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LONG-TERM OPERATION OF RESERVOIRS IN SERIES

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Abstract. In this paper we discuss the optimal long-term operation of multireservoir power system connected in series on a river for maximum total benefits from the system. The paper begins with the problem formulation, where the problem is posed as a mathematical problem. In the next section we develop a method to solve the problem using the minimum norm formulation in the framework of functional analysis optimization technique. We use a linear storage-elevation curve and a constant water conversion factor (MWh/m^3), for modeling the hydroelectric generation and the amount of water left in storage at the end of the optimization interval $[0, K]$, where K is number of the months.

1. Problem formulation

The system under study consists of n hydroelectric power plants in series on a river, *Figure 1.* We will number the installations from upstream to downstream. The problem of the power system of *Figure 1* is to determine the discharges $u_{ik}, i = 1, ..., n, k = 1, ..., K$, as functions of time under the following conditions:

(1)The expected value of the water in storage at the end of the last period studied is a maximum.

(2) The expected value of the MWh generated during the optimization interval is a maximum.

(3) The water conservation equation for each reservoir is adequately described by the following difference equations:

$$x_{ik} = x_{i,k-1} + I_{ik} + u_{i-1,k} - u_{ik} - s_{ik} + s_{i-1,k}, \quad i = 1, \ldots n, \quad k = 1, \ldots, K \quad (1)$$

where x_{ik} is the volume of water in the reservoir,

I_{ik} is the monthly inflow,

s_{ik} is the spillage.

(4) To satisfy the multipurpose stream use requirements the following operational constraints should be satisfied:

$$x_i^m \leq x_{ik} \leq x_i^M, \quad i = 1, \ldots, n, \quad k = 1, \ldots, K \quad (2)$$

$$u_{ik}^m \leq u_{ik} \leq u_{ik}^M, \quad i = 1, \ldots, n, \quad k = 1, \ldots, K \quad (3)$$

where x_i^M is the capacity of the reservoir, x_i^m is the minimum storage, u_{ik}^m is the minimum discharge through the turbine, and u_{ik}^M is the maximum discharge

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through the turbine. If $u_{ik} > u_{ik}^M$ and x_{ik} is equal to x_i^M then $s_{ik} = u_{ik} - u_{ik}^M$ is discharged through the spillways.

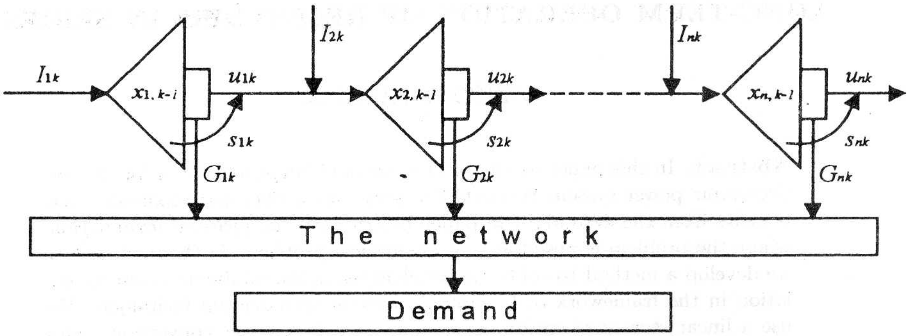

Figure 1

In mathematical terms, the long-term problem for the power system of *Figure 1* is to determine the discharges $u_{ij}, i = 1, \ldots, n, k = 1, \ldots, K$ that maximizes the profit function

$$J = \sum_{i=1}^{n} V_i(x_{i,K}) + \sum_{i=1}^{n} \sum_{k=1}^{K} c_k G_i(u_{ik}, x_{i,k-1}) \tag{4}$$

subject to satisfying the equality constrains given by (1) and the inequality constrains given by (2) and (3). Here $V_i(x_{iK})$ is the value of water left in storage in reservoir i at the end of the last period studied K, $Gi(x_{i,k-1}, u_{ik})$ is the generation of plant i during period k in MWh and c_k is the value *(in money)* of a MWh produced anywhere on the river in month k.

2. Modeling of the system

The conventional approach for obtaining the equivalent reservoir and hydroplant is based on the potential energy concept. Each reservoir on a river is mathematically represented by an equivalent potential energy balance equation.

The potential energy balance equation is obtained by multiplying both sides of the reservoir balance-of-water equation by the water conversion factors of at-site and downstream hydroplants. We may express the function $V_i(x_{iK})$ as:

$$V_i(x_{iK}) = \sum_{j=i}^{n} h_j x_{iK} \tag{5}$$

where h_j is the average water conversion factor (MWh/Mm^3) at side j. In the above equation we assumed that the cost of this energy is one $USA\$/MWh$ (the average cost during the year).

The generation of a hydroelectric plants is a nonlinear function of the water discharge u_{ik} and the net head, which itself is a function of the storage $x_{i,k-1}$.

We will assume a linear relation between the storage and the head (the storage-elevation curve is linear and the tailwater elevation is constant independently of the discharge). We may choose:

$$G_i(u_{ik}, x_{i,k-1}) = a_i u_{ik} + b_i u_{ik} x_{i,k-1} (MWh) \tag{6}$$

where a_i and b_i are constants for the reservoir i. Now, the profit functional in equation (4) becomes

$$J = \sum_{i=1}^{n} \sum_{j=1}^{n} h_j x_{iK} + \sum_{i=1}^{n} \sum_{k=1}^{K} (A_{ik} u_{ik} + u_{ik} B_{ik} x_{i,k-1}) \tag{7}$$

subject to satisfying the constraints given by (1)-(3), where $A_{ik} = a_i c_k$, $B_{ik} = b_i c_k, i = 1, \ldots, n$.

3. Mathematical

The reservoir dynamic equation is added to the profit functional using the unknown Lagrange multiplier lik, and the inequality constrains (2) – (3) are added using the Kuhn-Tucker multipliers, so that a modified functional is obtained:

$$\begin{aligned} J_0(u_{ik}, x_{i,k-1}) = & \sum_{i=1}^{n} \sum_{j=1}^{n} h_j x_{iK} + \sum_{i=1}^{n} \sum_{k=1}^{K} \{ A_{ik} u_{ik} + u_{ik} B_{ik} x_{i,k-1} + \\ & + \lambda_{ik} (-x_{ik} + x_{i,k-1} + u_{i-1,k} - u_{ik}) + \\ & + (e_{ik}^M - e_{ik}^m) x_{ik} + (f_{ik}^M - f_{ik}^m) u_{ik} \} \end{aligned} \tag{8}$$

Here terms explicitly independent of u_{ik} and x_{ik} are dropped. In the above equation E_{ik}^M, e_{ik}^M, f_{ik}^m and f_{ik}^M are Kuhn-Tucker multipliers. These are equal to zero if the constraints are not violated and greater than zero if the constraints are violated.

Let R denote a set of n reservoirs, and define the $n \times 1$ column vectors

$$H = col(H_i, i \in R), \text{ where } H_i = \sum_{j=1}^{n} h_j, \tag{9}$$

$$\begin{aligned} x(k) &= col(x_{ik}, i \in R), \\ u(k) &= col(u_{ik}, i \in R), \\ \lambda(k) &= col(\lambda_{ik}, i \in R), \\ \mu(k) &= col(\mu_{ik}, i \in R), \\ \psi(k) &= col(\psi_{ik}, i \in R), \end{aligned}$$

where $\mu_{ik} = e_{ik}^M - e_{ik}^m, \psi_{ik} = f_{ik}^M - f_{ik}^m$, $i \in R$, and the $n \times n$ diagonal matrix $B(k) = diag(B_{ik}, i \in R)$.

Furthermore, we define the $n \times n$ lower triangular matrix M by $m_{ii} = -1$, $i \in R$, $m_{j+1,j} = 1$, $j = 1, \ldots, n-1$.

Then the modified functional in equation (8) becomes

$$\begin{aligned} J_0(u(k), x(k-1)) = H^T x(K) + \sum_{k=1}^{K} \{ u^T(k)B(k)x(k-1) + \\ + (\lambda(k) + \mu(k)^T)x(k-1) - \lambda^T(k)x(k) + \\ + (A(k) + M^T\lambda(k) + M^T\mu(k) + \psi(k))^T u(k) \}. \end{aligned} \tag{10}$$

We will use the following identity:

$$\sum_{k=1}^{K} \lambda^T(k)x(k) = -\lambda^T(0)x(0) + \lambda^T(K)x(K) + \sum_{k=1}^{K} \lambda^T(k-1)x(k-1) \tag{11}$$

Then we can write the functional (10) as

$$\begin{aligned} J_0(u(k), x(k-1)) =& (H - \lambda(k))^T x(K) + \lambda^T(0)x(0) + \\ & + \sum_{k=1}^{K} \{ \mu^T(k)B(k)x(k-1) + \\ & + (\lambda(k) - \lambda(k-1) + \mu(k))^T x(k-1) + \\ & + (A(k) + M^T\lambda(k) + M^T\mu(k) + \psi(k))^T u(k) \} \end{aligned} \tag{12}$$

Define $2n \times 1$ column vectors

$$X(k) = col(x(k-1), u(k))$$

$$R(k) = col(\lambda(k) - \lambda(k-1) + \mu(k), A(k) + M^T\lambda(k) + M^T\mu(k) + \psi(k))$$

and $2n \times 2n$ matrix $L(k)$ by

$$L(k) = \begin{bmatrix} 0 & \frac{B(k)}{2} \\ \frac{B(k)}{2} & 0 \end{bmatrix}$$

Using these definitions, (12) becomes

$$\begin{aligned} J_1(x(K), X(K)) =& (H - \lambda(K))^T x(K) + \lambda^T(0)x(0) + \\ & + \sum_{k=1}^{K} \{ X(k)^T L(k) X(k) + R^T(k)X(k) \} \end{aligned} \tag{13}$$

Equation (13) is composed of a boundary term and a discrete integral part, which are independent of each other. To maximize J_0 in equation (13), one maximizes each term separately:

$$\begin{aligned} max J_1(x(K), X(k)) = \max_{x(K)} (H - \lambda(K))^T x(K) + \lambda^T(0)x(0)) + \\ + \max_{X(k)} \sum_{k=1}^{K} \{ X(k)^T L(k) X(k) + R^T(k)X(k) \} \end{aligned}$$

4. The Optimal Solution

There is exactly one optimal solution to the problem formulated in (14). The boundary part is optimized when $\lambda(k) - H = 0$ because $\delta x(K)$ is arbitrary and $x(0)$ is constant. We define a $2n \times 1$ vector $V(k)$ as

$$V(k) = L^{-1}(k)R(k). \tag{14}$$

Now, the discrete integral part of equation (14) can be written as

$$J_2(X(k)) = \sum_{k=1}^{K} \left\{ X(k) + \tfrac{1}{2}V(k))^T L(k)(X(k) + \tfrac{1}{2}V(k)) - \tfrac{1}{4}V^T(k)L(k)V(k) \right\}.$$

The last term in the above equation does not depend explicitly on $X(k)$, so it is enough only the equation

$$J_2(X(k)) = \sum_{k=1}^{K} (X(k) + \tfrac{1}{2}V(k))^T L(k)(X(k) + \tfrac{1}{2}V(k)) \tag{15}$$

to be considered. Equation (16) defines a norm. This norm is considered to be an element of Hilbert space because $X(k)$ is always positive. Equation (16) can be written as

$$J_2(X(k)) = \|X(k) + \tfrac{1}{2}V(k)\|_{L(k)}. \tag{16}$$

Maximization of $J2$ is mathematically equivalent to the minimization of the norm of equation (17). The minimum of the norm in equation (17) is clearly achieved when

$$\|X(k) + \frac{1}{2}V(k)\| = 0. \tag{17}$$

From the equations (15) and (18) we have that the optimal solution is given by

$$R(k) + 2L(k)X(k) = 0.$$

Writing the last equation explicitly and adding the reservoir dynamic equation, one obtains the long - term optimal equations as

$$\begin{aligned} &-x(k) + x(k-1) + I(k) + Mu(k) + Ms(k) = 0 \\ &\lambda(k) - \lambda(k-1) + \mu(k) + B(k)u(k) = 0 \\ &A(k) + M^T\lambda(k) + M^T\mu(k) + \psi(k) + B(k)x(k-1) = 0. \end{aligned} \tag{18}$$

We can now state the optimal solution of equations in component form as

$$-x_{ik} + x_{i,k-1} + I_{ik} + u_{i-1,k} - u_{ik} - s_{ik} + s_{i-1,k}0,\ i = 1,\dots,n, k = 1,\dots,K$$
$$\lambda_{ik} - \lambda_{i-1,k} + \mu_{ik} + c_k b_i u_{ik} = 0,\ i = 1,\dots,n,\ k = 1,\dots,K$$
$$c_k a_k + \lambda_{i+1,k} - \lambda_{ik} + \mu_{i+1,k} + \mu_{ik} + \psi_{ik} + c_k b_i x_{i,k-1} = 0,\ i = 1,\dots,n,\ k = 1,\dots,K.$$

Also we have the following limits on the variable:

$$\text{if } x_{ik} < x_i^m, \text{ then put } x_{ik} = x_i^m, \text{ if } x_{ik} > x_i^M \text{ then put } x_{ik} = x_i^M \qquad (19)$$
$$\text{if } u_{ik} < u_i^m, \text{ then put } u_{ik} = u_i^m, \text{ if } u_{ik} > u_i^M \text{ then put } u_{ik} = u_i^M$$

Relations (20) – (21) completely specify the optimal solution.

5. Practical Application

A computer program is written to solve equations (20) iteratively under the limits (21) using the steepest descend method. This program is applied to the system that consists of four reservoirs connected in series on a river. The characteristics of these reservoirs are given in *Table 1.* The optimization is done on a monthly time basis for a period of a year.

The expected natural inflows to the sites in the year on high flows, which we call *Year 1*, and the cost of the energy are given in *Table 2.* In *Table 3* we give the optimal discharges and the profits realized during the year of high flow. In *Table 4* we give the expected natural inflows to the sites in the year of low flows, which we call *Year 2.* Also, in *Table 5* we report the corresponding results obtained for the same system during the year of low flow.

Table 1

Site	Min. capacity x_i^m $[Mm^3]$	Max. capacity x_i^M $[Mm^3]$	Min. effective discharhe $u_{ik}^m[m^3/s]$	Max. effective discharge $u_{ik}^M[m^3/s]$	Reservoirs constants a_i $[MWh/Mm^3]$	b_i $[MWh/(Mm^3)^2]$
1	0	9628	0	270	11.80	1.300×10^{-3}
2	0	570	0	320	231.50	0.532×10^{-3}
3	0	50	0	320	215.82	12.667×10^{-3}
4	0	3420	0	380	473.00	11.173×10^{-3}

Table 2

Month k	I_{1k} $[Mm^3]$	I_{2k} $[Mm^3]$	I_{3k} $[Mm^3]$	I_{4k} $[Mm^3]$	c_k $[USA\$/MWh]$
1	828	380	161	1208	3.12
2	829	331	132	961	3.75
3	578	224	85	810	4.12
4	494	176	68	586	5.20
5	365	95	30	402	5.68
6	333	82	36	358	5.36
7	293	68	25	323	5.12
8	319	107	79	495	4.90
9	810	483	189	980	3.32
10	1287	781	225	1437	2.90
11	1150	591	146	1302	2.85
12	824	363	132	1145	3.18

Table 3

Month k	u_{1k} $[Mm^3]$	u_{2k} $[Mm^3]$	u_{3k} $[Mm^3]$	u_{4k} $[Mm^3]$	profits $[USA\$]$
1	82	268	528	708	980300
2	243	531	483	634	858930
3	1071	1295	1338	1454	1672310
4	1071	1217	1249	1567	2040840
5	968	1291	1334	1702	2471260
6	1065	1363	1413	1756	2556200
7	1039	1104	1290	1596	1852640
8	732	870	986	1120	1106010
9	548	945	1016	1321	1440070
10	306	764	1289	1349	1404850
11	0	319	467	673	723750
12	0	392	552	720	840200
				Total profits:	17947360

Table 4

Month k	I_{1k} $[Mm^3]$	I_{2k} $[Mm^3]$	I_{3k} $[Mm^3]$	I_{4k} $[Mm^3]$	c_k $[USA\$/MWh]$
1	568	207	129	918	3.12
2	442	193	87	592	3.75
3	460	171	54	508	4.12
4	305	127	43	411	5.20
5	224	86	33	272	5.68
6	205	73	20	223	5.36
7	208	61	12	177	5.12
8	261	90	34	245	4.90
9	498	301	108	730	3.32
10	742	474	125	930	2.90
11	624	369	91	870	2.85
12	527	229	87	882	3.18

Table 5

Month k	u_{1k} $[Mm^3]$	u_{2k} $[Mm^3]$	u_{3k} $[Mm^3]$	u_{4k} $[Mm^3]$	profits $[USA\$]$
1	0	215	334	425	502850
2	0	373	399	543	680040
3	771	942	1064	1180	1409300
4	660	967	1081	1221	1897510
5	768	1023	1152	1379	2171320
6	821	1065	1202	1382	2286370
7	889	1166	1244	1307	2045760
8	136	225	492	866	851270
9	0	398	484	787	712830
10	0	272	586	973	875890
11	0	0	122	0	43330
12	0	0	375	0	120370
				Total profits:	13596840

6. Conclusion

In this paper an application of the minimum norm theorem to the optimization of the total benefits from a multireservoir power system connected in series on a river is presended. It was found that this algorithm can deal with a large-scale power system with stochastic inflows. For obtaining suitable approximations new optimal equations are derived; if these equations are solved forward and backward in time, one can obtain the optimal long-term scheduling for maximum total benefits from any number of series reservoirs. A program simulation of the power system showed that the obtained mathematical formulation fitted quite well.

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