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PLURI-GREEN POTENTIALS IN THE UNIT BALL OF $\mathbb{C}^n$

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Dedicated to Academician Blagoj Popov on the Occasion of His 85^{th} Birthday

Abstract. In this paper, we present some results related to boundary behavior of pluri-Green potentials in the unit ball $\mathbb{B}$ of $\mathbb{C}^n$. Sufficient conditions for existence of radial and tangential limits of the pluri-Green potentials are given.

In the paper we also present some results related to exceptional sets of pluri-Green potentials in the unit ball of $\mathbb{C}^n$ in terms of the non-isotropic Hausdorff capacity.

1. Introduction

In this paper we study the boundary behavior of pluri-Green potentials V_μ on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$. These are functions of the form

$$V_\mu(z) = \int_{\mathbb{B}} \log \frac{1}{|\varphi_z(w)|} d\mu(w),$$

where μ is a nonnegative, regular Borel measure on $\mathbb{B}$ satisfying the condition

$$\int_{\mathbb{B}} (1 - |w|^2) d\mu(w) < \infty \tag{1.1}$$

and for a fixed $z \in \mathbb{B}$, φ_z denotes the holomorphic automorphism of $\mathbb{B}$ which satisfies $\varphi_z(0) = z$, $\varphi_z(z) = 0$ and $\varphi_z \circ \varphi_z(w) = w$, for every $w \in \mathbb{B}$.

Exceptional sets of Blaschke products, holomorphic functions, or Green potentials in the unit disc $\mathbb{U}$ of the complex plane $\mathbb{C}$ are described usually in terms of Hausdorff capacity (see [5], [6] and [13]). In higher dimension, the non-isotropic Hausdorff capacity is much more appropriate. For the description of exceptional sets of holomorphic functions, or invariant Green potentials see [2], [3], [4], [9] and [10]).

The paper was motivated by the works of Arsove and Huber [5], Cima and Stanton [8], Samuelsson [13], and Stoll [14], [15] and [16] in which boundary behavior of subharmonic functions in the unit disc $\mathbb{U}$ of $\mathbb{C}^n$, invariant, invariant Green potentials and $\mathcal{M}$ - subharmonic functions in the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ are considered.

The paper is organized as follows. In Section 2 we introduce the necessary terminology and notation. In Section 3 we give some preliminary results that will be used for the proofs of our main results that are given in Section 4.

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2. Notation and Terminology

Throughout this paper we use the notation and terminology of Rudin's book [12] and most of the general results used in our paper can be found in this book. For $n \geq 1$, $\mathbb{C}^n$ denotes the n - dimensional complex space, with the usual inner product $\langle z, w\rangle$ and norm $|z| = \sqrt{\langle z, z\rangle}$. $\mathbb{B}$ will denote the unit ball in $\mathbb{C}^n$ with $d\nu$ the Lebesque measure on $\mathbb{B}$, normalized so that $\nu(\mathbb{B}) = 1$, and $\mathbb{S} = \partial(\mathbb{B})$ will be the boundary of $\mathbb{B}$ with $d\sigma$ the surface area measure on $\mathbb{S}$, again normalized so that $\sigma(\mathbb{S}) = 1$.

The class of all non-decreasing, continuous functions h on $[0, \infty)$ such that $h(t) > 0$ for $t > 0$ and $h(t)/t$ is non-increasing on $(0, \infty)$ will be denoted by $\mathcal{H}$. The subclass of $\mathcal{H}$, consisting of the functions $h \in \mathcal{H}$ satisfying the condition $h(0) = 0$ is denoted by $\mathcal{H}_0$.

Let h be a non-decreasing function on $[0, \infty)$ vanishing at 0 and satisfying the condition $h(2x) \leq c\,h(x)$ for some positive constant c. Such function h will be referred as *allowed*.

It is easy to check that each $h \in \mathcal{H}_0$ is an allowed function.

For $\delta > 0$ and $\zeta \in \mathbb{S}$ let $Q(\zeta, \delta) = \{\eta \in \mathbb{S} : |1 - \langle \eta, \zeta\rangle| < \delta\}$. The set $Q(\zeta, \delta)$ is called a *Koranyi's ball centered at the point* ζ *and radius* δ.

For an allowed function h, the *non-isotropic Hausdorff capacity* H_h of a compact subset K of the sphere $\mathbb{S}$ is defined by

$$H_h(K) = \inf\left\{\sum_{j=1}^{\infty} h(r_j)\right\},$$

where the infimum is taken over all countable covers $\Big\{Q(\zeta_j, r_j) : j \in \mathbb{N}\Big\}$ of K of Koranyi's balls $Q(\zeta_j, r_j)$ centered at $\zeta_j \in \mathbb{S}$ and radius r_j. For an arbitrary set $A \subseteq \mathbb{S}$, the non-isotropic Hausdorff capacity is defined by

$$H_h(A) = \sup\Big\{H_h(K) : \text{K-compact subset of A}\Big\}.$$

For basic definitions, background and more information on the non-isotropic Hausdorff capacity we refer to [3], [9], [10] and [11].

For $c > 0$, $\tau \geq 1$ and $\zeta \in \mathbb{S}$, let

$$\mathcal{T}_{\tau,c}(\zeta) = \{z \in \mathbb{B} : |1 - \langle z, \zeta\rangle|^{\tau} < c(1 - |z|^2)\}.$$

When $\tau = 1$ (and $c > \frac{1}{2}$) we obtain the *admissible approach regions* which usually are denoted by D_c.

$\mathcal{M}$ will denote the *Mobius group of all holomorphic automorphisms* of the ball $\mathbb{B}$. By a Cartan's theorem it follows that $\psi \in \mathcal{M}$ if and only if $\psi = u \circ \varphi_a$ for a unique unitary transformation u on $\mathbb{C}^n$, where $a = \psi^{-1}(0)$.

For $r > 0$ let $\mathbb{B}_r = \{z \in \mathbb{C}^n : |z| < r\}$, and for $a \in \mathbb{B}$, $a \neq 0$ let $E(a) = \varphi_a(\mathbb{B}_{\frac{1}{2}})$. Since φ_a is an involution, $z \in E(a)$ if and only if $|\varphi_a(z)| < \frac{1}{2}$.

By λ we denote the measure on $\mathbb{B}$ defined by

$$d\lambda(z) = \frac{d\nu(z)}{(1 - |z|^2)^{n+1}}.$$

The measure λ is $\mathcal{M}$ - invariant, i.e. $\int\limits_{\mathbb{B}} f(z)d\lambda(z) = \int\limits_{\mathbb{B}} (f \circ \psi)(z)d\lambda(z)$ for all $f \in L^1(d\lambda)$ and $\psi \in \mathcal{M}$.

A function $f \in C^2(\mathbb{B})$ is called $\mathcal{M}$ - *harmonic* on $\mathbb{B}$ if $\widetilde{\Delta} f(z) = 0$, for all $z \in \mathbb{B}$, where the *Laplace-Beltrami operator* $\widetilde{\Delta}$ on $\mathbb{B}$ is given by

$$\widetilde{\Delta} f(z) = \frac{1}{n+1}\Delta(f \circ \varphi_z)(0) = \frac{4(1-|z|^2)}{n+1}\sum_{i,j=1}^{n}\big(\delta_{i,j} - z_i\overline{z_j}\big)\frac{\partial^2 f}{\partial z_j \partial \overline{z_i}},$$

and Δ is the usual Laplacian in $\mathbb{R}^{2n}$.

For each fixed $w \in \mathbb{B}$, the function $z \to \log\left(1/|\varphi_z(w)|\right)$ is called the *pluri-Green function* of $\mathbb{B}$ with *pole* at w.

An upper semi-continuous function $u : \mathbb{B} \to [-\infty, \infty)$, with $u \not\equiv -\infty$ is said to be $\mathcal{M}$ - *subharmonic* or *invariant subharmonic* if for each $a \in \mathbb{B}$ and for every r, $0 < r < 1$

$$u(a) \le \int_S u(\varphi_a(rt))d\sigma(t).$$

For a C^2 function u this is equivalent to $\widetilde{\Delta} u \ge 0$.

A function u is $\mathcal{M}$ - *superharmonic* if $-u$ is $\mathcal{M}$ - subharmonic.

If μ is a nonnegative, regular Borel measure on $\mathbb{B}$ then the function

$$V_\mu(z) = \int\limits_{\mathbb{B}} \log\frac{1}{|\varphi_z(w)|}d\mu(w)$$

is called the *invariant pluri-Green potential* of μ on $\mathbb{B}$, if for some $z_0 \in \mathbb{B}$ we have

$$\int\limits_{\mathbb{B}} \log\frac{1}{|\varphi_{z_0}(w)|}d\mu(w) < \infty.$$

For $\zeta \in \mathbb{S}$ and $t > 0$ let

$$B_t(\zeta) = \Big\{z \in \mathbb{B} : |1 - \langle z, \zeta\rangle| \le t\Big\},$$

and for a nonnegative, regular Borel measure μ on $\mathbb{B}$ satisfying the growth condition (1.1) let

$$r(\mu, \zeta, t) = \int\limits_{\mathbb{B}_t(\zeta)} (1 - |w|^2)d\mu(w).$$

For a regular Borel measure μ on $\mathbb{B}$ satisfying the growth condition (1.1), and for $h \in \mathcal{H}$, and $0 < \alpha < 1$ we introduce the sets

$$\mathcal{L}(\mu, h) = \Big\{\zeta \in S : \liminf_{r\to 1-}\frac{1-r}{h(1-r)}V_\mu(r\zeta) = +\infty\Big\},$$

$$\mathcal{L}_\alpha(\mu, h) = \left\{\zeta \in S : \liminf_{\substack{z\to\zeta\\ z\in D_\alpha(\zeta)}}\frac{|1-\langle z,\zeta\rangle|}{h(|1-\langle z,\zeta\rangle|)}V_\mu(z) = +\infty\right\},$$

$$\underline{\mathcal{R}}(\mu, h) = \Big\{\zeta \in S : \liminf_{t\to 0+}\frac{r(\mu,\zeta,t)}{h(t)} = +\infty\Big\},$$

$$\overline{\mathcal{R}}(\mu,h)=\left\{\zeta\in S:\limsup_{t\to 0^+}\frac{r(\mu,\zeta,t)}{h(t)}=+\infty\right\},$$
$$\mathcal{L}_0(\mu,h)=\left\{\zeta\in S:\liminf_{r\to 1-}\frac{1-r}{h(1-r)}V_\mu(r\zeta)>0\right\}.$$

A final remark on notation: throughout this paper we will use the same letter C or $C(a,b)$ to denote various absolute positive constants or positive constants which depend only on the indicated variables, but not necessarily the same on any two occurrences.

3. Preliminary Results

In this section several preliminary results are given that will be used for the proof of the main theorems.

(i) The following integral formulas, the proofs of which can be found in [12] will be useful throughout:

$$\int_{\mathbb{C}^n} f d\nu = 2n\int_0^\infty\left(r^{2n-1}\int_{\mathbb{S}} f(r\zeta)d\sigma(\zeta)\right)dr \tag{3.1}$$

$$\int_{\mathbb{S}} f d\sigma=\int_{\mathbb{S}}\left(\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}\zeta)d\theta\right)d\sigma(\zeta). \tag{3.2}$$

(ii) The following identity from [12] will be used also: For all $a,\ z,\ w\in\mathbb{B}$ we have

$$1-|\varphi_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2}. \tag{3.3}$$

Theorem 3.1. *(See [1]) Let μ be a nonnegative, regular Borel measure on $\mathbb{B}$ and let*

$$V_\mu(z)=\int_{\mathbb{B}}\log\frac{1}{|\varphi_w(z)|}d\mu(w).$$

Then V_μ is plurisuperharmonic on $\mathbb{B}$ if and only if

$$\int_{\mathbb{B}}(1-|w|^2)d\mu(w)<\infty.$$

Furthermore, if this is the case, then $\lim\limits_{r\to 1-}\int_{\mathbb{S}} V_\mu(rt)d\sigma(t)=0.$

Lemma 3.2. *([16, Lemma 6.15]) If $K\subseteq\mathbb{B}$ is a compact set, then there exists a constant C_K and r_0, $0<r_0<1$, such that for all $w\in K$ and all $|z|\geq r_0$ we have*

$$\log\frac{1}{|\varphi_w(z)|}\leq C_K(1-|z|^2)(1-|w|^2).$$

Lemma 3.3. *(See [1])* $\int_{\mathbb{B}}\left(\log\frac{1}{|z|}\right)^q d\lambda(z)<\infty$ *if and only if* $q>n$.

For $z\in\mathbb{B}$, c real, and $\alpha>n$ consider the integral:

$$J_{c,\alpha}(z)=\int_{\mathbb{B}}\frac{(1-|w|^2)^\alpha}{|1-\langle z,w\rangle|^{\alpha+c}}d\lambda(w).$$

The following asymptotic estimate is well known ([12], Proposition 1.4.10) and will play an important role throughout the paper:

Proposition 3.4. *For* $\alpha > n$, $z \in \mathbb{B}$

$$J_{c,\alpha}(z) \approx \begin{cases} (1-|z|^2)^{-c}, & c > 0 \\ \log \frac{1}{1-|z|^2}, & c = 0 \\ 1, & c < 0 \end{cases} \tag{3.4}$$

The notation $a(z) \approx b(z)$ means that the ratio $a(z)/b(z)$ has a finite limit as $|z| \to 1$.

The next, *"Frostman type"* theorem (Theorem 1 in [9], proved for $h(t) = t^m$), is the key for the proof of our main results. The extension to arbitrary allowed h ([3]) posses no difficulty.

Theorem 3.5. *Let h be an allowed function. For a compact set $K \subseteq \mathbb{S}$, $H_h(K) > 0$ if and only if K contains the support of a positive measure $\nu \not\equiv 0$ on $\mathbb{S}$ satisfying*

$$\nu(Q(\zeta,\delta)) \le C\, h(\delta) \tag{3.5}$$

for all $\delta > 0$ and $\zeta \in \mathbb{S}$ and an absolute constant C.

The following covering lemma ([16, Lemma 5.2.3]) also will be needed.

Proposition 3.6. *Suppose that E is the union of a finite collection $\Big\{Q(\zeta_i,\delta_i)\Big\}$ of Koranyi's balls. Then there exists a finite disjoint sub-collection $\Big\{Q(\zeta_{i_k},\delta_{i_k})\Big\}_{k=1}^{m}$ such that*

$$E \subseteq \bigcup_{k=1}^{m} Q(\zeta_{i_k}, 9\delta_{i_k})$$

and

$$\sigma(E) \le C_n \sum_{k=1}^{m} \sigma(Q(\zeta_{i_k},\delta_{i_k})),$$

where C_n is a constant depending only on n.

Now, we proceed with several other lemmas, analogous to those in Samuelsson's paper [13].

Lemma 3.7. *Let t be a fixed number such that $0 < t < \frac{1}{3}$ and let $I_t = [1-3t, 1-2t]$. For $\zeta \in \mathbb{S}$ and $z \in \mathbb{B}$ let*

$$G_{\zeta,t}(z) = \int_{I_t} \log \frac{1}{|\varphi_z(r\zeta)|}\, dr.$$

Then there exists positive constants C_1 and C_2, independent of ζ, z and t such that

$$G_{\zeta,t}(z) \le C_1(1-|z|^2), \tag{3.6}$$

and

$$G_{\zeta,t}(z) \le C_2\, t^2\, \frac{1-|z|}{|1-\langle z,\zeta\rangle|^2} \tag{3.7}$$

for all $z \in \mathbb{B}$.

Proof. First, suppose that $\zeta = e_1 = (1,\cdots,0)$. For $z \in \mathbb{B}$, by the identity (3.1) we have

$$G_{e_1,t}(z)=\frac{1}{2}\int\limits_{I_t}\log\left\{1+\frac{(1-r^2)(1-|z|^2)}{|1-\langle z,re_1\rangle|^2-(1-r^2)(1-|z|^2)}\right\}dr$$
$$\leq\frac{1}{2}\int\limits_{I_t}\log\left\{1+\frac{(1-r^2)(1-|z|^2)}{|\,|z|-r|^2}\right\}dr.$$

<u>Case 1^0</u>. If $\min\{|\,|z|-r|:r\in I_t\}\geq t$, then by the inequality $\log x\leq 1-x$ we have

$$\frac{1}{2}\int\limits_{I_t}\log\left\{1+\frac{12t(1-|z|)}{t^2}\right\}dr\leq 6\frac{1-|z|}{t}\int\limits_{I_t}dr\leq 6(1-|z|),$$

which proves (3.6) for $\zeta=e_1$ in this case.

<u>Case 2^0</u>. If $\min\{|\,|z|-r|:r\in I_t\}\leq t$, then $1-|z|=1-r-(|z|-r)>1-r-t\geq t$. Therefore in this case we have

$$G_{e_1,t}(z)\leq\frac{1}{2}\int\limits_{I_t}\log\left\{1+\frac{6t(1-|z|^2)}{|\,|z|-r|^2}\right\}dr\leq\frac{1}{2}\int\limits_{I_t}\log\left\{1+\left(\frac{4(1-|z|)}{|\,|z|-r|}\right)^2\right\}dr$$
$$\leq\int\limits_{|z|}^{\infty}\log\left\{1+\left(\frac{4(1-|z|)}{|\,|z|-r|}\right)^2\right\}dr\leq 4(1-|z|)\int\limits_{0}^{\infty}\frac{1}{x^2}\log(1+x^2)\,dx\leq C(1-|z|^2),$$

which proves (3.6) for $\zeta=e_1$.

To prove (3.7) for $\zeta=e_1$, first notice that for $z=(z_1,\cdots,z_n)\in\mathbb{B}$ we have

$$|1-\langle z,re_1\rangle|^2-(1-r^2)(1-|z|^2)\geq|z_1-r|^2. \tag{3.8}$$

The last inequality is easily verified. Indeed, using the obvious inequality $|z|\geq|z_1|$ we have

$$\begin{aligned}|1-\langle z,re_1\rangle|^2-(1-r^2)(1-|z|^2)&=|1-rz_1|^2-(1-r^2)(1-|z|^2\\&\geq|1-rz_1|^2-(1-r^2)(1-|z_1|^2\\&=|r-z_1|^2.\end{aligned}$$

Now, let $z=(z_1,\cdots,z_n)\in\mathbb{B}$ be such that $|1-z_1|>4t$. Then for $r\in I_t$ we have

$$\begin{aligned}|z_1-r|&\geq|1-z_1|-(1-r)\geq|1-z_1|-3t\\&>|1-z_1|-\tfrac{3}{4}|1-z_1|=\tfrac{1}{4}|1-z_1|.\end{aligned}$$

Therefore from (3.8) it follows that

$$G_{e_1,t}(z)=\frac{1}{2}\int\limits_{I_t}\log\left\{1+\frac{(1-r^2)(1-|z|^2)}{|1-\langle z,re_1\rangle|^2-(1-r^2)(1-|z|^2)}\right\}dr$$
$$\leq\frac{1}{2}\int\limits_{I_t}\log\left\{1+\frac{12t(1-|z|)}{|z_1-r|^2}\right\}dr\leq\frac{1}{2}\int\limits_{I_t}\frac{(12t)(1-|z|)}{|z_1-r|^2}dr$$
$$\leq\frac{1}{2}\frac{(12t)(16t)(1-|z|)}{|1-z_1|^2}=96t^2\frac{1-|z|}{|1-\langle z,e_1\rangle|^2}.$$

If $|1-z_1|<4t$, then by (3.6) for $\zeta=e_1$, we have

$$G_{e_1,t}(z)\leq C_1(1-|z|^2)=C_1(1-|z|^2)\frac{|1-z_1|^2}{|1-z_1|^2}$$

$$\leq 2C_1 16t^2 \frac{1-|z|}{|1-\langle z, e_1\rangle|^2} = C_2 t^2 \frac{1-|z|}{|1-\langle z, \zeta\rangle|^2}.$$

This completes the proof of (3.7) for $\zeta = e_1$. Finally, let $\zeta \in \mathbb{S}$ be arbitrary, and let φ be the unitary transformation on $\mathbb{C}^n$ such that $\varphi(e_1) = \zeta$. From the identity (3.1) we have $|\varphi_z(\varphi(w))| = |\varphi_{\varphi^{-1}(z)}(w)|$ for every $z \in \mathbb{B}$ and every $w \in \mathbb{C}^n$, and so the proof of (3.6) and (3.7) for the arbitrary ζ reduces to the one for $\zeta = e_1$.

Lemma 3.8. *Let* t *and* I_t *be as in Lemma* 3.6 *and let* $\zeta \in \mathbb{S}$ *and* $h \in \mathcal{H}$ *be such that*

$$r(\mu, \zeta, x) \leq h(x)$$

for all $x > 0$ *and every nonnegative regular Borel measure* μ *on* $\mathbb{B}$ *satisfying the growth condition* (1.1). *Then there exists an absolute constant* C *which does not depend on* t *such that*

$$\inf_{r\in I_t}\left\{V_\mu(r\zeta)\right\} \leq C\,\frac{h(t)}{t},$$

Proof. From the definition of $G_{\zeta,t}$ and Fubini's theorem we have the following:

$$\int_{\mathbb{B}} G_{\zeta,t}(z)d\mu(z) = \int_{I_t}\left(\int_{\mathbb{B}} \log\frac{1}{|\varphi_z(r\zeta)|}d\mu(z)\right)dr = \int_{I_t}\left(\int_{\mathbb{B}} \log\frac{1}{|\varphi_{r\zeta}(z)|}d\mu(z)\right)dr$$

$$= \int_{I_t} V_\mu(r\zeta)dr \geq \int_{I_t}\left(\inf_{s\in I_t}\left\{V_\mu(s\zeta)\right\}\right)dr = t\inf_{s\in I_t} V_\mu(s\zeta).$$

Thus $\inf\left\{V_\mu(r\zeta) : r\in I_t\right\} \leq \frac{1}{t}\int_{\mathbb{B}} G_{\zeta,t}(z)\,d\mu(z)$, for every t, $0<t<\frac{1}{3}$. Therefore from the estimates (3.6) and (3.7) in Lemma 3.7 it follows that

$$\inf_{r\in I_t}\left\{V_\mu(r\zeta)\right\} \leq \frac{1}{t}\int_{\mathbb{B}} G_{\zeta,t}(z)d\mu(z) = \frac{1}{t}\int_{\mathbb{B}_t(\zeta)} G_{\zeta,t}(z)d\mu(z) + \frac{1}{t}\int_{\mathbb{B}\setminus\mathbb{B}_t(\zeta)} G_{\zeta,t}(z)d\mu(z)$$

$$\leq \frac{C_1}{t}\int_{\mathbb{B}_t(\zeta)}(1-|z|^2)d\mu(z) + C_2 t\int_{\mathbb{B}\setminus\mathbb{B}_t(\zeta)}\frac{(1-|z|^2)}{|1-\langle z,\zeta\rangle|^2}d\mu(z),$$

i.e.,

$$\inf_{r\in I_t}\left\{V_\mu(r\zeta)\right\} \leq \frac{C_1}{t}\int_{\mathbb{B}_t(\zeta)}(1-|z|^2)d\mu(z) + C_2 t\int_{\mathbb{B}\setminus\mathbb{B}_t(\zeta)}\frac{1-|z|^2}{|1-\langle z,\zeta\rangle|^2}d\mu(z).$$

If for $j = 1, 2, \cdots$ we set $A_{j,t}(\zeta) = \left\{z\in\mathbb{B} : 2^{j-1}t < |1-\langle z,\zeta\rangle| \leq 2^j t\right\}$, then from the last estimate, and the definition (3.3) of $r(\mu,\zeta,t)$ we have

$$\inf_{r\in I_t}\left\{V_\mu(r\zeta)\right\} \leq \frac{C_1}{t}\,r(\mu,\zeta,t) + C_2 t\sum_{j=1}^{\infty}\int_{A_{j,t}(\zeta)}\frac{1-|z|}{|1-\langle z,\zeta\rangle|^2}d\mu(z)$$

$$\leq \frac{C_1}{t}\,r(\mu,\zeta,t) + C_2 t\sum_{j=1}^{\infty}\int_{A_{j,t}(\zeta)}\frac{1-|z|}{t^2 2^{2j-2}}d\mu(z)$$

$$\leq \frac{C_1}{t}\,r(\mu,\zeta,t) + \frac{C_2}{t}\sum_{j=1}^{\infty}2^{2-2j}r(\mu,\zeta,2^j t) = \frac{C}{t}\,h(t).$$

By hypothesis, $r(\mu, \zeta, x) \le h(x)$ for every $x > 0$, and therefore from above it follows that

$$
\begin{aligned}
\inf_{r \in I_t} \left\{ V_\mu(r\zeta) \right\} &\le \frac{C_1}{t}\, h(t) + \frac{C_2}{t} \sum_{j=1}^{\infty} 2^{-2j+2} h(2^j t) \\
&\le \frac{C_1}{t}\, h(t) + \frac{C_2}{t} \sum_{j=1}^{\infty} 2^{-2j+2} 2^j h(t) \\
&= \frac{C_1 + C_2^{'}}{t}\, h(t).
\end{aligned}
$$

Above we used the fact that $h(kx) \le k\, h(x)$, for every $k \ge 1$ and every $x > 0$. This fact easily follows from the hypothesis that the function $h(t)/t$ is non-increasing on the interval $(0, \infty)$.

Lemma 3.9. *Let $h \in \mathcal{H}_0$. For a nonnegative, regular Borel measure μ on $\mathbb{B}$ which satisfies the growth condition* (1.1) *let*

$$
\overline{\mathcal{R}_0}(\mu, h) = \left\{ \zeta \in \mathbb{S} : \limsup_{t \to 0+} \frac{r(\mu, \zeta, t)}{h(t)} > 0 \right\}.
$$

Then $H_h(\overline{\mathcal{R}_0}(\mu, h)) = 0$.

Proof. Since H_h is sub-additive it is enough to show that $H_h(\overline{\mathcal{R}_a}(\mu, h)) = 0$ for all $a > 0$, where for $a > 0$

$$
\overline{\mathcal{R}_a}(\mu, h) = \left\{ \zeta \in \mathbb{S} : \limsup_{t \to 0+} \frac{r(\mu, \zeta, t)}{h(t)} > a \right\}.
$$

Let $\epsilon > 0$ be given. From the growth condition (1.1) and the regularity of the measure μ it follows that there exists a compact set $K \subseteq \mathbb{B}$ such that

$$
\int_{\mathbb{B} \setminus K} (1 - |w|^2) d\mu(w) < \epsilon. \tag{3.9}
$$

Let $F \subseteq \overline{\mathcal{R}_a}$, be a compact set and let

$$
\mathcal{F}_a = \left\{ Q(\zeta, 2t) : \frac{r(\mu, \zeta, t)}{h(t)} > a,\ \zeta \in F,\ 0 < t < \rho(K) \right\},
$$

where $\rho(K) = \inf\{1 - |z| : z \in K\}$. Notice that $\mathcal{F}_a$ is a covering of F by Koranyi's balls. Since F is a compact set, there exist points $\zeta_1, \cdots, \zeta_m \in S$ and positive numbers $t_1, \cdots, t_m$ such that

$$
F \subseteq \bigcup_{j=1}^{m} Q(\zeta_j, 2t_j) \text{ and } \quad r(\mu, \zeta_j, t_j) > a\, h(t_j), \quad 0 < t_j < \rho(K),\ j = 1, \cdots, k.
$$

By the covering Lemma 2.6, there exists a finite disjoint sub-collection of $\mathcal{F}_a$, which for convenience we denote by $\{Q(\eta_j, 2t_j) : j = 1, \cdots, k\}$, such that

$$
F \subseteq \bigcup_{j=1}^{k} Q(\eta_j, 18t_j) \text{ and } \quad r(\mu, \eta_j, t_j) > a\, h(t_j), \quad 0 < t_j < \rho(K),\ j = 1, \cdots, k.
$$

Now let ν be any positive measure on $\mathbb{S}$ satisfying the condition $\nu(Q(\zeta, \delta) \le C\, h(\delta)$ for all $\zeta \in \mathbb{S}$ and every $\delta > 0$. Then we have the following estimates:

$$\nu(F) \le \sum_{j=1}^{k} \nu(Q(\eta_j, 18t_j)) \le C \sum_{j=1}^{k} h(18t_j) \le C \sum_{j=1}^{k} h(t_j)$$
$$\le \frac{C}{a} \sum_{j=1}^{k} r(\mu, \eta_j, t_j) = \frac{C}{a} \sum_{j=1}^{k} \int\limits_{\mathbb{B}_{t_j}(\eta_j)} (1-|w|^2) d\mu(w),$$

i.e.,

$$\nu(F) \le \frac{C}{a} \sum_{j=1}^{k} \int\limits_{\mathbb{B}_{t_j}(\eta_j)} (1-|w|^2) d\mu(w). \tag{3.10}$$

If $z \in \mathbb{B}$ is such that $|1 - \langle z, \eta_j \rangle| \le t_j$, for some $j = 1, \cdots, k$, then for this z we have $1 - |z| \le |1 - \langle z, \eta_j\rangle| \le t_j < \rho(K)$. Therefore

$$\bigcup_{j=1}^{k} \Big\{ z \in \mathbb{B} : \, |1 - \langle z, \eta_j \rangle| \le t_j \Big\} \subseteq \mathbb{B} \setminus K,$$

i.e.,

$$\bigcup_{j=1}^{k} \mathbb{B}_{t_j}(\eta_j) \subseteq \mathbb{B} \setminus K. \tag{3.11}$$

From the trivial inequality $|1 - \langle z, \zeta \rangle| \ge \frac{1}{2} |1 - \langle \frac{z}{|z|}, \zeta \rangle|$ for $z \in \mathbb{B}$ and $\zeta \in \mathbb{S}$ we have the following set inclusion:

$$\big\{ z \in \mathbb{B} : \, |1 - \langle z, \eta_j \rangle| \le t_j \big\} \subseteq \Big\{ z \in \mathbb{B} : \, |1 - \langle \frac{z}{|z|}, \eta_j \rangle| \le 2t_j \Big\}.$$

Because $\big\{ Q(\eta_j, 2t_j) : \, j = 1, \cdots, k \big\}$ is a pairwise disjoint family of Koranyi's balls, from the last inclusion it follows that the family $\big\{ \mathbb{B}_{t_j}(\eta_j) : \, j = 1, \cdots, k \big\}$ is also pairwise disjoint. Therefore, from (3.10) and (3.11) we have

$$\nu(F) \le C \sum_{j=1}^{k} \int\limits_{\mathbb{B}_{t_j}(\eta_j)} (1-|z|^2) d\mu(z) = C \int\limits_{\bigcup_{j=1}^{k} \mathbb{B}_{t_j}(\eta_j)} (1-|z|^2) d\mu(z)$$
$$\le C \int\limits_{\mathbb{B} \setminus K} (1-|z|^2) d\mu(z) < C\epsilon.$$

Since $\epsilon > 0$ was arbitrary we have $\nu(F) = 0$ and hence Theorem 3.5 implies the result.

Lemma 3.10. *Let* $\alpha > \frac{1}{2}$ *be fixed. Then there exist two positive constants* C_1 *and* C_2 *such that*

(a) $$C_1 \liminf_{t \to 0+} \tfrac{r(\mu,\zeta,t)}{h(t)} \le \liminf_{\substack{z \to \zeta \\ z \in D_\alpha(\zeta)}} \frac{|1 - \langle z, \zeta \rangle|}{h(|1 - \langle z, \zeta \rangle|)} V_\mu(z),$$

and

(b) $$\liminf_{r \to 1-} \tfrac{1-r}{h(1-r)} V_\mu(r\zeta) \le C_2 \limsup_{t \to 0+} \tfrac{r(\mu,\zeta,t)}{h(t)}$$

for all $\zeta \in S$, $h \in \mathcal{H}$, *and all nonnegative, regular Borel measures* μ *on* $\mathbb{B}$ *satisfying the growth condition* (1.1).

4. Main Results

In this Section we present our main results.

Theorem 4.1. *Suppose* $f(z) \geq 0$, $1 < p < \frac{n}{n-1}$ *and* $f \in L^p(d\lambda)$. *Then the pluri-Green potential* V_f *is continuous on the closed ball* $\overline{\mathbb{B}}$.

Proof. We will follow the corresponding proof of Theorem 1 in [8] for the case of the invariant Green potential.

If q is the conjugate exponent of p, then $q > n$. Since V_f is the convolution of the functions $f \in L^p(d\lambda)$ and $\log(1/|z|) \in L^q(d\lambda)$ (Lemma 2.2), we have that V_f is continuous on $\mathbb{B}$. Now we show that $V_f(z) \to 0$ uniformly as $|z| \to 1$. For $0 < r < 1$, let $f_r = \chi_{r\mathbb{B}} f$ and let

$$V_r(z) = \int\limits_{\mathbb{B}} \log \frac{1}{|\varphi_w(z)|} f_r(z) d\lambda(z).$$

Because f_r has a compact support we have that $V_r(z) \to 0$ uniformly as $|z| \to 1$. Applying Hölder's inequality we have

$$|V_f(z) - V_r(z)| \leq \left\{ \int\limits_{\mathbb{B}} \left(\log \frac{1}{|\varphi_w(z)|} \right)^q d\lambda(w) \right\}^{\frac{1}{q}} \left\{ \int\limits_{\mathbb{B}} |f(w) - f_r(w)|^p d\lambda(w) \right\}^{\frac{1}{p}},$$

and hence by the $\mathcal{M}$ - invariance of λ, $\|V_f - V_r\| \leq C\|f - f_r\|_p$. Therefore $V_r(z) \to V_f(z)$ uniformly as $r \to 1$, from which the result follows. $\square$

Remark: The assumptions for p in Theorem 4.1 are best possible. Indeed, suppose first that $p = 1$. If we had $V_f(0) = -\int\limits_{\mathbb{B}} \log(|z|) f(z) d\lambda(z) < \infty$ for every $f \in L^1(d\lambda)$, then by the Riesz representation theorem (the dual of $L^1(d\lambda)$ we would have $\log(1/|z|) \in L^\infty(d\lambda)$ which is impossible by Lemma 3.2. Therefore $V_f(0) = \infty$ for some $f \in L^1(d\lambda)$ and so V_f is discontinuous on $\mathbb{B}$ for that f.

Now let $n \neq 1$ and $p = n/(n-1)$. If we had $V_f(0) < \infty$ for every $f \in L^{\frac{n}{n-1}}(d\lambda)$, then again by the Riesz representation theorem we would have $\log(1/|z|) \in L^n(d\lambda)$ which again is impossible by Lemma 3.2.

As immediate consequences of Lemma 3.9 and Lemma 3.10 we have the following results

Theorem 4.2. *If* μ *is a nonnegative, Borel regular measure on* $\mathbb{B}$ *satisfying the condition* (1.1), *then the set* $\mathcal{L}_0(\mu, 1)$ *is empty. If additionally* $h \in \mathcal{H}_0$, *then* $H_h(\mathcal{L}_0(\mu, h)) = 0$.

Theorem 4.3. *Let* $h \in \mathcal{H}$, $0 < \alpha < 1$ *and let* μ *be a nonnegative, Borel regular measure on* $\mathbb{B}$ *satisfying the growth condition* (1.1). *Then*

$$\underline{\mathcal{R}}(\mu, h) \subseteq \mathcal{L}_\alpha(\mu, h) \subseteq \mathcal{L}(\mu, h) \subseteq \overline{\mathcal{R}}(\mu, h).$$

For a nonnegative, regular Borel measure μ on $\mathbb{B}$ satisfying the growth condition (1.1) let F_μ be the function on $\mathbb{B}$ defined by

$$F_\mu(z) = (1 - |z|^2) \int\limits_{\mathbb{B}} \frac{1 - |w|^2}{|1 - \langle z, w \rangle|^2} d\mu(w).$$

The next several propositions will be needed for the proofs of our next results. They provide sufficient conditions for the existence of the $\mathcal{T}_\tau$ - limit of F_μ at a point $\zeta \in \mathbb{S}$.

We omit their proofs since they are almost identical to those of Proposition 1, Proposition 2 and Proposition 3 given in [16], pp. 148 − 151.

Proposition 4.4. *Let* μ *be a nonnegative, regular Borel measure on* $\mathbb{B}$ *satisfying the growth condition* (1.1). *If* $\zeta \in \mathbb{S}$ *is such that*

$$\int\limits_{\mathcal{T}_{\tau,c}(\zeta)} \frac{(1-|w|^2)^{\alpha}}{|1-\langle w,\zeta\rangle|^{\tau\alpha}} d\mu(w) < \infty,$$

and

$$\int\limits_{\mathbb{B}\setminus\mathcal{T}_{\tau,c}(\zeta)} \frac{(1-|w|^2)^{\gamma}}{|1-\langle w,\zeta\rangle|^{\tau\gamma}} d\mu(w) < \infty$$

for some c, α *and* γ *with* $c>0$, $\alpha>0$, $\gamma<1$, *and* $\tau \geq 1$, *then* F_μ *has* $\mathcal{T}_\tau$ *- limit* 0 *at the point* ζ.

Proposition 4.5. *Let* f *be a nonnegative measurable function on* $\mathbb{B}$. *If* $\zeta \in \mathbb{S}$ $c>0$, *and* $\tau \geq 1$ *are such that*

$$\int\limits_{\mathcal{T}_{\tau,c}(\zeta)} \frac{(1-|w|^2)^{\alpha}}{|1-\langle w,\zeta\rangle|^{\tau\alpha}} f^p(w) d\lambda(w) < \infty,$$

and

$$\int\limits_{\mathbb{B}\setminus\mathcal{T}_{\tau,c}(\zeta)} \frac{(1-|w|^2)^{\gamma}}{|1-\langle w,\zeta\rangle|^{\tau\gamma}} f^p(w) d\lambda(w) < \infty$$

for some p, $1<p<\frac{n}{n-1}$, *and some* α, *and* γ, *with* $0<\alpha<n+p-np$, *and* $0<\gamma<n+p-np$, *then* F_f *has* $\mathcal{T}_\tau$ *- limit* 0 *at the point* ζ.

Proposition 4.6. Let μ be a nonnegative, regular Borel measure on $\mathbb{B}$ satisfying

$$\int\limits_{\mathbb{B}} (1-|w|^2)^{\beta} d\mu(w) < \infty, \tag{4.1}$$

for some β, $0<\beta<1$, and let $1 \leq \tau \leq \frac{n}{\beta}$. Then

(a) for any γ, $\beta<\gamma\leq 1$,

$$H_{\beta\tau}\left(\left\{\zeta\in\mathbb{S} : \int\limits_{\mathbb{B}\setminus\mathcal{T}_{\tau,c}(\zeta)} \frac{(1-|w|^2)^{\gamma}}{|1-\langle w,\zeta\rangle|^{\tau\gamma}} d\mu(w) = \infty\right\}\right) = 0,$$

and

(b) for any α, $0<\alpha<\beta$,

$$H_{\beta\tau}\left(\left\{\zeta\in\mathbb{S} : \int\limits_{\mathcal{T}_{\tau,c}(\zeta)} \frac{(1-|w|^2)^{\alpha}}{|1-\langle w,\zeta\rangle|^{\tau\alpha}} d\mu(w) = \infty\right\}\right) = 0.$$

From the above propositions we have the following

Corollary 4.7. *Let* μ *and* β *satisfy the condition 4.1 in Proposition* 4.6. *Then for each* τ, $1\leq\tau\leq\frac{n}{\beta}$, *there exists a set* $E_\tau \subseteq \mathbb{S}$ *with* $H_{\beta\tau}(E_\tau)=0$ *such that* F_μ

has $\mathcal{T}_\tau$ *- limit* 0 *at all points* $\zeta \in S \setminus E_\tau$.

Corollary 4.8. *Let* μ *and* β *be as in Proposition 4.5. Then for each* τ, $1 \le \tau \le \frac{n}{\beta}$, $H_{\beta\tau}\left(\{\zeta \in \mathbb{S} : \mu(\mathcal{T}_{\tau,c}(\zeta)) = \infty\}\right) = 0$.

Proof. Since $|1 - \langle w, \zeta\rangle|^\tau < c(1 - |w|^2)$ for all $w \in \mathcal{T}_{\tau,c}(\zeta)$,

$$\int\limits_{\mathcal{T}_{\tau,c}(\zeta)} \frac{(1 - |w|^2)^\beta}{|1 - \langle w, \zeta\rangle|^{\tau\beta}} d\mu(w) \ge C\,\mu(\mathcal{T}_{\tau,c}(\zeta)).$$

The result now is an immediate consequence of part (b) in Proposition 4.5.

Theorem 4.9. *Let* f *be a non-negative measurable function on* $\mathbb{B}$ *which satisfies*

$$\int\limits_{\mathbb{B}} (1 - |z|^2)^\alpha f^p(z) d\lambda(z) < \infty, \tag{4.2}$$

for some p, $1 < p < \frac{n}{n-1}$ *and some* α, , $0 < \alpha < n + p - np$. *Then for each* τ, $1 \le \tau \le \frac{n}{\alpha}$ *there exists a set* $E_\tau \subseteq \mathbb{S}$ *with* $H_{\alpha\tau}(E_\tau) = 0$, *such that* V_f *has* $\mathcal{T}_\tau$ *- limit* 0, *at all points* $\zeta \in \mathbb{S} \setminus E_\tau$.

Proof. For a function f which satisfies (4.2) let

$$V_1(z) = \int\limits_{E(z)} \log \frac{1}{|\varphi_z(w)|} f(w) d\lambda(w), \quad \text{and} \quad V_2(z) = \int\limits_{\mathbb{B}\setminus E(z)} \log \frac{1}{|\varphi_z(w)|} f(w) d\lambda(w).$$

Recall that, $E(z) = \left\{w \in \mathbb{B} : |\varphi_z(w)| < \frac{1}{2}\right\}$. From the inequality $\log x \le 1 - x$ for $x > 0$ and from the identity (2.3) in **(ii)** it follows that

$$\log \frac{1}{|\varphi_z(w)|} \le C\, \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle w, z\rangle|^2}$$

for all $w \in \mathbb{B} \setminus E(z)$. Thus,

$$V_2(z) \le C(1 - |z|^2) \int\limits_{\mathbb{B}} \frac{1 - |w|^2}{|1 - \langle w, z\rangle|^2} f(w) d\lambda(w) = C\, F_f(z), \tag{4.3}$$

for all $z \in \mathbb{B}$.

Let μ be the measure defined by $d\mu(w) = f^p(w) d\lambda(w)$. Then from Propositions 4.4 and 4.5 it follows that there exists a set $E_2 \subseteq \mathbb{S}$, such that $H_{\alpha\tau}(E_2) = 0$ and F_f has $\mathcal{T}_\tau$ - limit 0 at all points $\zeta \in \mathbb{S} \setminus E_2$. Therefore from (4.3) it follows that that V_2 has $\mathcal{T}_\tau$ - limit 0 at all points $\zeta \in \mathbb{S} \setminus E_2$.

Now consider the function V_1. Taking $d\mu(w) = f^p(w) d\lambda(w)$ in Corollary 4.7, we have that there exists a set $E_1 \subseteq \mathbb{S}$ such that $H_{\alpha\tau}(E_1) = 0$ and

$$\int\limits_{\mathcal{T}_{\tau,c'}(\zeta)} f^p(w) d\lambda(w) < \infty \tag{4.4}$$

for all $\zeta \in \mathbb{S} \setminus E_1$ and any $c' > 0$. Suppose that $c > 0$ and $z \in \mathcal{T}_{\tau,c}(\zeta)$. By [16, Lemma 8.17] we have that $E(z) \subseteq \mathcal{T}_{\tau,c'}(\zeta) \cap A_r$ for every $c' > c2^{\tau+1}$ and $r \ge 2|z|^2 - 1$.

If q is the conjugate exponent of p then from the above set inclusion, as well as the $\mathcal{M}$ - invariance of λ and the Holder's inequality we have

$$\begin{aligned} V_1(z) &\le \left\{ \int\limits_{E(z)} \left(\log \frac{1}{|\varphi_z(w)|}\right)^q d\lambda(w) \right\}^{\frac{1}{q}} \left\{ \int\limits_{E(z)} f^p(w) d\lambda(w) \right\}^{\frac{1}{p}} \\ &= \left\{ \int\limits_{\{w:\, |w|<1/2\}} \left(\log \frac{1}{|\varphi_z(w)|}\right)^q d\lambda(w) \right\}^{\frac{1}{q}} \left\{ \int\limits_{E(z)} f^p(w) d\lambda(w) \right\}^{\frac{1}{p}} \\ &\le C \left\{ \int\limits_{\mathcal{T}_{\tau,c'}(\zeta)\cap A_r} f^p(w) d\lambda(w) \right\}^{\frac{1}{p}}, \end{aligned}$$

where C is a constant which is independent of z.

Therefore, for all $z \in \mathbb{B}$ we have

$$V_1(z) \le C \left\{ \int\limits_{\mathcal{T}_{\tau,c'}(\zeta)\cap A_r} f^p(w) d\lambda(w) \right\}^{\frac{1}{p}}.$$

As a consequence of (4.4) the above integral tends to 0 as $r \to 1$. So V_1 has $\mathcal{T}_\tau$ - limit 0 at all points $\zeta \in \mathbb{S} \setminus E_1$. Now if we take $E_\tau = E_1 \cup E_2$, then $H_{\beta\tau}(E_\tau) = 0$ and V_f has $\mathcal{T}_\tau$ - limit 0 at all points $\zeta \in \mathbb{S} \setminus E_\tau$.

Theorem 4.10. *Let* $\left\{ z_j : j = 1, 2, \cdots \right\}$ *be a sequence in* $\mathbb{B}$ *satisfying*

$$\sum_{j=1}^{\infty} (1 - |z_j|^2)^\alpha < \infty$$

for some α, $0 < \alpha < 1$, *and let* μ *be the measure on* $\mathbb{B}$ *given by* $\mu = \sum \delta_{z_j}$, *where* δ_{z_j} *is the unit point-mass measure concentrated at the point* z_j. *Then for each* τ, $1 \le \tau \le \frac{n}{\alpha}$, *there exists a set* $E_\tau \subseteq \mathbb{S}$ *with* $H_{\alpha\tau}(E_\tau) = 0$, *and such that* V_μ *has* $\mathcal{T}_\tau$ *- limit* 0 *at all points* $\zeta \in \mathbb{S} \setminus E_\tau$.

Proof. As in Theorem 4.8, for the measure μ, let

$$V_1(z) = \int\limits_{E(z)} \log \frac{1}{|\varphi_z(w)|} f(w) d\lambda(w), \quad \text{and} \quad V_2(z) = \int\limits_{\mathbb{B}\setminus E(z)} \log \frac{1}{|\varphi_z(w)|} f(w) d\lambda(w).$$

Similarly like in Theorem 4.9, for the functions V_2 and F_μ, we have $V_2(z) \le C\, F_\mu(z)$, for all $z \in \mathbb{B}$, where C is a constant independent on z.

By Corollary 4.7 it follows that there exists a set $E_1 \subseteq \mathbb{S}$ such that $H_{\beta\tau}(E_1) = 0$ and F_μ has $\mathcal{T}_\tau$ - limit 0 at all points $\zeta \in \mathbb{S} \setminus E_1$. Therefore V_2 has $\mathcal{T}_\tau$ - limit 0 at all points $\zeta \in \mathbb{S} \setminus E_1$. Further, by Corollary 4.8 we have that there exists a set $E_2 \subseteq \mathbb{S}$ such that $H_{\alpha\tau}(E_2) = 0$ and $\mu\left(\mathcal{T}_{\tau,c}(\zeta)\right) < \infty$ for all $\zeta \in \mathbb{S} \setminus E_2$.

Now, it is obvious that $\mu\left(\mathcal{T}_{\tau,c}(\zeta)\right) < \infty$ if and only if $\mathcal{T}_{\tau,c}(\zeta)$ contains only a finite number of terms of the sequence $\{z_k\}$. But for those ζ for which $\mu\left(\mathcal{T}_{\tau,c}(\zeta)\right) < \infty$ we clearly have that V_1 has $\mathcal{T}_\tau$ - limit 0 at all points $\zeta \in \mathbb{S} \setminus E_2$. Finally taking $E_\tau = E_1 \cup E_2$ the result follows.

References

[1] K. Adzievski, *Boundary behavior of pluri-Green potentials on the unit ball B in $\mathbb{C}^n$* Complex Variables **32** (1995), 32-43.

[2] P. Ahern, *Exceptional sets for holomorphic Sobolev functions*, Michigan Math. J. **35** (1988), 29–41.

[3] P. Ahern and W. Cohn, *Exceptional sets for Hardy Sobolev functions*, $p > 1$, Indiana Univ. Math. J. **38.No.2** (1989), 417–453

[4] P. Ahern and W. Cohn, *Weighted maximal functions and derivatives of invariant Poisson integrals of potentials*, Pac. J. Math. **163. No.1** (1994), 1–16.

[5] M. Arsove and A. Huber, *On the existence of non-tangential limits of subharmonic functions*, Journal London Mathematical Society **42** (1967), 125–132.

[6] G. T. Cargo, *Angular and tangential limits of Blashcke products and their successive derivatives*, Canad. J. Math. **14** (1962), 334–348.

[7] C. Cascante and J. Ortega, *A characterization of tangential exceptional sets for H^p_α, $\alpha = n$* Proc. R. Soc. Edinb., Sect. A **126A** (1996), 625–642.

[8] J. Cima and C. S. Stanton, *Admissible limits of* **M**-*subharmonic functions* **32** (1985), Michigan Mathematical Journal, 211–220.

[9] W. Cohn, *Non-isotropic Hausdorff measure and exceptional sets for holomorphic Sobolev functions*, Illinois J. Math. **33. No.4** (1989), 673–690.

[10] Ph. Jamming and M. M. Roginskaya, *Boundary behavior of $\mathcal{M}$-harmonic functions and non-isotropic Hausdorff measure*, Monatshefte für Mathematik **134** (2002), 217–226

[11] M. M. Roginskaya, *Asymptotical properties of harmonic and $\mathcal{M}$-harmonic functions near the boundary of the unit sphere*, Zapiski nauchnih seminarov POMI *270* (2000), 309–316. (Russian)

[12] W. Rudin, *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Springer-Verlag, New York, 1980.

[13] Å. Samuelsson, *On radial zeros of Blaschke products*, Ark. Mat. **Band 7 nr 33** (1969), 477–494.

[14] M. Stoll, *Boundary limits and non-integrability of $\mathcal{M}$-subharmonic functions on the unit ball of $\mathbb{C}^n$ $(n \geq 1)$*, Trans. Amer. Math. Soc. **349, Number 9** (1997), 3773–3785.

[15] M. Stoll, *Non-isotropic Hausdorff capacity of exceptional sets of invariant potentials*, Potential Analysis **4** (1995), 141-155.

[16] M. Stoll, *Invariant Potential Theory in the Unit Ball of $\mathbb{C}^n$*, London Mathematical Society Lecture Note Series 199, Cambridge University Press, 1994.

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