Math. Maced.
Vol. 4 (2006)
7-10

ON ONE CONJECTURE CONCERNING SYMMETRIC PRODUCTS OF MANIFOLDS

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Abstract. In this paper it is given a partial proof of the following conjecture about symmetric products of manifolds.

Let M be a 2-dimensional manifold and $m \geq 2$.

(i) If $0 \leq t \leq m$, and $M^{(m)} \cong R^{2m-t} \times (S^1)^t$, then M is homeomorphic to $C \setminus \mathcal{A}_t$, where $\mathcal{A}_t$ is a set of t distinct complex numbers.

(ii) If $m+1 \leq t \leq 2m$, then there does not exist a manifold M such that $M^{(m)} \cong R^{2m-t} \times (S^1)^t$.

This conjecture is proved for $t \in \{0, 1, m+1, \cdots, 2m\}$, and it is still open for $t \in \{2, \cdots, m\}$.

The theory of the symmetric products on manifolds is related to the fully commutative vector valued groups. These kinds of groups were introduced in [1] and [2] and mainly they were studied in the monograph [15] and in the recent paper [16]. We will not give the definition of such algebraic structures because here we will need only the application of the symmetric products of manifolds. The main examples of continuous (topological) fully commutative vector valued groups are the so called affine and projective $com(m+k,m)$-groups ([15, 16]). These affine and projective $com(m+k,m)$-groups are constructed over the set $M = C^* \setminus \mathcal{A}_t$ where $C^* = C \cup \{\infty\}$ and $\mathcal{A}_t$ denotes any set of t distinct complex numbers. More generally one can consider topological $com(m+k,m)$-groups, but all known $com(m+k,m)$-groups are isomorphic to the affine $com(m+k,m)$-groups. Indeed, it is introduced [15] the following conjecture.

Conjecture 1. *Each connected, locally Euclidean topological $com(m+k,m)$-group is isomorphic to an affine $com(m+k,m)$-group.*

This conjecture is difficult for proving, but also there is a weaker one introduced in [15]. In order to present it, we refer to the theorem 6.1 [18] which states that if $M \cong C \setminus \mathcal{A}_t$, $(0 \leq t \leq m)$, then

$$M^{(m)} \cong C^{m-t} \times (C \setminus \{0\})^t \cong R^{2m-t} \times (S^1)^t.$$

If (M, f) is a topological $com(m+k,m)$-group, then $M^{(m)}$ admits commutative Lie group [15] and thus $M^{(m)}$ is homeomorphic to one of the spaces $R^{2m-p} \times (S^1)^p$, $0 \leq p \leq 2m$. Since $M^{(m)}$ is a manifold, M must be a 2-dimensional manifold. But all known examples are constructed on spaces homeomorphic to $C \setminus \mathcal{A}_t$ for $0 \leq t \leq m$. Thus in [15] is given the following conjecture which is the first step in proving the Conjecture 1 and now we will prove it partially.

2000 Mathematics Subject Classification. 55R10.

Key words and phrases. symmetric products of manifolds, com(n,m)-groups.

Conjecture 2. *Let M be a 2-dimensional manifold and $m \geq 2$.*

(i) If $0 \leq t \leq m$, and $M^{(m)} \cong R^{2m-t} \times (S^1)^t$, then M is homeomorphic to $C \setminus \mathcal{A}_t$, where $\mathcal{A}_t$ is a set of t distinct complex numbers.

(ii) If $m+1 \leq t \leq 2m$, then there does not exist a manifold M such that $M^{(m)} \cong R^{2m-t} \times (S^1)^t$.

Proof of (ii). First note that the required manifold M must be orientable because $M^{(m)} = R^{2m-t} \times (S^1)^t$ is orientable manifold and M must be connected, too. Moreover, $dim H_1(M, Z) = dim \pi_1(M^{(m)}) = t < \infty$ and hence M should be homeomorphic to an orientable compact surface without finite number of points or an orientable compact surface.

Before we continue the proof, we will prove that for each compact polyhedron M the relationship between $\chi(M)$ and $\chi(M^{(m)})$ is given by

$$\chi(M^{(m)}) = \binom{m + \chi(M) - 1}{m}.$$

The Poincare polynomial of a symmetric product of a compact polyhedron is given in [7]. If $B_0, B_1, B_2, \cdots$ are the Betti numbers of a space M, then the Poincare polynomial of the mth symmetric product of M is the coefficient in front of t^m in the power series expansion of

$$\frac{(1+xt)^{B_1}(1+x^3t)^{B_3}\cdots}{(1-t)^{B_0}(1-x^2t)^{B_2}(1-x^4t)^{B_4}\cdots}.$$

Now if we put $x=-1$, then the Euler characteristic of $M^{(m)}$ is the coefficient in front of t^m. Indeed, the previous expression for $x=-1$ becomes

$$(1-t)^{-B_0}(1-t)^{B_1}(1-t)^{-B_2}(1-t)^{B_3}\cdots =$$

$$= (1-t)^{-\chi(M)} = \sum_{m=0}^{\infty}(-1)^m t^m \binom{-\chi(M)}{m} = \sum_{m=0}^{\infty} t^m \binom{m+\chi(M)-1}{m},$$

and hence we obtain the required relationship between $\chi(M)$ and $\chi(M^{(m)})$.

It is easy to prove the conjecture for $t = 2m$. If $M^{(m)} = (S^1)^{2m}$, then M must be compact and $H_1(M, Z) = \pi_1(M^{(m)}) = \pi_1((S^1)^{(2m)}) = Z^{2m}$. This implies that M is a sphere with m handles. In this case $\chi(M) = 2 - 2m$ and we get

$$\chi(M^{(m)}) = \binom{m+\chi(M)-1}{m} = \binom{m+2-2m-1}{m} = \binom{1-m}{m} =$$

$$= \frac{(1-m)(-m)(-m-1)\cdots(-2m+2)}{m!} = (-1)^m\binom{2m-2}{m} \neq 0,$$

while $\chi((S^1)^{2m}) = 0$, which is a contradiction.

Suppose that $m+1 \leq t \leq 2m-1$. If $M^{(m)} \cong R^{2m-t} \times (S^1)^t$, then M is a noncompact orientable manifold and $\chi(M^{(m)}) = 0$. Let us suppose that M is a sphere with α handles and without β points, where $\alpha \geq 0$ and $\beta > 0$ because M is not compact. We will obtain a contradiction. Note that in this case $\chi(M) = 2 - 2\alpha - \beta$. It is known that

$$dim H_1(M) = 2\alpha + \beta - 1 = 1 - \chi(M)$$

and we obtain

$$dimH_1(M) = dim\pi_1(M^{(m)}) = dim\pi_1(R^{2m-t} \times (S^1)^t) = t$$

and hence $1-\chi(M) = t \geq m+1$, i.e. $-\chi(M) \geq m$. Hence we obtain a contradiction

$$0 = \chi(M^{(m)}) = (-1)^m \binom{-\chi(M)}{m} \neq 0$$

because $-\chi(M) \geq m$. This proves (ii).

It is easy to prove the conjecture for $t = 0$. Let $M^{(m)} = R^{2m}$. Then M is orientable and not compact manifold and moreover $H_1(M, Z) = \pi_1(M^{(m)}) = \pi_1(R^{2m}) = \{0\}$. Thus we get that $M \cong R^2$. We will prove now the conjecture for $t = 1$, such that the case $2 \leq t \leq m$ will stay unsolved.

Assume that $M^{(m)} \cong S^1 \times R^{2m-1}$. Then M must be noncompact and orientable manifold and

$$dimH_1(M, Z) = dim\pi_1(M^{(m)}) = dim\pi_1(S^1 \times R^{2m-1}) = 1.$$

Moreover, since $\chi(M^{(m)}) = 0$ and $\chi(M^{(m)}) = \binom{m+\chi(M)-1}{m}$ it follows that $0 \leq m + \chi(M) - 1 \leq m - 1$, i.e. $1 - m \leq \chi(M) \leq 0$. Assume that M is a sphere with α handles and without β points, where $\alpha \geq 1$ and $\beta \geq 1$. Then

$$1 = dimH_1(M, Z) = 2\alpha + \beta - 1 \geq 2\alpha \geq 2 > 1.$$

This contradiction yields $\alpha = 0$, i.e. M must be a complex plane without β points. In this case $\beta = 1 - \chi(M) \leq m$ implies

$$(C \setminus A_\beta)^{(m)} \cong (S^1)^\beta \times R^{2m-\beta}$$

and since $(C \setminus A_\beta)^{(m)} \cong S^1 \times R^{2m-1}$ we obtain $\beta = 1$, i.e. $M \cong S^1 \times R$.

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