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TENSOR LINEAR GROUPS

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Abstract. Associative transvections among tensor spaces are discussed, when $\mu=2\nu$ or $\mu=2^{\nu}$. The transvections can be treated as group operations givning rise to constructions of linear groups. In the case $\mu=2^{\nu}$ each $\mathcal{T}^{\mu}_{\mu}(\mathbb{C}^n)$ enables us to obtain a finite sequence of tensor linear groups.

Let $\mathbb{C}^n$ be the complex n-dimensional vector space, $\mathbb{N}$ the set of natural numbers. Let $u^{\alpha_1\cdots\alpha_\mu}_{\beta_1\cdots\beta_\mu}$ $(\mu=2^{\nu},\ \nu\in\mathbb{N})$ be the natural coordinate system in $\mathcal{T}^{\mu}_{\mu}(\mathbb{C}^n)=\mathbb{C}^n\otimes\cdots\otimes\mathbb{C}^n\otimes\mathbb{C}^{n*}\otimes\cdots\mathbb{C}^{n*}$ ($\mathbb{C}^n, \mathbb{C}^{n*}$ taken μ-times each). We consider first, the case $\nu=1$. Let $e_1,\dots,e_n$ be the natural basis of $\mathbb{C}^n$ and consider arbitrary elements $a=a^{\alpha\beta}_{\gamma\eta}e_\alpha\otimes e_\beta\otimes e^\gamma\otimes e^\eta$, $b=b^{\alpha\beta}_{\gamma\eta}e_\alpha\otimes e_\beta\otimes e^\gamma\otimes e^\eta$.
We define the transvections $\varphi, \underset{1}{\varphi}, \underset{2}{\varphi}: \mathcal{T}^2_2\times\mathcal{T}^2_2\to\mathcal{T}^2_2$ as

$$\varphi(a,b)=c,\quad \underset{l}{\varphi}(a,b)=\underset{l}{c}\qquad (l=1,2)$$

where

$$c^{\alpha\beta}_{\gamma\eta}=u^{\alpha\beta}_{\gamma\eta}\cdot\varphi(a,b)=a^{\alpha\sigma}_{\gamma\rho}b^{\rho\beta}_{\sigma\eta} \tag{1}$$

$$\underset{1}{c}{}^{\alpha\beta}_{\gamma\eta}=u^{\alpha\beta}_{\gamma\eta}\cdot\underset{1}{\varphi}(a,b)=a^{\alpha\beta}_{\sigma\rho}b^{\sigma\rho}_{\gamma\eta} \tag{1'}$$

$$\underset{2}{c}{}^{\alpha\beta}_{\gamma\eta}=u^{\alpha\beta}_{\gamma\eta}\cdot\underset{2}{\varphi}(a,b)=a^{\sigma\rho}_{\gamma\eta}b^{\alpha\beta}_{\sigma\rho}\,, \tag{1"}$$

the notation $u^{\alpha\beta}_{\gamma\eta}(a)=a^{\alpha\beta}_{\gamma\eta}$, $u^{\alpha\beta}_{\gamma\eta}(b)=b^{\alpha\beta}_{\gamma\eta}$, $u^{\alpha\beta}_{\gamma\eta}\cdot\varphi(a,b)=\varphi^{\alpha\beta}_{\gamma\eta}(a,b)$ having been used. The mappings $\varphi,\underset{1}{\varphi},\underset{2}{\varphi}$ are associative and define a left and a right action of $\mathcal{T}^2_2(\mathbb{C}^n)$ on itself, L_a and R_b respectively,

$$L_a(b)=R_b(a)=\varphi(a,b),\quad \underset{l}{L}_a(b)=\underset{l}{R}_b(a)=\underset{l}{\varphi}(a,b)\qquad l=1,2.$$

If we look for corresponding identities, from

$$L_a b=b,\quad R_a b=b$$

we obtain

$$a^{\alpha\beta}_{\gamma\eta} = \delta^{\alpha}_{\eta}\delta^{\beta}_{\gamma} . \tag{2}$$

We shall denote the identity elements simply by 1. The cases

$$\underset{l}{L}{}_a b = b, \quad \underset{l}{R}{}_a b = b \qquad l = 1, 2$$

lead to equal identity elements,

$$1^{\alpha\beta}_{\gamma\eta} = \delta^{\alpha}_{\gamma}\delta^{\beta}_{\eta} . \tag{2'}$$

Consider the mappings

$$\underset{\sim}{i}, \underset{l}{i} : \mathcal{T}^2_2(\mathbb{C}^n) \to \mathcal{T}^1_1(\mathbb{C}^{n^2}) \qquad l = 1, 2$$

defined as

$$\underset{\sim}{i}(a) = \begin{pmatrix} a^{11}_{11} & \dots & a^{1n}_{11} & \dots\dots & a^{11}_{1n} & \dots & a^{1n}_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a^{11}_{n1} & \dots & a^{1n}_{n1} & \dots\dots & a^{11}_{nn} & \dots & a^{1n}_{nn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a^{n1}_{11} & \dots & a^{nn}_{11} & \dots\dots & a^{n1}_{1n} & \dots & a^{nn}_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a^{n1}_{n1} & \dots & a^{nn}_{n1} & \dots\dots & a^{n1}_{nn} & \dots & a^{nn}_{nn} \end{pmatrix} \tag{3}$$

$$\underset{1}{i}(a) = \begin{pmatrix} a^{11}_{11} & \dots & a^{11}_{1n} & \dots\dots & a^{11}_{n1} & \dots & a^{11}_{nn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a^{1n}_{11} & \dots & a^{1n}_{1n} & \dots\dots & a^{1n}_{n1} & \dots & a^{1n}_{nn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a^{n1}_{11} & \dots & a^{n1}_{1n} & \dots\dots & a^{n1}_{n1} & \dots & a^{n1}_{nn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a^{nn}_{11} & \dots & a^{nn}_{1n} & \dots\dots & a^{nn}_{n1} & \dots & a^{nn}_{nn} \end{pmatrix} \tag{3'}$$

$$\underset{2}{i}(a) = \begin{pmatrix} a^{11}_{11} & \dots & a^{1n}_{11} & \dots\dots & a^{n1}_{11} & \dots & a^{nn}_{11} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a^{11}_{1n} & \dots & a^{1n}_{1n} & \dots\dots & a^{n1}_{1n} & \dots & a^{nn}_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a^{11}_{n1} & \dots & a^{1n}_{n1} & \dots\dots & a^{n1}_{n1} & \dots & a^{nn}_{n1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a^{11}_{nn} & \dots & a^{1n}_{nn} & \dots\dots & a^{n1}_{nn} & \dots & a^{nn}_{nn} \end{pmatrix} \tag{3''}$$

$$a \in \mathcal{T}^2_2(\mathbb{C}^n).$$

We shall denote $\underset{\sim}{i}(a) = \underset{\sim}{a}$, $\underset{l}{i}(a) = \underset{l}{a}$, hence

$$a^{\alpha\beta}_{\gamma\eta} = \underset{\sim}{a}^{(\alpha-1)n+\gamma}_{(\eta-1)n+\beta} \tag{4}$$

$$a^{\alpha\beta}_{\gamma\eta} = \underset{1}{a}^{(\alpha-1)n+\beta}_{(\gamma-1)n+\eta} \tag{4'}$$

$$a^{\alpha\beta}_{\gamma\eta} = \underset{2}{a}^{(\gamma-1)n+\eta}_{(\alpha-1)n+\beta} \tag{4''}$$

The maps $\underset{\sim}{\xi}', \underset{k}{\xi}'$ (k=1,2) are one-to-one and onto. We define an element $a \in \mathcal{T}^2_2(\mathbb{C}^n)$ as $\underset{\sim}{\xi}'$-regular (resp. $\underset{k}{\xi}'$-regular) if and only if the corresponding $\underset{\sim}{a}$ ($\underset{k}{a}$ resp.) is regular. In the same way we define $a \in \mathcal{T}^2_2(\mathbb{C}^n)$ as $\underset{\sim}{\xi}'$-symmetric, $\underset{\sim}{\xi}'$-unitary, $\underset{\sim}{\xi}'$-hermitian and so on (or $\underset{k}{\xi}'$-symmetric, $\underset{k}{\xi}'$-unitary, $\underset{k}{\xi}'$-hermitian and so on) if the corresponding $\underset{\sim}{a}$ (or $\underset{k}{a}$ resp.) has the corresponding property. In the same way we define

$$^{t}a = \underset{\sim}{\xi}'^{-1}({}^{t}\underset{\sim}{a}), \tag{5}$$

$$\underset{l}{^{t}}a = \underset{l}{\xi}'^{-1}({}^{t}\underset{l}{a}) \qquad (l=1,2), \tag{5'}$$

$$a \in \mathcal{T}^2_2(\mathbb{C}^n).$$

We set

$$GL^{(2,2)}_{n^2}(\mathbb{C}) = \{a \in \mathcal{T}^2_2(\mathbb{C}^n);\ \underset{\sim}{a} \in GL_{n^2}(\mathbb{C})\},$$

$$\underset{k}{GL}^{(2,2)}_{n^2}(\mathbb{C}) = \{a \in \mathcal{T}^2_2(\mathbb{C}^n);\ \underset{k}{a} \in GL_{n^2}(\mathbb{C})\} \qquad k=1,2.$$

For each $a \in GL^{(2,2)}_{n^2}(\mathbb{C})$ (or $a \in \underset{k}{GL}^{(2,2)}_{n^2}(\mathbb{C})$, k=1,2), we define a^{-1} (or $\underset{k}{a}^{-1}$, k=1,2) as

$$a^{-1} = \underset{\sim}{\xi}'^{-1}(\underset{\sim}{a}^{-1}), \tag{6}$$

$$\underset{k}{a}^{-1} = \underset{k}{\xi}'^{-1}(\underset{k}{a}^{-1}) \qquad k=1,2. \tag{6'}$$

We may state now the following

<u>Theorem A</u>. The ordered pairs $(GL^{(2,2)}_{n^2}(\mathbb{C}), \varphi)$, $(\underset{k}{GL}^{(2,2)}_{n^2}(\mathbb{C}), \underset{k}{\varphi})$ k=1,2, are groups isomorphic with the complex general linear group $GL_{n^2}(\mathbb{C})$.

Each of the operations $\varphi, \underset{k}{\varphi}$ (k=1,2) leads to a natural definition of the exponential mapping in $\mathcal{T}^2_2(\mathbb{C}^n)$. We set

$$\exp a = \underset{\sim}{i}^{-1}(\exp \underset{\sim}{a}),$$
$$\underset{k}{\exp} a = \underset{k}{i}^{-1}(\exp \underset{\tilde{k}}{a}) \quad (k = 1, 2),$$

and it is easily seen that each variant has all the standard properties of the exponential mapping defined on spaces of linear transformations.

For the corresponding canonical forms of the groups $GL^{(2,2)}_{n^2}(\mathbb{C})$, $\underset{k}{GL}^{(2,2)}_{n^2}(\mathbb{C})$ we obtain (the proof shall be given in a more general case)

$$\vartheta = \overset{\vee}{u}{}^{\alpha\sigma}_{\gamma\rho}\, du^{\rho\beta}_{\sigma\eta}\, \underset{o}{\partial}{}^{\gamma\eta}_{\alpha\beta}, \tag{7}$$

$$\underset{1}{\vartheta} = \underset{1}{\overset{\vee}{u}}{}^{\alpha\beta}_{\sigma\rho}\, du^{\sigma\rho}_{\gamma\eta}\, \underset{o}{\partial}{}^{\gamma\eta}_{\alpha\beta}, \tag{7'}$$

$$\underset{2}{\vartheta} = \underset{2}{\overset{\vee}{u}}{}^{\sigma\rho}_{\gamma\eta}\, du^{\alpha\beta}_{\sigma\rho}\, \underset{o}{\partial}{}^{\gamma\eta}_{\alpha\beta} \tag{7''}$$

respectively, where $\underset{o}{\partial}{}^{\alpha\beta}_{\gamma\eta} = (\partial/\partial u^{\gamma\eta}_{\alpha\beta})_1$ and the functions $\overset{\vee}{u}{}^{\alpha\beta}_{\gamma\eta}$ are defined by

$$\overset{\vee}{u}{}^{\alpha\beta}_{\gamma\eta}(a) = \overset{\vee}{u}{}^{\alpha\beta}_{\gamma\eta}(a^{-1}),$$
$$\underset{k}{\overset{\vee}{u}}{}^{\alpha\beta}_{\gamma\eta}(a) = u^{\alpha\beta}_{\gamma\eta}(\underset{k}{a}^{-1}) \qquad k = 1, 2.$$

If we extend the domains of φ, $\underset{k}{\varphi}$ and $\underset{\sim}{i}, \underset{k}{i}$ (k=1,2) to $\mathcal{T}^2_2 \times \mathcal{T}^1_1$, $\mathcal{T}^2_2 \times \mathcal{T}^2$, $\mathcal{T}^2_2 \times \mathcal{T}_2$ and $\mathcal{T}^1_1, \mathcal{T}^2, \mathcal{T}_2$ resp. by

$$\varphi^{\alpha\beta}_{\gamma\eta}(\mathcal{A}, a) = u^{\alpha\sigma}_{\gamma\rho}(\mathcal{A})\, u^{\rho}_{\sigma}(a) \tag{8}$$

$$\underset{1}{\varphi}{}^{\alpha\beta}_{\gamma\eta}(\mathcal{A}, b) = u^{\alpha\beta}_{\sigma\rho}(\mathcal{A})\, u^{\sigma\rho}(b) \tag{8'}$$

$$\underset{2}{\varphi}{}^{\alpha\beta}_{\gamma\eta}(\mathcal{A}, c) = u^{\sigma\rho}_{\gamma\eta}(\mathcal{A})\, u_{\sigma\rho}(c) \tag{8''}$$

$$\underset{\sim}{i}(a) = \underset{\sim}{a} = {}^t(u^1_1(a), \ldots, u^1_n(a), \ldots\ldots\ldots, u^n_1(a), \ldots, u^n_n(a)) \in \mathbb{C}^{n^2} \tag{9}$$

$$\underset{1}{i}(b) = \underset{1}{b} = {}^t(u^{11}(b), \ldots, u^{1n}(b), \ldots\ldots\ldots, u^{n1}(b), \ldots, u^{nn}(b)) \in \mathbb{C}^{n^2} \tag{9'}$$

$$\underset{2}{i}(c) = \underset{2}{c} = {}^t(u_{11}(c), \ldots, u_{1n}(c), \ldots\ldots\ldots, u_{n1}(c), \ldots, u_{nn}(c)) \in \mathbb{C}^{n^2} \tag{9''}$$

$$\mathcal{A} \in \mathcal{T}^2_2(\mathbb{C}^n),\ a \in \mathcal{T}^1_1(\mathbb{C}^n),\ b \in \mathcal{T}^2(\mathbb{C}^n),\ c \in \mathcal{T}_2(\mathbb{C}^n)$$

we have

$$\varphi(\mathcal{A}, a) = \underset{\sim}{i}^{-1}(\underset{\sim}{\mathcal{A}}, \underset{\sim}{a}), \tag{10}$$
$$\underset{1}{\varphi}(\mathcal{A}, b) = \underset{1}{i}^{-1}(\underset{\sim}{\mathcal{A}}, \underset{1}{b}), \tag{10'}$$
$$\underset{2}{\varphi}(\mathcal{A}, c) = \underset{2}{i}^{-1}(\underset{\sim}{\mathcal{A}}, \underset{2}{c}) \tag{10''}$$

where in the brackets on the right-hand sides are usual matrix multiplications.

Various possibilities appear if we proceed to extend these operations to corresponding tensor spaces of higher ranks. In the case of $\mathcal{T}^4_4(\mathbb{C}^n)$ and further we shall limit the examination to transvections analogous with φ.

With $\mathcal{S}, \mathcal{T} \in \mathcal{T}^4_4(\mathbb{C}^n)$ we extend φ as we define

$$u^{\alpha\beta\gamma\eta}_{\kappa\lambda\mu\tau} \cdot \varphi(\mathcal{S}, \mathcal{T}) = u^{\alpha\beta\sigma\rho}_{\kappa\lambda\varepsilon\delta}(\mathcal{S})\, u^{\varepsilon\delta\gamma\eta}_{\sigma\rho\mu\tau}(\mathcal{T}). \tag{11}$$

We have

$$\varphi(\varphi(\mathcal{S}, \mathcal{T}), \mathcal{U}) = \varphi(\mathcal{S}, \varphi(\mathcal{T}, \mathcal{U}))$$

and

$$u^{\alpha\beta\gamma\eta}_{\kappa\lambda\mu\tau}(1) = \delta^{\alpha}_{\mu}\delta^{\beta}_{\tau}\delta^{\gamma}_{\kappa}\delta^{\eta}_{\lambda} \tag{12}$$

for the identity element. We define

$$\underset{\sim}{\iota} : \mathcal{T}^4_4(\mathbb{C}^n) \to \mathcal{T}^2_2(\mathbb{C}^{n^2})$$

by

$$u^{\alpha\beta\gamma\eta}_{\kappa\lambda\mu\nu}(\mathcal{T}) = u^{(\alpha-1)n+\kappa \quad (\gamma-1)n+\mu}_{(\lambda-1)n+\beta \quad (\nu-1)n+\eta}(\underset{\sim}{\iota}\mathcal{T}), \quad \mathcal{T} \in \mathcal{T}^4_4(\mathbb{C}^n). \tag{13}$$

We define $\mathcal{T}$ to have a property P if $\underset{\sim}{\mathcal{T}} = \underset{\sim}{\iota}\mathcal{T}$ has that property. We set

$$GL^{(4,4)}_{n^4}(\mathbb{C}) = \{\mathcal{U} \in \mathcal{T}^4_4(\mathbb{C}^n);\ \underset{\sim}{\mathcal{U}} \in GL^{(2,2)}_{n^4}(\mathbb{C}^2)\}.$$

For each $\mathcal{U} \in GL^{(4,4)}_{n^4}(\mathbb{C})$ we define the inverse element $\mathcal{U}^{-1}$ by

$$\mathcal{U}^{-1} = \underset{\sim}{\iota}^{-1}(\underset{\sim}{\mathcal{U}}^{-1}). \tag{14}$$

We set also

$$\exp \mathcal{U} = \underset{\sim}{\iota}^{-1}(\exp \underset{\sim}{\mathcal{U}}), \quad \mathcal{U} \in \mathcal{T}^4_4(\mathbb{C}^n). \tag{15}$$

The following statement is obvious.

<u>Theorem B</u>. The set $GL^{(4,4)}_{n^4}(\mathbb{C})$ with the operation φ is a group isomorphic with $GL_{n^4}(\mathbb{C})$.

The following statement can be proved in a straight-forward way.

Theorem C. The exponential mapping has the properties

$$(c_1) \qquad (\exp \mathcal{U})^{-1} = \exp \mathcal{U}^{-1}$$

$$(c_2) \qquad {}^t(\exp \mathcal{U}) = \exp {}^t\mathcal{U}$$

$$(c_3) \qquad \exp(\mathcal{S}\mathcal{U}\mathcal{S}^{-1}) = \mathcal{S}(\exp \mathcal{U})\mathcal{S}^{-1}$$

$$\mathcal{S}, \mathcal{U} \in \mathcal{T}_4^4(\mathbb{C}^n).$$

We shall look for the canonical form of $GL_{n^4}^{(4,4)}(\mathbb{C})$, the line of considerations being valid for arbitrary $\mathcal{T}_\mu^\mu(\mathbb{C}^n)$, when $\mu = 2^\nu$, $\nu \in \mathbb{N}$ We set $\partial_{\kappa\lambda\mu\tau}^{\alpha\beta\gamma\eta} = \partial/\partial u_{\alpha\beta\gamma\eta}^{\kappa\lambda\mu\tau}$, $\overset{o}{\partial}{}_{\kappa\lambda\mu\tau}^{\alpha\beta\gamma\eta} = (\partial/\partial u_{\alpha\beta\gamma\eta}^{\kappa\lambda\mu\tau})_1$. Let $\mathcal{X}$ be an arbitrary tangent vector of $GL_{n^4}(\mathbb{C})$ at $\mathcal{T}$. For each $\mathcal{S} \in GL_{n^4}^{(4,4)}(\mathbb{C})$ we have

$$(dL_{\mathcal{S}}\mathcal{X})u_{\lambda\kappa\mu\tau}^{\alpha\beta\gamma\eta} = \mathcal{X}(u_{\lambda\kappa\mu\tau}^{\alpha\beta\gamma\eta} \circ \mathcal{S}).$$

Since

$$(u_{\lambda\kappa\mu\tau}^{\alpha\beta\gamma\eta} \circ \varphi)\mathcal{U} = u_{\lambda\kappa\varepsilon\delta}^{\alpha\beta\tau\rho}(\mathcal{S})\, u_{\sigma\rho\mu\tau}^{\varepsilon\delta\gamma\eta}(\mathcal{U}), \quad \mathcal{U} \in GL_{n^4}^{(4,4)}(\mathbb{C}),$$

we have

$$du_{\lambda\kappa\mu\tau}^{\alpha\beta\gamma\eta}(dL_{\mathcal{S}}\mathcal{X}) = u_{\lambda\kappa\varepsilon\delta}^{\alpha\beta\sigma\rho}(\mathcal{S})\, du_{\sigma\rho\mu\tau}^{\varepsilon\delta\gamma\eta}(\mathcal{X}).$$

For each $\mathcal{U} \in GL_{n^4}^{(4,4)}(\mathbb{C})$ we set $\check{u}_{\lambda\kappa\mu\tau}^{\alpha\beta\gamma\eta}(\mathcal{U}) = u_{\lambda\kappa\mu\tau}^{\alpha\beta\gamma\eta}(\mathcal{U}^{-1})$, hence $\mathcal{S} = \mathcal{T}^{-1}$ leads to

$$\vartheta = \check{u}_{\lambda\mu\varepsilon\delta}^{\alpha\beta\sigma\rho}\, du_{\sigma\rho\nu\tau}^{\varepsilon\delta\gamma\eta}\, \overset{o}{\partial}{}_{\alpha\beta\gamma\eta}^{\lambda\mu\nu\tau}. \tag{16}$$

We shall consider further only tensor spaces $\mathcal{T}_\mu^\mu(\mathbb{C}^n)$ when $\mu = 2^\nu$ for some $\nu \in \mathbb{N}$. We set $\nu = 3$ and considered $\mathcal{T}_8^8(\mathbb{C}^n)$ to obtain a group $GL_{n^8}^{(8,8)}(\mathbb{C})$. For each $\mathcal{P} \in \mathcal{T}_8^8(\mathbb{C}^n)$, $\underset{\sim}{\mathcal{P}} = \underset{\sim}{\iota}\mathcal{P} \in \mathcal{T}_4^4(\mathbb{C}^{n^2})$ is defined by

$$u_{\pi\rho\sigma\tau\varphi\chi\psi\omega}^{\alpha\beta\gamma\delta\varepsilon\kappa\lambda\mu}(\mathcal{P}) = u_{(\rho-1)n+\beta\ \ (\tau-1)n+\delta\ \ (\chi-1)n+\kappa\ \ (\omega-1)n+\mu}^{(\alpha-1)n+\pi\ \ (\gamma-1)n+\sigma\ \ (\varepsilon-1)n+\varphi\ \ (\lambda-1)n+\psi}(\underset{\sim}{\mathcal{P}}).$$

We set $\underset{\sim}{\iota} \circ \underset{\sim}{\iota} = \underset{\sim}{\iota}^2$, $\underset{\sim}{\iota}^2 \circ \underset{\sim}{\iota} = \underset{\sim}{\iota}^3$ and can proceed further

$$\xi^2 : \mathcal{T}_8^8(\mathbb{C}^n) \to \mathcal{T}_2^2(\mathbb{C}^{n^4})$$

as

$$u^{\alpha\beta\gamma\delta\varepsilon\kappa\lambda\mu}_{\pi\rho\sigma\tau\varphi\chi\psi\omega}(\mathcal{P}) = u^{(\alpha-1)n^2+(\pi+\rho-2)n+\beta \quad (\varepsilon-1)n^2+(\varphi+\chi-2)n+\kappa}_{(\tau-1)n^2+(\delta+\gamma-2)n+\sigma \quad (\omega-1)n^2+(\mu+\lambda-2)n+\psi}(\xi^2\mathcal{P}).$$

In the same way

$$\xi^3 : \mathcal{T}_8^8(\mathbb{C}^n) \to \mathcal{T}_1^1(\mathbb{C}^{n^8}),$$

$$u^{\alpha\beta\gamma\delta\varepsilon\kappa\lambda\mu}_{\pi\rho\sigma\tau\varphi\chi\psi\omega}(\mathcal{P}) = u^{(\alpha-1)n^3+(\pi+\rho+\tau-3)n^2+(\beta+\gamma+\delta-3)n+\sigma}_{(\omega-1)n^3+(\varepsilon+\lambda+\mu-3)n^2+(\varphi+\chi+\psi-3)n+\kappa}(\xi^3\mathcal{P}).$$

We proceed successively and define the group $GL_{n^\mu}^{(\mu,\mu)}(\mathbb{C})$, $\mu=2^\nu$. Consider $\mathcal{T}_\mu^\mu(\mathbb{C}^n)$, $\mu=2^\nu$ with the mapping

$$\varphi : \mathcal{T}_\mu^\mu(\mathbb{C}^n) \times \mathcal{T}_\mu^\mu(\mathbb{C}^n) \to \mathcal{T}_\mu^\mu(\mathbb{C}^n) \qquad (17)$$

defined as

$$u^{\alpha_1\cdots\alpha_\tau\beta_1\cdots\beta_\tau}_{\gamma_1\cdots\gamma_\tau\eta_1\cdots\eta_\tau}\cdot\varphi(\mathcal{A},\mathcal{B}) = u^{\alpha_1\cdots\alpha_\tau\sigma_1\cdots\sigma_\tau}_{\gamma_1\cdots\gamma_\tau\mu_1\cdots\mu_\tau}(\mathcal{A})\,u^{\mu_1\cdots\mu_\tau\beta_1\cdots\beta_\tau}_{\sigma_1\cdots\sigma_\tau\eta_1\cdots\eta_\tau}(\mathcal{B}) \qquad (18)$$

where $\tau=\mu/2$. The mapping φ is associative and defines a left and a right action on $\mathcal{T}_\mu^\mu$ with equal identity elements

$$u^{\alpha_1\cdots\alpha_\tau\beta_1\cdots\beta_\tau}_{\gamma_1\cdots\gamma_\tau\eta_1\cdots\eta_\tau}(1) = \delta^{\alpha_1}_{\eta_1}\cdots\delta^{\alpha_\tau}_{\eta_\tau}\delta^{\beta_1}_{\gamma_1}\cdots\delta^{\beta_\tau}_{\gamma_\tau}. \qquad (19)$$

We define inductively the mapping

$$\xi : \mathcal{T}_\mu^\mu(\mathbb{C}^n) \to \mathcal{T}_{\mu/2}^{\mu/2}(\mathbb{C}^{n^2}) \qquad (\mu=2^\nu,\ \nu\in\mathbb{N}) \qquad (20)$$

giving rise to the sequence of linear groups

$$GL_{n^\mu}^{(1,1)}(\mathbb{C}) = GL_{n^\mu}(\mathbb{C}),$$

$$GL_{n^\mu}^{(2,2)}(\mathbb{C}),$$

$$GL_{n^\mu}^{(4,4)}(\mathbb{C}),$$

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$$GL_{n^\mu}^{(\mu,\mu)}(\mathbb{C}) = \{\mathcal{A}\in\mathcal{T}_\mu^\mu(\mathbb{C}^n);\ \xi\mathcal{A}\in GL_{n^\mu}^{(\mu/2,\mu/2)}(\mathbb{C})\}.$$

The group $GL_{n^\mu}^{(\mu,\mu)}(\mathbb{C})$ can be directly defined as

$$GL_{n^\mu}^{(\mu,\mu)}(\mathbb{C}) = \{ \mathcal{A} \in \mathcal{T}_\mu^\mu(\mathbb{C}^n) : \xi^\nu(\mathcal{A}) \in GL_{n^\mu}(\mathbb{C}) \}$$

where $\xi^\nu = \xi^{\nu-1} \circ \xi$. Theorem C can be naturally extended to include all the groups of the sequence, an element $\mathcal{A} \in \mathcal{T}_\mu^\mu(\mathbb{C}^n)$ is defined to have a property P if $\xi\mathcal{A}$ possesses the one, and the canonical form of $GL_{n^\mu}^{(\mu,\mu)}(\mathbb{C})$ can be written as

$$\vartheta = \check{u}^{\alpha_1 \ldots \alpha_r \sigma_1 \ldots \sigma_r}_{\beta_1 \ldots \beta_r \rho_1 \ldots \rho_r} \, du^{\rho_1 \ldots \rho_r \kappa_1 \ldots \kappa_r}_{\sigma_1 \ldots \sigma_r \lambda_1 \ldots \lambda_r} \, \partial^{\beta_1 \ldots \beta_r \lambda_1 \ldots \lambda_r}_{\alpha_1 \ldots \alpha_r \kappa_1 \ldots \kappa_r}$$

with

$$\check{u}^{\alpha_1 \ldots \alpha_\mu}_{\beta_1 \ldots \beta_\mu}(\mathcal{C}) = u^{\alpha_1 \ldots \alpha_\mu}_{\beta_1 \ldots \beta_\mu}(\mathcal{C}^{-1}) \qquad \mathcal{C} \in GL_{n^\mu}^{(\mu,\mu)}(\mathbb{C}),$$

$$\partial^{\alpha_1 \ldots \alpha_\mu}_{\beta_1 \ldots \beta_\mu} = (\partial / \partial^{\beta_1 \ldots \beta_\mu}_{\alpha_1 \ldots \alpha_\mu})_1 .$$

We conclude with the following obvious statement:

Theorem D. Each $\mathcal{T}_\mu^\mu(\mathbb{C}^n)$, $n = 2^\nu$ is a Lie algebra with the bracket operation $[\mathcal{A},\mathcal{B}] = \varphi(\mathcal{A},\mathcal{B}) - \varphi(\mathcal{B},\mathcal{A})$. The corresponding Lie group is $GL_{n^\mu}^{(\mu,\mu)}(\mathbb{C})$.

R E F E R E N C E S

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ТЕНЗОРСКИ ЛИНЕАРНИ ГРУПИ

О. Јотов

Р е з и м е

Од множеството различни контракции меѓу тензорски простори од тип (r,s) се издвојува класа асоцијативни контракции при $r = s = 2^\nu$, $\nu \in \mathbb{N}$. Се покажува дека таква функција може да се третира како групна операција за соодветно подмножество на $\mathcal{T}_\mu^\mu(\mathbb{C}^n)$. Дефиниционата област на експоненцијалната функција природно се проширува на тензорските простори и се добива низа од тензорски групи изоморфни со соодветни комплексни линеарни групи.