Математички Билтен
Книга 9-10 (XXXV-XXXVI)
1985-1986 (5-10)
Скопје, Југославија

ON SIMILARITY AND QUASISIMILARITY OF UNILATERAL OPERATOR VALUED WEIGHTED SHIFTS[1)]

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In this paper we consider the problem of similarity and quasisimilarity of unilateral operator valued weighted shifts.

Troughout this paper a separable complex Hilbert space is denoted by H; (x,y) denotes the scalar product of the vectors x and y in H; by B(H) is denoted the algebra of bounded linear operators on H.

$H^{(1)}$ is a notation for the space of infinite sequences of vectors $(x_n)_{n=0}^{\infty}$, $x_n \in H$ such that

$$\sum_{n=0}^{\infty} \|x_n\|^2 < \infty$$

with a scalar product defined by

$$(x,y) = \sum_{n=0}^{\infty} (x_n, y_n)$$

where $x=(x_n)\in H^{(1)}$ and $y=(y_n)\in H^{(1)}$.

Let $(A_n)_{n=0}^{\infty}$ be a uniformly bounded sequence of bounded linear operators on H. The operator on $H^{(1)}=H\oplus H\oplus \dots$ given by

$$A(x_0,x_1,x_2,\dots) = (0,A_0x_0,A_1x_1,\dots)$$

is called the unilateral operator valued weighted shift with weights $(A_n)_{n=0}^{\infty}$.

Two operators S and T are called similar if there exists an invertible operator X such that SX=XT. A bounded operator X is quasi-invertible if X is injective and has a dense range (i. e. Ker X=KerX*={0}).

The bounded operators A and B are quasisimilar if there exist quasi-invertible operators X and Y such that AX=XB and YA=BY.

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1) This paper was communicated on the 8-th Congress of the Yugoslav Mathematical Society.

In the sequel we consider operator valued weighted shifts with invertible weights.

The following result is presented in [6].

Lemma. If A and B are unilateral operator valued weighted shifts with $(A_k)_{k=o}^{\infty}$, $(B_k)_{k=o}^{\infty}$ respectively and $X \in B(H^{(1)})$ with matrix $(X_{ij})_{i,j=o,1,\ldots,}$ then AX=XB if and only if

$$X_{ij} = \begin{cases} 0 & : i < j \\ A_{i-1} \ldots A_{i-j} X_{i-j,o} B_o^{-1} B_1^{-1} \ldots B_j^{-1} & : i \geq j \end{cases} \qquad (0)$$

Remark. The matrix (X_{ij}) is called a lower triangular if $X_{ij}=o$ for $i < j$.

We state the following theorem.

Theorem 1. Two unilateral operator valued weighted shifts with weights $(A_k)_{k=o}^{\infty}$, and $(B_k)_{k=o}^{\infty}$ respectively, are similar if and only if there exists an invertible operator X_o on H such that

$$\| A_{k-1} \ldots A_o X_o B_o^{-1} B_1^{-1} \ldots B_{k-1}^{-1} \| < \infty \qquad (1)$$

and

$$\| B_{k-1} \ldots B_o X_o^{-1} A_o^{-1} A_1^{-1} \ldots A_{k-1}^{-1} \| < \infty \qquad (2)$$

Proof. Let us assume that the operators A and B are similar. Then there exists an invertible operator X on $H^{(1)}$ such that AX=XB. According to the lemma we have that the operator X has a lower triangular matrix of the form

$$\begin{bmatrix} X_o & 0 & 0 & \bullet \\ X_1 & A_o X_o B_o^{-1} & 0 & \bullet \\ X_2 & A_1 X_1 B_1^{-1} & A_1 A_o X_o B_o^{-1} B_1^{-1} & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \qquad (3)$$

where for a convenience we set $X_o=X_{oo}$, $X_1=X_{1o}$, $X_i=X_{io}, \ldots$.

We claim that the operator X_o is invertible. By assumption the operator X is invertible and the equality AX=XB holds.

From this equality using the invertibility of X we obtain

$$X^{-1}A = BX^{-1}.$$

Applying the lemma to the operators X and X^{-1} we obtain that the operators X and X^{-1} are lower triangular. Therefore X and $(X^{-1})^*$ are upper triangular i.e.

$$X = \begin{bmatrix} X_o & X_1 & . & . & . \\ 0 & . & . & . & . \\ . & . & . & . & . \end{bmatrix}, \quad (X^{-1})^* = \begin{bmatrix} Y_o^* & Y_1^* & . & . & . \\ 0 & . & . & . & . \\ . & . & . & . & . \end{bmatrix}$$

The subspace $H_o = H\oplus O\oplus O\oplus O\oplus \ldots$ is invariant under the operators X^* and $(X^{-1})^*$. Let $X_o^* = X^*|H_o$ and $Y_o^* = (X^{-1})^*|H_o$.

Therefore we have

$$X_o^* Y_o^* = (X^*|H_o)(X^{-1})^*|H_o = X^*(X^{-1})^*|H_o = I|H_o.$$

We are going to prove the inequality (1). First we use the matrix form (3), for the operator X.

For $f_k \in H$, we denote

$$\hat{f}_k = O\oplus\ldots O\oplus f_k\oplus O\oplus\ldots$$

So, $\hat{f}_k$ is a vector with components all equal to zero except the one on the k-th position and let P_k be the orthogonal projection onto the subspace $H_k = O\oplus O\oplus O\oplus, \ldots O\oplus H_k\oplus O\oplus O\ldots$.

Using the triangular representation (3) we obtain

$$\|A_{k-1}\ldots A_1 A_o X_o B_1^{-1} B_1^{-1}\ldots B_{k-1}^{-1} f_k\| =$$
$$= \|P_k X \hat{f}_k\| \leq \|X \hat{f}_k\| \leq \|X\| \; \|\hat{f}_k\| = \|X\| \, \|f_k\|$$

Thus, we can conclude that

$$\|A_{k-1}\ldots A_o X_o B_o^{-1} B_1^{-1} .. B_{k-1}^{-1}\| \leq \|X\| .$$

If we apply the same technique to the operator equality $BX^{-1} = X^{-1}A$ we obtain the condition (2).

Suppose that the conditions (1) and (2) hold.

We will show that there exists an invertible operator X on $H^{(1)}$ such that AX=XB. Let $X = (X_i)_{i=o}^{\infty}$ be a diagonal operator with diagonal elements

$X_o = I$, $X_1 = A_o B_o^{-1}$, $X_2 = A_1 A_o B_o^{-1} B_1^{-1}$, $X_i = A_{i-1} A_{i-2}\ldots A_1 A_o B_o^{-1} B_1^{-1}\ldots B_{i-1}^{-1}$

let $f = (f_o, f_1, f_2, \ldots) \in H^{(1)}$. Then we have

$$X(f_o, f_1, f_2, \ldots) = (f_o, A_o B_o^{-1} f_1, A_1 A_o B_o^{-1} B_1^{-1} f_2, A_2 A_1 A_o B_o^{-1} B_1^{-1} B_2^{-1} f_3, \ldots)$$

and so

$$AXf = AX(f_o, f_1, f_2, f_3, \ldots) =$$
$$= (0, A_o f_o, A_1 A_o B_o^{-1} f_1, A_2 A_1 A_o B_o^{-1} B_1^{-1} f_2, A_3 A_2 A_1 A_o B_o^{-1} B_1^{-1} B_2^{-1} f_3, \ldots).$$

On the other hand we have

$$Bf = (0, B_o f_o, B_1 f_1, B_2 f_2, \ldots),$$

and so

$$XBf = (0, A_o B_o^{-1} B_o f_o, A_1 A_o B_o^{-1} B_1^{-1} B_1 f_1, A_2 A_1 A_o B_o^{-1} B_1^{-1} B_2^{-1} B_2 f_2, \ldots) =$$
$$= (0, A_o f_o, A_1 A_o B_o^{-1} f_1, A_2 A_1 A_o B_o^{-1} B_1^{-1} f_2, \ldots)$$

Thus we prove that the equality $AX=XB$ holds.

Conditions (1) and (2) imply that the operator X has an inverse X^{-1}. Moreover the operator X^{-1} is bounded. So X is invertible on H and a similarity of operators A and B is proved.

The answer to the question of quasisimilarity of two operator valued weighted shifts is much more difficult than the similarity. Below we give a sufficient condition for quasisimilarity with respect to the weights.

We look at the diagonal elements of a matrix (3) in the proof of theorem 1.

Theorem 2. Suppose that A and B are unilateral operator valued weighted shifts with invertible weights $(A)_{i=o}^{\infty}$ and $(B_i)_{i=o}^{\infty}$ and suppose that there exist operators X_o and Y_o which are quasi-invertible and such that

$$\| A_{k-1} \ldots A_1 A_o X_o B_o^{-1} B_1^{-1} \ldots B_{k-1}^{-1} \| < \infty \qquad (1)$$

and

$$\| B_{k-1} \ldots B_o Y_o A_o^{-1} \ldots A_{k-1}^{-1} \| < \infty \qquad (2)$$

Then the operators A and B are quasisimilar.

Proof. We define the operators X and Y as follows

$$X = \begin{bmatrix} X_o & 0 & 0 & 0 \\ \bullet & A_o X_o B_o^{-1} & 0 & \bullet \\ \bullet & \bullet & A_1 A_o X_o B_o^{-1} B_1^{-1} & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \qquad (3)$$

and

$$Y = \begin{bmatrix} Y_o & 0 & 0 & 0 \\ \bullet & B_oY_oA_o^{-1} & 0 & \bullet \\ \bullet & \bullet & B_1B_oY_oA_o^{-1} & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \tag{4}$$

The conditions (1), (2) imply that the operators X and Y are bounded linear operators on $H^{(1)}$.

The operators X and Y are one-to-one. Let $x,y \in H^{(1)}$ and $x \neq y$. Then there exists an index k such that $x_k \neq y_k$. As the operators $A_{k-1} \ldots A_oX_oB_o^{-1}B_1^{-1} \ldots B_{k-1}^{-1}$ and $B_{k-1} \ldots B_oY_oA_o^{-1} \ldots A_{k-1}^{-1}$ are one-to-one, we have $Xx \neq Xy$ and $Yx \neq Yy$.

The operators X and Y have dense ranges. Let $x=(x_o,x_1,\ldots,x_n,\ldots) \in H^{(1)}$ and $\varepsilon > 0$.

By the hypothesis the operators $A_{k-1} \ldots A_oX_oB_o^{-1} \ldots B_{k-1}^{-1}$, $k=0,1,2,\ldots$ have dense ranges in H. So for a $x_k \in H$, there exists $y_k \in H$ such that

$$\|A_{k-1} \ldots A_oX_oB_o^{-1} \ldots B_{k-1}^{-1}y_k - x_k\|^2 < \frac{\varepsilon}{2^{k+1}}$$

Therefore, for $y=(y_o,y_1,\ldots) \in H^{(1)}$ we have

$$\|Xy-x\|^2 = \Sigma\|A_{k-1} \ldots A_oX_oB_o^{-1} \ldots B_{k-1}^{-1}y_k - x_k\|^2 < \varepsilon \sum_{k=o}^{\infty} \frac{1}{2^{k+1}} = \varepsilon$$

In the case of semilarity it can be proved that the operator Y has a dense range in $H^{(1)}$. Now, it is easy to show that AX=XB and YA=BY.

Fialkow [1] gave a necessary and sufficient condition for similarity and quasisimilarity of bilateral scalar weighted shifts. The case of unilateral scalar weighted shifts had been resolved by Kelley (see Halmos [4], Problem 76.).

It was proved by Lambert that the notions of similarity and quasisimilarity coincide in the case of invertible unilateral scalar weighted shifts.

The theorems 1 and 2 are closely connected to the results which are presented in the paper [6].

R E F E R E N C E S

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СЛИЧНОСТ И КВАЗИСЛИЧНОСТ НА ЕДНОСТРАНИТЕ ОПЕРАТОРСКО-ТЕЖИНСКИ ШИФТОВИ

Новак Ивановски и Марија Оровчанец

Р е з и м е

Во овој труд се разгледува проблемот за сличност и квазисличност на едностраните тежински шифтови чии тежини се инвертибилни оператори.