Math. Maced.
Vol. 6 (2008)
55-60

TAUBERIAN THEOREM OF LITTLEWOOD TYPE FOR SERIES IN BESSEL-MAITLAND FUNCTIONS

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Dedicated to Academician Blagoj Popov on the Occasion of His 85^{th} Birthday

Abstract. A Tauberian theorem of Littlewood type for the summation of divergent series defined by means of the Bessel-Maitland functions is proved.

1. Introduction

The origin of the summability theory of divergent series can be found back to the time of Leonard Euler. He was the first who discussed that the question must be "how to define the sum of divergent series", but not "what equals this sum". He wrote ([2], p. 78, (110.)): "I say that the whole difficulty lies in the sense of the word "sum". Indeed, if the phrase "sum of series" means, as usual, the result of the summation of all its terms, then a sum can be obtained only for convergent series. If the word "sum" is understood only in such a close sense, then the divergent series have not any sum, at all. If another sense, different from the usual, is given to the word "sum", one can get out of these difficulties. Moreover, if the series is convergent, this "new" sum is required to be the same as the usual one. Since the divergent series do not have any sum, the new sum makes no problems". Euler's point of view for divergent series is completely actual: a divergent series has no sum in the usual sense, but it is possible to introduce a new definition of a "sum" (i.e., definition of the summation method of series), applied to all the convergent and some divergent series. Moreover, this requires the new sum of the convergent series to be the same as in a usual sense (i.e., the method to be regular).

Let us consider the numerical series

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{C}, \quad n = 0, 1, 2, ... \tag{1.1}$$

To define its Abel summability ([3], p. 20, 1.3 (2)), we consider also the power series $\sum\limits_{n=0}^{\infty} a_n z^n$.

Definition 1.1. The series (1.1) is called A - summable if the series $\sum\limits_{n=0}^{\infty} a_n z^n$ converges

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2000 *Mathematics Subject Classification.* 30B10, 30B30, 33C10, 33C20, 33C60.

Key words and phrases. Bessel-Maitland functions, Abelian and Tauberian theorems, summation of divergent series.

[1]This work is performed in the frames of the Research Project "Transform Methods, Special Functions, Operational Calculi and Applications" under the auspices of the Institute of Mathematics and Informatics - Bulgarian Academy of Sciences.

in the disk $D = \{z : z \in \mathbb{C}, |z| < 1\}$ and moreover there exists $\lim\limits_{z\to 1-0} \sum\limits_{n=0}^{\infty} a_n z^n = S$. The complex number S is called A-sum of the series (1.1) and the usual notation of that is

$$\sum_{n=0}^{\infty} a_n = S \quad (A).$$

Remark 1.1. The A-summation is regular. It means that if the series (1.1) is convergent, then it is A-summable, and its A-sum is equal to its usual sum.

Remark 1.2. The A-summability of the series (1.1) does not imply in general its convergence. But, with additional conditions on the growth of the general term of the series (1.1), the convergence can be ensured.

A classical result in this direction is given by the following theorem ([3], Theorem 85).

Theorem 1.1. (Tauber). Let the series (1.1) be A-summable,

$$\sum_{n=0}^{\infty} a_n = S \quad (A) \quad \text{and} \quad \lim_{n\to\infty} n a_n = 0.$$

Then the series $\sum\limits_{n=0}^{\infty} a_n$ converges with a sum S.

At first sight it seems that the condition $a_n = o(1/n)$ is essential. Nevertheless, Littlewood succeeded to weaken it and obtain the following strengthened version of the Tauber theorem ([3], 7.6.6; Theorem 90).

Theorem 1.2. (Littlewood). Let the series (1.1) be A-summable,

$$\sum_{n=0}^{\infty} a_n = S \quad (A) \qquad \text{and} \quad a_n = O(1/n).$$

Then the series $\sum\limits_{n=0}^{\infty} a_n$ converges with a sum S.

2. Series in Bessel-Maitland functions

Let $J_\nu^\mu(z)$ be the so-called Bessel-Maitland function:

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(\nu + \mu k + 1)} = \psi(\mu, \nu + 1; -z), \qquad \mu > -1,$$

as a varinat of the Wright function $\psi(\alpha, \beta' z)$, see more details for example in: Marichev [6], p. 110; Kiryakova [5], p. 352; Kilbas, Srivastava and Trujillo [4], p. 55, (1.11.10). This generalization of the Bessel functions of first kind $J_\nu(z)$ with one more additional index μ have been misnamed in the literature as Bessel-Maitland, or Wright function (in the name of E. M. (Maitland) Wright).

Denote for a sake of brevity :

$$\widetilde{J}_n^\mu(z) = z^n J_n^\mu(z), \quad n = 0, 1, 2, ...$$

and consider series of the form

$$\sum_{n=0}^{\infty} a_n \widetilde{J}_n^\mu(z), \quad z \in \mathbb{C}, \quad \mu > 0. \tag{2.1}$$

In studying the behaviour of such series on the boundary of its domain of convergence in the complex plane we recall that Cauchy-Hadamard and Abel type theorems are valid for such type of series as proven in [8].

Theorem 2.1 (of Cauchy-Hadamard type). The domain of convergence of the series (2.1) is the disk $\{z : z \in \mathbb{C},\ |z| < R,\ R = 1/\Lambda\}$, where

$$\Lambda = \limsup_{n\to\infty} (\ |a_n|\ /\ \Gamma(n+1)\ )^{1/n}. \tag{2.2}$$

The cases $\Lambda = 0$ and $\Lambda = \infty$ can be included in the general case, provided $1/\Lambda$ means ∞, respectively 0.

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angular domain where $2\varphi < \pi$ and vertex at the point $z = z_0$, symmetric with respect to the line through the points 0 and z_0.

Theorem 2.2 (of Abel type). Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, and Λ is defined by (2.2), $0 < \Lambda < \infty$. Let $K = \{z : z \in \mathbb{C},\ |z| < R, R = 1/\Lambda\}$. If $f(z)$ is the sum of the series (2.1) in the disk K and this series converges at the point z_0 of the boundary of K, then

$$\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n^{\mu}(z_0), \quad z \in g_\varphi, \quad |z| < R.$$

3. $(J, z_0; \mu)$ - SUMMATION

Let $z_0 \in \mathbb{C}$, $\quad |z_0| = R$, $\quad 0 < R < \infty$, $\quad J_n^{\mu}(z_0) \neq 0$ for $n = 0, 1, 2, \ldots$.
For a sake of brevity, denote

$$J_{n,\mu}^{*}(z; z_0) = \frac{\widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}.$$

Definition 3.1. The series (1.1) is said to be $(J, z_0; \mu)$ - summable if the series

$$\sum_{n=0}^{\infty} a_n J_{n,\mu}^{*}(z; z_0) \tag{3.1}$$

converges in the disk $|z| < R$ and, moreover, there exists the limit

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n J_{n,\mu}^{*}(z; z_0), \tag{3.2}$$

provided z remains on the segment $[0, z_0)$.

Remark 3.1. Every $(J, z_0; \mu)$ - summation is regular, and this property is just a particular case of Theorem 2.2.

In [8] a Tauber type theorem for $(J, z_0; \mu)$ - summation is given, namely:

Theorem 3.1. (of Tauber type). If the series (1.1) is $(J, z_0; \mu)$ - summable and

$$\lim_{n\to\infty} n a_n = 0,$$

then it is convergent.

4. An asymptotic formula

The asymptotic formula with respect to the index

$$J_n^\mu(z) = \frac{1}{\Gamma(n+1)}(1+\theta_n^\mu(z)), \quad z \in \mathbb{C}, \ \mu > 0 \quad \theta_n^\mu(z) \to 0 \ \text{ as } \ n \to \infty \tag{4.1}$$

is given in [7] for the Bessel-Maitland functions. The functions $\theta_n^\mu(z)$ are entire functions. The convergence of $\{\theta_n^\mu(z)\}$ is uniform on the compact subsets of the complex plane $\mathbb{C}$. Considering explicitly $\theta_n^\mu(z)$, we can make this result sharper, as follows.

Theorem 4.1. Let $K \subset \mathbb{C}$ be a nonempty compact set. Then there exists a constant C, $0 < C < \infty$, such that for each $n \in \mathbb{N}_0$ and each $z \in K$ the following inequality holds

$$|\theta_n^\mu(z)| \le C\, \Gamma(n+1)/\Gamma(n+1+\mu). \tag{4.2}$$

Proof. Let $z \in \mathbb{C}$. Due to (4.1) we can write

$$\theta_n^\mu(z) = \sum_{k=1}^{\infty} \frac{\Gamma(n+1)(-z)^k}{k!\,\Gamma(n+\mu k+1)},$$

Denoting

$$w_k(n,\mu) = \frac{\Gamma(n+\mu+1)}{\Gamma(n+\mu k+1)}, \quad u_k(z;n,\mu) = \frac{w_k(n,\mu)}{k!}\,(-z)^k$$

we represent $\theta_n^\mu(z)$ in the form

$$\theta_n^\mu(z) = \frac{\Gamma(n+1)}{\Gamma(n+\mu+1)} \sum_{k=1}^{\infty} u_k(z;\, n,\mu). \tag{4.3}$$

After that we estimate $w_k(n,\mu)$ and the module of $\theta_n^\mu(z)$.

To this end we remember that Euler's Γ-function is positive on the interval $(0,\infty)$, decreasing in the interval $(0,\alpha_0)$, increasing in (α_0,∞) and

$$\min_{\alpha \in (0,\infty)} \Gamma(\alpha) = \Gamma(\alpha_0) > 0, \quad \alpha_0 \in (1,2).$$

First, let $n=0$ and $1 < \mu+1 < \alpha_0$. Then $\Gamma(\mu+1) < \Gamma(1) = 1$, and $\Gamma(\alpha_0) \le \Gamma(\mu k+1)$, whence $w_k(0,\mu) < 1/\Gamma(\alpha_0)$. In the case $\alpha_0 \le \mu+1$, inequality $w_k(0,\mu) \le 1$ holds. So, we obtain the estimate

$$|u_k(z;n,\mu)| \le c_0 \frac{|z|^k}{k!}, \quad c_0 = \begin{cases} \frac{1}{\Gamma(\alpha_0)}, & if\ \mu+1<\alpha_0 \\ 1, & if\ \mu+1 \ge \alpha_0 \end{cases}$$

for the absolute value of $u_k(z;n,\mu)$ and therefore, in view of (4.3), we have

$$|\theta_0^\mu(z)| \le \frac{c_0}{\Gamma(1+\mu)}\,(\exp|z|-1) \tag{4.4}$$

for the module of $\theta_0^\mu(z)$.

Further, let $n \in \mathbb{N}$. Then inequality $w_k(n,\mu) \le 1$ holds and therefore

$$|\theta_n^\mu(z)| \le \frac{\Gamma(n+1)}{\Gamma(n+1+\mu)}\,(\exp|z|-1) \tag{4.5}$$

on the whole complex plane. Then, the estimate (4.2) follows immediately from (4.4) and (4.5). □

Remark 4.1. Stirling's formula gives that

$$\frac{\Gamma(n+1)}{\Gamma(n+1+\mu)} = O\left(\frac{1}{n^\mu}\right) \quad n \in \mathbb{N}.$$

Remark 4.2. The uniform convergence of $\{\theta_n^\mu(z)\}$ on the compact subsets of $\mathbb{C}$ follows from (4.2) as well.

Remark 4.3. Formula (4.1) has been used in the proof of Theorems 2.1, 2.2, 3.1. We apply (4.2) essentially in proving the corresponding strengthened version of Tauber theorem for series in Bessel-Maitland functions.

5. A Littlewood type theorem

A Littlewood generalization of the $o(n)$ version of Tauber type theorem (Theorem 3.1) is given in this part. Similar theorem is proved in [1] (a generalization of a Tauber type theorem, proved in [10]) for summation by means of Laguerre polynomials and in [9] for series in Bessel functions of first kind.

Theorem 5.1 (of Littlewood type). If the series (1.1) is $(J, z_0; \mu)$ - summable and

$$a_n = O(1/n) \tag{5.1}$$

then the series (1.1) converges.

Proof. Let z belong to the segment $[0, z_0]$. By using the asymptotic formula (4.1) for the Bessel-Maitland functions, we obtain:

$$a_n J^*_{n,\mu}(z; z_0) = a_n \left(\frac{z}{z_0}\right)^n \frac{1+\theta_n^\mu(z)}{1+\theta_n^\mu(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \left(1 + \widetilde{\theta}_n^\mu(z; z_0)\right),$$

where $\widetilde{\theta}_n^\mu(z; z_0) = \dfrac{\theta_n^\mu(z) - \theta_n^\mu(z_0)}{1+\theta_n^\mu(z_0)}$. Then $\widetilde{\theta}_n^\mu(z; z_0) = O(1/n^\mu)$, due to (4.2) and Remark 4.1.

Let us write (3.1) in the form

$$\sum_{n=0}^{\infty} a_n J^*_{n,\mu}(z; z_0) = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \left(1 + \widetilde{\theta}_n^\mu(z; z_0)\right). \tag{5.2}$$

Denoting $w_n(z) = a_n \left(\frac{z}{z_0}\right)^n \widetilde{\theta}_n^\mu(z; z_0)$ we consider the series $\sum\limits_{n=0}^{\infty} w_n(z)$. Since $|w_n(z)| \le |a_n|\,|\widetilde{\theta}_n^\mu(z; z_0)|$ and according to condition (5.1) and Theorem 4.1, there exists a constant $\widetilde{C}$, such that $|w_n(z)| \le \widetilde{C}/n^{1+\mu}$ for $n \in \mathbb{N}$. Since $\sum\limits_{n=1}^{\infty} 1/n^{1+\mu}$ converges, the series $\sum\limits_{n=0}^{\infty} w_n(z)$ is also convergent, even absolutely and uniformly on the segment $[0, z_0]$. Therefore (since $\lim\limits_{z \to z_0} w_n(z) = w_n(z_0) = a_n \widetilde{\theta}_n^\mu(z_0; z_0) = 0$)

$$\lim_{z \to z_0} \sum_{n=0}^{\infty} w_n(z) = \sum_{n=0}^{\infty} \lim_{z \to z_0} w_n(z) = 0.$$

Obviously, the assumption that the series (1.1) is $(J, z_0; \mu)$ - summable implies the existence of the limit (3.2). Then, having in mind that (5.2) can be written in the form

$$\sum_{n=0}^{\infty} a_n J^*_{n,\mu}(z; z_0) = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n + \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \widetilde{\theta}_n^\mu(z; z_0),$$

we conclude that there exists the limit

$$\lim_{z\to z_0}\sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \tag{5.3}$$

and, moreover,

$$\lim_{z\to z_0}\sum_{n=0}^{\infty} a_n J^*_{n,\mu}(z;z_0) = \lim_{z\to z_0}\sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n .$$

From the existence of the limit (5.3) it follows that the series (1.1) is A-summable. Then according to Theorem 1.2, the series (1.1) converges. □

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