 1$. Then the condition II_{10} implies

$$\| x-y, z \|^2 = \| x + \tfrac{1}{\alpha}(-\alpha y) + (-1-\tfrac{1}{\alpha})0, z \|^2$$
$$= \tfrac{\alpha+1}{\alpha} \| x, z \|^2 + \tfrac{\alpha+1}{\alpha^2} \| -\alpha y, z \|^2 - \tfrac{1}{\alpha} \| x-(-\alpha y), z \|^2$$
$$= \tfrac{\alpha+1}{\alpha} \| x, z \|^2 + (\alpha+1) \| y, z \|^2 - \tfrac{1}{\alpha} \| x+\alpha y, z \|^2 .$$

and

$$\| x+y, z \|^2 = \| -x - \tfrac{1}{\alpha}(\alpha y) + (1+\tfrac{1}{\alpha})0, z \|^2$$
$$= \tfrac{\alpha+1}{\alpha} \| x, z \|^2 + \tfrac{\alpha+1}{\alpha^2} \| \alpha y, z \|^2 - \tfrac{1}{\alpha} \| x-\alpha y, z \|^2$$
$$= \tfrac{\alpha+1}{\alpha} \| x, z \|^2 + (\alpha+1) \| y, z \|^2 - \tfrac{1}{\alpha} \| x-\alpha y, z \|^2 .$$

Further, since $\| x+y, z \| = \| x-y, z \|$, the last two equalities imply that

$$\| x+\alpha y, z \|^2 = \| x-\alpha y, z \|^2, \text{ i.e. } \| x+\alpha y, z \| = \| x-\alpha y, z \| .$$

The latter means that the condition II_4 is satisfied. ■

Lemma 9. Let L be 2-normed space. Then the condition II_{10} implies the condition II_2.

Proof. Let $\| x+y, z \| = \| x-y, z \|$, $x,y,z \in L$. Then since the proof of Theorem 2 we get

$$\| x+y, z \|^2 = 2 \| x, z \|^2 + 2 \| y, z \|^2 - \| x-y, z \|^2 ,$$

and since $\| x+y, z \| = \| x-y, z \|$, the last equality is equivalent with

$$\| x+y, z \|^2 = \| x, z \|^2 + \| y, z \|^2 .$$

The latter means that the condition II_2 is satisfied. ■

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