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CONDITIONS FOR EXISTENCE QUASI-PERIODIC SOLUTIONS WITH A CONSTANT QUASI-PERIOD FOR DIFFERENTIAL EQUATION OF FOURTH ORDER

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Abstract. In this paper, using the same reducible method as in the papers [1], [2], [3] and [4], we give some conditions for existence of quasi-periodic solution with a constant quasi-period for the ordinary differential equation of fourth order (2.1) and find this solution.

1. INTRODUCTION

In this part we give some previous results ([2],[3]) which will be needed in the following part of this paper.

Definition 1.1. *We say that* $y = \varphi(x)$, $x \in I \subseteq D_\varphi \subset R$ *is a quasi-periodic function (QPF) if there are a function* $\omega = \omega(x)$ *and a coefficient* $\lambda = \lambda(\omega)$ $(\lambda > 0, \lambda \neq 1)$ *such that the relation*

$$\varphi(x + \omega) = \lambda\varphi(x), \quad x, x + \omega \in I$$

is satisfied. The function $\omega(x)$ *is called a quasi-period (QP) and* λ *is said to be a quasi-periodic coefficient (QPC) of the function* $\varphi(x)$.

Let the equation

$$y''' + a(x)y'' + b(x)y' + c(x)y = d(x) \tag{1.1}$$

where $a(x), b(x), c(x), d(x)$ are three times differentiable functions at $I \subseteq D_a \cap D_b \cap D_c \cap \ldots \cap D_y$, be given. The following theorem holds.

Theorem 1.1. [3] *Let the coefficients* $a(x), b(x), c(x), d(x)$ *in (1.1) be QPF with a constant QP* $\overline{\omega}$ *and QPC* $\mu, \nu, \eta, \varsigma$ *respectively, such that* $\mu \neq \nu$, $\nu \neq \eta, \mu \neq \eta, \mu \neq \varsigma, \nu \neq \varsigma$, $\mu, \nu, \eta, \varsigma \neq \lambda$. *Equation (1.1) has QPS* $y = \dfrac{d(x)}{c(x)}$ *with QP* $\overline{\omega}$ *and QPC* $\lambda = \dfrac{\varsigma}{\eta}$, *if the relations*

$$\left(\frac{d}{c}\right)''' + \frac{\mu - \nu}{\mu - 1} \cdot \left(\frac{d}{c}\right)' \cdot b = 0, \quad \textit{and} \tag{1.2}$$

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$$\left(\frac{d}{c}\right)''' - \frac{\mu-\nu}{\nu-1}\cdot\left(\frac{d}{c}\right)''\cdot a = 0 \tag{1.3}$$

are satisfied.

Let the equation

$$y'' + f(x)y' + g(x)y = h(x) \tag{1.4}$$

where $f(x), g(x), h(x)$ are two times differentiable functions at $I \subseteq D_f \cap D_g \cap D_h \cap D_y$, be given. The following theorem holds.

Theorem 1.2. [2] *Let the coefficients $f(x), g(x), h(x)$ in (1.4) be QPF with a constant QP ϖ and QPC μ, ν, η respectively, such that $\mu \neq \nu$, $\nu \neq \eta$, $\mu \neq \eta$, and $\mu \neq \lambda$, $\nu \neq \lambda$, $\eta \neq \lambda$, $\nu \neq 1$. Then the equation (1.4) has QPS $y = \dfrac{h(x)}{g(x)}$ with QP ϖ and QPC $\lambda = \dfrac{\eta}{\nu}$, if the relation*

$$\left(\frac{h}{g}\right)'' + \left(\frac{h}{g}\right)' f = 0 \quad (f = f(x),\ \ g = g(x),\ \ h = h(x)) \tag{1.5}$$

is satisfied.

Remark 1.1. We note that the Definition 1.1. for quasi-periodicity of functions, as well as the reducible method to a given differential equation with respect to the quasi-periodic solution, have been introduced by the authors. The authors have not found this definition and the applied reducible method in any available literature. Both of them have already been used in authors' previous papers [1],[2],[3] and [4].

2. MAIN RESULTS

Let

$$L(x) \equiv y^{iv} + a(x)y''' + b(x)y'' + c(x)y' + d(x)y - e(x) = 0 \tag{2.1}$$

be a given differential equation where $a(x), b(x), c(x), d(x), e(x)$, are four times differentiable functions on $I \subseteq D_a \cap D_b \cap \cdots \cap D_e \cap D_y$.

Lemma 2.1. *If $y = y(x)$ is a quasi-periodic solution (QPS) for the eq.(2.1) with a constant QP ϖ and QPC $\lambda(\lambda > 0, \lambda \neq 1)$, then the eq.(2.1) is reduced to the linear differentail equation of third order with respect to QPS $y = y(x)$:*

$$\begin{aligned}(a(t)-a(x))\,y'''(x) + (b(t)-b(x))\,y''(x) + (c(t)-c(x))\,y'(x) + \\ + (d(t)-d(x))\,y(x) = \frac{1}{\lambda}e(t) - e(x)/_{t=x+\varpi}\end{aligned} \tag{2.2}$$

Proof. By the system

$$\begin{cases} L(x) = 0 \\ t = x + \overline{\omega} \\ L(t) = 0 \\ y(t) = \lambda y(x) \\ \dfrac{d^k}{dx^k} y(t) = \lambda y^{(k)}(x), \ k = 1, 2, 3, 4 \end{cases} \tag{2.3}$$

we can eliminate $y^{iv}(t)$, $y^{iv}(x)$, $y(t)$ and so we obtain the eq.(2.2). □

Theorem 2.1. *Let the eq.(2.1) have QPS with a constant QP $\overline{\omega}$ and QPC λ $(\lambda > 0, \lambda \neq 1)$. If the coefficients $a(x), b(x), c(x), d(x), e(x)$ for (2.1) are QPF with the same QP $\overline{\omega}$ and QPC $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ respectively, then the eq.(2.1) is reduced to the equation*

$$(\lambda_1 - 1)a(x)y''' + (\lambda_2 - 1)b(x)y'' + (\lambda_3 - 1)c(x)y' + (\lambda_4 - 1)d(x)y = \frac{1}{\lambda}(\lambda_5 - \lambda)e(x). \tag{2.4}$$

Proof. Substituting in (2.2) $a(t) = \lambda_1 a(x)$, $b(t) = \lambda_2 b(x)$, $c(t) = \lambda_3 c(x)$, $d(t) = \lambda_4 d(x)$, $e(t) = \lambda_5 e(x)/_{t=x+\overline{\omega}}$, we obtain (2.4). □

Theorem 2.2. *Let the coefficients $a = a(x), b = b(x), c = c(x), d = d(x), e = e(x)$ for the eq.(2.1) be QPF with the same constant QP $\overline{\omega}$ and QPC $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ respectively, such that $\lambda_1 \neq 1$, $\lambda_2 = \lambda_1$, $\lambda_3 = 1$, $\lambda_4 \neq \lambda_5$. Then, the eq.(2.1) has QPS $y = \dfrac{e}{d}$ if the relations*

$$a \cdot \left(\frac{e}{d}\right)''' + b \cdot \left(\frac{e}{d}\right)'' = 0 \tag{2.5}$$

$$\left(\frac{e}{d}\right)^{iv} + c \cdot \left(\frac{e}{d}\right)' = 0 \tag{2.6}$$

are satisfied.

Proof. Let $y = y(x)$ be QPS for the eq.(2.1) with QP $\overline{\omega}$ and QPC $\lambda \neq \lambda_5$. Using the Theorem 2.1. we reduce the eq.(2.1) to the equation of third order (2.4). If $\lambda_1 \neq 1$ we can write the eq.(2.4) in the form

$$y''' + By'' + Cy' + Dy = E \tag{2.7}$$

where

$$B = \frac{\lambda_2 - 1}{\lambda_1 - 1} \cdot \frac{b}{a}, \ C = \frac{\lambda_3 - 1}{\lambda_1 - 1} \cdot \frac{c}{a}, \ D = \frac{\lambda_4 - 1}{\lambda_1 - 1} \cdot \frac{d}{a}, \ E = \frac{\lambda_5 - \lambda}{\lambda(\lambda_1 - 1)} \cdot \frac{e}{a},$$

and such that

$$B(x + \omega) = \frac{\lambda_2}{\lambda_1} B(x), \ \ C(x + \omega) = \frac{\lambda_3}{\lambda_1} C(x),$$

$$D(x + \omega) = \frac{\lambda_4}{\lambda_1} D(x), \ \ E(x + \omega) = \frac{\lambda_5}{\lambda_1} E(x).$$

Since B, C, D, E are QPF, the eq.(2.7) has QPS $y = \frac{e}{d}$ if B, C, D, E satisfy the relations [3]:

$$\left(\frac{\lambda_2}{\lambda_1} - 1\right) \cdot B \cdot \left(\frac{E}{D}\right)'' + \left(\frac{\lambda_3}{\lambda_1} - 1\right) \cdot C \cdot \left(\frac{E}{D}\right)' = 0$$

$$\left(\frac{E}{D}\right)''' - B \cdot \left(\frac{E}{D}\right)'' + C \cdot \left(\frac{E}{D}\right)' = 0$$

i.e.

$$(\lambda_2 - \lambda_1)(\lambda_2 - 1) \cdot b \cdot \left(\frac{e}{d}\right)'' + (\lambda_3 - \lambda_1)(\lambda_3 - 1) \cdot c \cdot \left(\frac{e}{d}\right)' = 0 \tag{2.8}$$

$$(\lambda_1 - 1) \cdot a \cdot \left(\frac{e}{d}\right)''' + (\lambda_2 - 1) \cdot b \cdot \left(\frac{e}{d}\right)'' + (\lambda_3 - 1) \cdot c \cdot \left(\frac{e}{d}\right)' = 0 \tag{2.9}$$

Since $y = \frac{e}{d}$ is also a solution for the eq.(2.1) it has to satisfy this one, from where we have the relation

$$\left(\frac{e}{d}\right)^{iv} + a \cdot \left(\frac{e}{d}\right)''' + b \cdot \left(\frac{e}{d}\right)'' + c \cdot \left(\frac{e}{d}\right)' = 0 \tag{2.10}$$

From (2.8) follows: if $\lambda_2 = \lambda_1$ then $\lambda_3 = 1$ (it means $c = c(x)$ is a periodic function) or $\lambda_3 = \lambda_1$.

Let $\lambda_2 = \lambda_1$ and $\lambda_3 = 1$. Then, from (2.9) and (2.10), we obtain the relations (2.5) and (2.6).

Let $\lambda_2 = \lambda_1$ and $\lambda_3 = \lambda_1$. Since $\lambda_1 \neq 1$, from (2.9) follows $a \cdot \left(\frac{e}{d}\right)''' + b \cdot \left(\frac{e}{d}\right)'' + c \cdot \left(\frac{e}{d}\right)' = 0$, but then, from (2.10), we obtain $\left(\frac{e}{d}\right)^{iv} = 0$, what means that the unique QPS for the eq.(2.1) is $y = 0$. □

Corrolary 2.1. *The equation $y^{iv} + c(x)y' + d(x)y = e(x)$ has the same QPS as the eq.(2.1).*

Proof. By the condition (2.5) follows that if $a(x) = 0$ then $b(x) = 0$, or vice versa, and $c(x)$ does not depend on $a(x)$ and $b(x)$. □

Example 2.1. The equation $y^{iv} + e^{2x}y''' + e^{2x}(\text{tg}x - 1)y'' + \frac{4\text{tg}x}{\text{tg}x + 1}y' + e^{-x}y = \sin x$ satisfies the conditions of the Theorem 2.2., so it has QPS $y = \frac{\sin x}{e^{-x}} = e^x \sin x$ with QP $\omega = 2\pi$ and QPC $\lambda = e^{2\pi}$. According to the Corollary 2.1. the equation $y^{iv} + \frac{4\text{tg}x}{\text{tg}x + 1}y' + e^{-x}y = \sin x$ also has QPS $y = e^x \sin x$.

Theorem 2.3. *Let the coefficients $a = a(x)$, $b = b(x)$, $c = c(x)$, $d = d(x)$, $e = e(x)$ for the eq.(2.1) be QPF with the same constant QP ϖ and QPC $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ respectively. Then*

1^o *If* $\lambda_1 = 1$, $\lambda_2 \neq 1$, $\lambda_3 = \lambda_2$, $\lambda_4 \neq 1$, $\lambda_4 \neq \lambda_2$ *then the eq.(2.1) has QPS* $y = \frac{e}{d}$ *when the relations*

$$b \cdot \left(\frac{e}{d}\right)'' + c \cdot \left(\frac{e}{d}\right)' = 0 \tag{2.11}$$

$$\left(\frac{e}{d}\right)^{iv} + a \cdot \left(\frac{e}{d}\right)''' = 0 \tag{2.12}$$

are satisfied.

2^o *If* $\lambda_1 = 1$, $\lambda_2 \neq \lambda_1$, $\lambda_3 = 1$, *then the eq.(2.1) has QPS* $y = \frac{e}{d}$ *when the relations*

$$\begin{cases} b \equiv 0 \\ \left(\frac{e}{d}\right)^{iv} + a \cdot \left(\frac{e}{d}\right)''' + c \cdot \left(\frac{e}{d}\right)' = 0 \end{cases} \tag{2.13}$$

are satisfied.

Proof. Let $y = y(x)$ be QPS for the eq.(2.1) with QP $\overline{\omega}$ and QPC $\lambda \neq \lambda_5$. If $\lambda_1 = 1$ (it means $a(x)$ is a periodic function), using the Theorem 2.1., we can reduce the eq. (2.1) to the equation of second order

$$(\lambda_2 - 1)by'' + (\lambda_3 - 1)cy' + (\lambda_4 - 1)dy = \frac{1}{\lambda}(\lambda_5 - \lambda)e.$$

If $\lambda_2 \neq 1$, we can write the last equation in the form

$$y'' + fy' + gy = h \tag{2.14}$$

where

$$f = \frac{\lambda_3 - 1}{\lambda_2 - 1} \cdot \frac{c}{b}, \quad g = \frac{\lambda_4 - 1}{\lambda_2 - 1} \cdot \frac{d}{b}, \quad h = \frac{\lambda_5 - \lambda}{\lambda(\lambda_2 - 1)} \cdot \frac{e}{b}.$$

Since f, g, h are QPF with QP $\overline{\omega}$ and QPC $\lambda_f = \frac{\lambda_3}{\lambda_2}$, $\lambda_g = \frac{\lambda_4}{\lambda_2}$, $\lambda_h = \frac{\lambda_5}{\lambda_2}$, the eq.(2.14) can be reduced to the equation [2]:

$$\left(\frac{\lambda_3}{\lambda_2} - 1\right) f \cdot y' + \left(\frac{\lambda_4}{\lambda_2} - 1\right) g \cdot y = \frac{1}{\lambda}\left(\frac{\lambda_5}{\lambda_2} - \lambda\right) \cdot h$$

i.e.

$$(\lambda_3 - \lambda_2)(\lambda_3 - 1)cy' + (\lambda_4 - \lambda_2)(\lambda_4 - 1)dy = \frac{1}{\lambda^2}(\lambda_5 - \lambda\lambda_2)(\lambda_5 - \lambda)e. \tag{2.15}$$

From (2.15) follows: if $(\lambda_3 - \lambda_2)(\lambda_3 - 1) = 0$ then $(\lambda_4 - \lambda_2)(\lambda_4 - 1)d \cdot y = \frac{1}{\lambda^2}(\lambda_5 - \lambda\lambda_2)(\lambda_5 - \lambda)e$. Since for $\lambda = \frac{\lambda_5}{\lambda_4}$ follows $\frac{(\lambda_5 - \lambda\lambda_2)(\lambda_5 - 1)}{\lambda^2(\lambda_4 - \lambda_2)(\lambda_4 - 1)} = 1$, we obtain the solution $y = \frac{e}{d}$. As $y = \frac{e}{d}$ is a solution for (2.14) and (2.1), the both of the relations

$$(\lambda_2 - 1) \cdot b \cdot \left(\frac{e}{d}\right)'' + (\lambda_3 - 1) \cdot c \cdot \left(\frac{e}{d}\right)' = 0, \tag{2.16}$$

$$\left(\frac{e}{d}\right)^{iv} + a \cdot \left(\frac{e}{d}\right)''' + b \cdot \left(\frac{e}{d}\right)'' + c \cdot \left(\frac{e}{d}\right)' = 0, \tag{2.17}$$

have to be satisfied. Now we have:
1^0 If $\lambda_2 = \lambda_3 \neq 1$ (i.e. b and c are QPF), then from (2.16) and (2.17) follows the relations (2.11) and (2.12).
2^0 If $\lambda_3 = 1$ (i.e. c is a periodic function), since $\lambda_2 \neq 1$, from (2.16) and (2.17) follows the relations (2.13). □

Corrolary 2.2. *The equation $y^{iv} + ay''' + d(x)y = e(x)$ has the same QPS as the eq.(2.1).*

Proof. By the condition (2.11) follows that if $b(x) = 0$ then $c(x) = 0$, or vice versa, and $a(x)$ does not depend on $b(x)$ and $c(x)$. □

Example 2.2. The equation

$$y^{iv} - \frac{2}{1+\mathrm{tg}x}y''' + e^x y'' + \frac{2e^x\mathrm{tg}x}{1-\mathrm{tg}x}y' + e^x y = e^{2x}\cos x$$

satisfies the conditions of the Theorem 2.3., so it has QPS $y = \dfrac{e^{2x}\cos x}{e^x} = e^x \cos x$ with QP $\omega = 2\pi$ and QPC $\lambda = \dfrac{e^{4\pi}}{e^{2\pi}} = e^{2\pi}$. According to the Corollary 2.2. the equation $y^{iv} - \dfrac{2}{1+\mathrm{tg}x}y''' + e^x y = e^{2x}\cos x$ also has QPS $y = e^x \cos x$.

Theorem 2.4. *Let the coefficients $a = a(x)$, $b = b(x)$, $c = c(x)$, $d = d(x)$, $e = e(x)$ for the eq.(2.1) be QPF with the same constant QP $\overline{\omega}$ and QPC λ_1, λ_2, λ_3, λ_4, λ_5 respectively such that $\lambda_1 = \lambda_2 = 1$ (i.e. $a(x)$ and $b(x)$ are periodic functions with a period $\overline{\omega}$) and $\lambda_3 \neq 1$. Then $y = \dfrac{e}{d}$ is QPS for eq.(2.1) if the relations*

$$\begin{cases} c \equiv 0 \\ \left(\dfrac{e}{d}\right)^{iv} + a\cdot\left(\dfrac{e}{d}\right)''' + b\cdot\left(\dfrac{e}{d}\right)'' = 0 \end{cases} \tag{2.18}$$

are satisfied.

Proof. It can be proved in a similar manner as the previous theorems, reducing the eq. (2.1) to the equation of first order and analyzing this one [1]. □

Example 2.3. The equation $y^{iv} + \sin x \cdot y''' + 2(\sin x - 2)y'' + e^x y = e^{-x}$ satisfies the conditions of the Theorem 2.4. and has QPS $y = e^{-2x}$.

Theorem 2.5. *Let the coefficients $a = a(x)$, $b = b(x)$, $c = c(x)$, $d = d(x)$, $e = e(x)$ for the eq.(2.1) be QPF with the same constant QP $\overline{\omega}$ and QPC λ_1, λ_2, λ_3, λ_4, λ_5 respectively, such thar $\lambda_1 = \lambda_2 = \lambda_3 = 1$ (i.e. $a(x)$, $b(x)$ and $c(x)$ are periodic functions with a period $\overline{\omega}$) and $\lambda_4 \neq 1$. Then $y = \dfrac{e}{d}$ is QPS for eq.(2.1) if the relation*

$$\left(\frac{e}{d}\right)^{iv} + a\cdot\left(\frac{e}{d}\right)''' + b\cdot\left(\frac{e}{d}\right)'' + c\cdot\left(\frac{e}{d}\right)' = 0$$

is satisfied.

Proof. It can be proved using Theorem 2.1. and reducing the eq.(2.1) to the algebraic equation of first order. □

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