Математички Билтен ISSN 0351-336X (print)
37(LXIII) No. 1 ISSN 1857-9914 (online)
2013(49-60) UDC: 512.643.9
Скопје, Македонија

DRAZIN'S PSEUDO-INVERSE OF RIGHT ANGLE SINGULAR MATRIX

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Abstract. Throughout theoretical investigation that is done in this paper for Drazin's pseudo inverse in associative ring we have build such an inverse in specific ring. Upper right angle matrices are singular matrices, their determinant is zero which means they don't have inverse matrices. Properties of Drazin's pseudo inverse enable us to find so called pseudo inverse of those matrices. In this paper we have construct Drazin's pseudo inverse for singular matrices in real square matrix ring. We will build the construction according to definition of pseudo-inverse given by Drain. In case when elements of any ring R are non-singular then the pseudo-inverse of Drazin becomes inverse that satisfies the condition $\forall a \in R$, $\exists a^{-1} \in R$, $a \cdot a^{-1} = e$, where e is identity element in R.

1. Introduction

Penrose in [6, $Theorem$ 1] gives the definition for generalized inverse and their main properties [6, $Proposition$ 1]. In Draizin's paper [4]is given the definition for pseudo-inverse of an element in any given associative ring. In this paper we will adjust the definition for a concrete matrix ring called upper right matrix (we name in this way in order to make the difference from rectangular matrix). An element a from any algebra $\mathcal{A}$ is called relatively regular, if there exists $x \in \mathcal{A}$ such that $axa = a$. If a is relatively regular element, then it has a generalized inverse, which is an element $b \in \mathcal{A}$ that satisfies the equations $aba = a \; and \; bab = b$ (see [5]). If b is generalized inverse of a, then a is generalized inverse of b.

2. The construction of drazin's pseudo inverse of upper right angle singular matrix

Definition 1. *Matrices* $[a_{ij}]_{n\times n}$, *where* $a_{ij} = 0$, $a_{nn} \neq 0$, $a_{i,j-1} = 0$, *for* $2 \leq i$, $j \leq n$, *we will call as right angle matrices of* $n - th \, order$.

Definition 2. *A matrix* $X \in \mathcal{A}$ *is a Drazin's pseudo inverse of* $A \in \mathcal{A}$ *if and only if the following conditions are satisfied*

(1) $A^2X = A$
(2) $XAX = X$
(3) $AX = XA$,

$\forall A \in \mathcal{A}$. The matrix X we will call as Drazin's pseudo-inverse matrix of A from $\mathcal{A}$.

In progress, we get the form of matrix
From $A^2 \cdot X = A$, while comparing the elements in respective positions, except the position of a_{1n}, we get

$$\begin{aligned} x_{11} &= \frac{1}{a_{11}},\ x_{nn} = \frac{1}{a_{nn}},\ x_{i,j-1} = \frac{a_{1,j-1}}{a_{11}^2}, \quad and \\ x_{j,n} &= \frac{a_{j,n}}{a_{nn}^2}, \quad j = 2, 3, 4, \ldots, n-1. \end{aligned} \tag{1}$$

As we see from 3) X should be commutative with any $A \in \mathcal{A}$, and $AX = XA$, equating the elements in l_{1n} position in both sides, we get

$$a_{11}x_{1n} + a_{12}x_{2n} + \ldots + a_{1n}x_{nn} = a_{1n}x_{11} + a_{2n}x_{12} + \ldots + a_{nn}x_{1n} \tag{2}$$

Having into consideration conditions (0.1) *and* (0.2), for x_{1n} we get

$$x_{1n} = \frac{\left(a_{1n}\frac{1}{a_{11}} + a_{2n}\frac{a_{12}}{a_{11}^2} + L + a_{n-1,n}\frac{a_{1,n-1}}{a_{11}^2} - a_{1,2}\frac{a_{2n}}{a_{nn}^2} - a_{13}\frac{a_{3n}}{a_{nn}^2} - L - a_{1n}\frac{1}{a_{nn}}\right)}{a_{11} - a_{nn}}.$$

From which we obtain

$$x_{1n} = \frac{\left(a_{1n}a_{11} + a_{2n}a_{12} + L + a_{n-1,n}a_{1,n-1}\right)a_{nn}^2 - \left(a_{12}a_{2n} + a_{13}a_{3n} + L + a_{1n}a_{nn}\right)a_{11}^2}{a_{11}^2 a_{nn}^2\left(a_{11} - a_{nn}\right)}$$

or

$$x_{1n} = \frac{a_{nn}^2 \sum\limits_{i=1}^{n-1} a_{1i}a_{in} - a_{11}^2 \sum\limits_{j=2}^{n} a_{1j}a_{jn}}{a_{11}^2 a_{nn}^2\left(a_{11} - a_{nn}\right)} \tag{3}$$

Proposition 1. *For the upper right-angle matrix*

$$A = \begin{pmatrix} a_1 & a_2 & \ldots & a_{1n} \\ 0 & 0 & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{nn} \end{pmatrix},$$

the matrix

$$X = \begin{pmatrix} \frac{1}{a_{11}} & \frac{a_{12}}{a_{11}^2} & \dots & \frac{a_{nn}^2 \sum_{i=1}^{n-1} a_{1i}a_{in} - a_{11}^2 \sum_{j=2}^{n} a_{1j}a_{jn}}{a_{11}^2 a_{nn}^2 (a_{11}-a_{nn})} \\ 0 & 0 & \dots & \frac{a_{2n}}{a_{nn}^2} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{a_{nn}} \end{pmatrix}$$

is Drazin's pseudo inverse matrix for singular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

for $a_{11} \neq 0$, $a_{nn} \neq 0$ *and* $a_{11} \neq a_{nn}$.

Proof. To prove this proposition it is necessary and it suffices to prove whether three conditions of definition hold. □

(1) Let us prove the first condition $A^2 \cdot X = A$

$$A^2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \vdots & a_{1n-2} & a_{1n-1} & a_{1n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{2n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{2n} \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{n-2n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{n-1n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{nn} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11}^2 & a_{11}a_{12} & \vdots & a_{11}a_{1n-1} & T_2 \\ 0 & 0 & \vdots & 0 & a_{2n}a_{nn} \\ \dots & \dots & \vdots & \dots & \dots \\ 0 & 0 & \vdots & 0 & a_{n-1n}a_{nn} \\ 0 & 0 & \vdots & 0 & a_{nn}^2 \end{pmatrix},$$

where

$$T_2 = \sum_{j=1}^{n} a_{1j}a_{jn}.$$

So

$$A^2X = \begin{pmatrix} a_{11}^2 & a_{11}a_{12} & \vdots & a_{11}a_{1n-1} & T_2 \\ 0 & 0 & \vdots & 0 & a_{2n}a_{nn} \\ \dots & \dots & \vdots & \dots & \dots \\ 0 & 0 & \vdots & 0 & a_{n-1n}a_{nn} \\ 0 & 0 & \vdots & 0 & a_{nn}^2 \end{pmatrix}.$$

$$\cdot \begin{pmatrix} \frac{1}{a_{11}} & \frac{a_{12}}{a_{11}^2} & \vdots & \frac{a_{nn}^2\sum_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2(a_{11}-a_{nn})} \\ 0 & 0 & \vdots & \frac{a_{2n}}{a_{nn}^2} \\ \dots & \dots & \vdots & \dots \\ 0 & 0 & \vdots & \frac{1}{a_{nn}} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & \vdots & a_{1n-2} & a_{1n-1} & T_1 \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{2n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{3n} \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{n-2n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{n-1n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{nn} \end{pmatrix}$$

where

$$\begin{aligned} T_1 &= a_{11}^2\frac{a_{nn}^2\sum_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)}+a_{11}a_{12}\frac{a_{2n}}{a_{nn}^2}+a_{11}a_{13}\frac{a_{3n}}{a_{nn}^2}+L+ \\ &+ a_{11}a_{1n-1}\frac{a_{n-1n}}{a_{nn}^2}+T_2\frac{1}{a_{nn}}=a_{1n}. \end{aligned}$$

So,

$$\frac{a_{nn}^2\sum_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)}+\frac{a_{11}}{a_{nn}^2}\sum_{j=2}^{n-1}a_{1j}a_{jn}+\frac{1}{a_{nn}}\sum_{j=1}^{n}a_1a_{jn}=a_{1n}.$$

We multiply both sides of the last equation with $a_{nn}^2(a_{11}-a_{nn})$ and we get

$$a_{nn}^2\sum_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum_{j=2}^{n}a_{1j}a_{jn}+a_{11}(a_{11}-a_{nn})\sum_{j=2}^{n-1}a_{1j}a_{jn}+$$

$$+a_{nn}(a_{11}-a_{nn})\sum_{j=1}^{n}a_{1j}a_{jn}=a_{1n}a_{nn}^2(a_{11}-a_{nn}).$$

After all, we obtain $a_{1n}a_{nn}^2(a_{11}-a_{nn})=a_{1n}a_{nn}^2(a_{11}-a_{nn})$. Which proves $A^2\cdot X=A$.

(2) Let us prove the second condition $XAX=X$

$$XAX=\begin{pmatrix} \frac{1}{a_{11}} & \frac{a_{12}}{a_{11}^2} & \vdots & \frac{a_{nn}^2\sum\limits_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2(a_{11}-a_{nn})} \\ 0 & 0 & \vdots & \frac{a_{2n}}{a_{nn}^2} \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & \frac{1}{a_{nn}} \end{pmatrix}\cdot$$

$$\cdot\begin{pmatrix} a_{11} & a_{12} & a_{13} & \vdots & a_{1n-2} & a_{1n-1} & a_{1n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{2n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{3n} \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{n-2n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{n-1n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{nn} \end{pmatrix}\cdot$$

$$\cdot\begin{pmatrix} \frac{1}{a_{11}} & \frac{a_{12}}{a_{11}^2} & \vdots & \frac{a_{nn}^2\sum\limits_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2(a_{11}-a_{nn})} \\ 0 & 0 & \vdots & \frac{a_{2n}}{a_{nn}^2} \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & \frac{1}{a_{nn}} \end{pmatrix}=$$

$$\begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n-1}}{a_{11}} & \sum\limits_{j=1}^{n-1}\frac{a_{1j}a_{jn}}{a_{11}^2}+\frac{\sum\limits_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2(a_{11}-a_{nn})}\cdot a_{nn} \\ 0 & 0 & 0 & \cdots & 0 & \frac{a_{2n}}{a_{nn}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{a_{3n}}{a_{nn}} \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}\cdot$$

$$\cdot\begin{pmatrix} \frac{1}{a_{11}} & \frac{a_{12}}{a_{11}^2} & \vdots & \frac{a_{nn}^2\sum\limits_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2(a_{11}-a_{nn})} \\ 0 & 0 & \vdots & \frac{a_{2n}}{a_{nn}^2} \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & \frac{1}{a_{nn}} \end{pmatrix} =$$

Finally, we obtain

$$= \begin{pmatrix} \frac{1}{a_{11}} & \frac{a_{2n}}{a_{11}^2} & \cdots & L \\ 0 & 0 & \cdots & \frac{a_{2n}}{a_{nn}^2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{pmatrix},$$

where

$$\begin{aligned} L &= \frac{a_{nn}^2\sum\limits_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)}+\frac{1}{a_{11}^2a_{nn}^2}\sum_{j=2}^{n-1}a_{1j}a_{jn}+ \\ &+ \left(\sum_{j=2}^{n-1}\frac{a_{1j}a_{jn}}{a_{11}^2}+\frac{a_{nn}^2\sum\limits_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)}\cdot a_{nn}\right)\frac{1}{a_{nn}}. \end{aligned}$$

For $L=\frac{a_{nn}^2\sum\limits_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2(a_{11}-a_{nn})}$ this matrix will be equal with X. Therefore we should prove that

$$\frac{a_{nn}^2\sum\limits_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)}+\frac{1}{a_{11}^2a_{nn}^2}\sum_{j=2}^{n-1}a_{1j}a_{jn}+\frac{1}{a_{11}^2a_{nn}}\sum_{j=2}^{n-1}a_{1j}a_{jn}=0.$$

Hence we obtain

$$\frac{1}{a_{11}a_{nn}^2}\sum_{j=2}^{n-1}a_{1j}a_{jn}+\frac{1}{a_{11}^2a_{nn}}\sum_{j=2}^{n-1}a_{1j}a_{jn}=\frac{a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}-a_{nn}^2\sum\limits_{i=1}^{n-1}a_{1i}a_{in}}{a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)}.$$

Multiplying both sides of the equation with $a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)$, we get

$$a_{11}\left(a_{11}-a_{nn}\right)\sum_{j=2}^{n-1}a_{1j}a_{jn}+a_{nn}\left(a_{11}-a_{nn}\right)\sum_{j=1}^{n-1}a_{1j}a_{jn}=$$

$$=a_{11}^2\sum_{j=1}^{n}a_{1j}a_{jn}-a_{nn}^2\sum_{i=1}^{n-1}a_{1i}a_{in}$$

$$a_{11}^2\sum_{j=2}^{n-1}a_{1j}a_{jn}-a_{11}a_{nn}\sum_{j=2}^{n-1}a_{1j}a_{jn}+a_{11}a_{nn}\sum_{j=1}^{n-1}a_{1j}a_{jn}-a_{nn}^2\sum_{j=1}^{n-1}a_{1j}a_{jn}=$$

$$=a_{11}^2\sum_{j=1}^{n}a_{1j}a_{jn}-a_{nn}^2\sum_{i=1}^{n-1}a_{1i}a_{in}$$

$$a_{11}^2\sum_{j=2}^{n-1}a_{1j}a_{jn}-a_{11}a_{nn}\sum_{j=2}^{n-1}a_{1j}a_{jn}+a_{11}a_{nn}\sum_{j=1}^{n-1}a_{1j}a_{jn}-a_{nn}^2\sum_{j=1}^{n-1}a_{1j}a_{jn}=$$

$$=a_{11}^2\sum_{j=1}^{n}a_{1j}a_{jn}+a_{11}^2a_{1n}a_{nn}-a_{nn}^2\sum_{i=1}^{n-1}a_{1i}a_{in}.$$

After possible cancellations, we obtain

$$a_{11}^2a_{1n}a_{nn}=a_{11}^2a_{1n}a_{nn}.$$

Which yields the proof of second condition.

(3) Let us prove the third condition $AX=XA$

$$\begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n-1}}{a_{11}} & \frac{a_{nn}^2\sum\limits_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2(a_{11}-a_{nn})}\cdot a_{11}+\sum\limits_{j=2}^{n-1}\frac{a_{1j}a_{jn}}{a_{nn}^2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{a_{2n}}{a_{nn}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{a_{3n}}{a_{nn}} \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}=$$

$$= \begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n-1}}{a_{11}} & \sum\limits_{j=1}^{n-1}\frac{a_{1j}a_{jn}}{a_{11}^2} + \frac{\sum\limits_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2(a_{11}-a_{nn})}\cdot a_{nn} \\ 0 & 0 & 0 & \cdots & 0 & \frac{a_{2n}}{a_{nn}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{a_{3n}}{a_{nn}} \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

for equality of those matrices, we need

$$\frac{a_{nn}^2\sum\limits_{i=1}^{n-1}a_{1i}a_{in} - a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)}\cdot a_{11} + \sum_{j=2}^{n-1}\frac{a_{1j}a_{jn}}{a_{nn}^2} =$$

$$= \sum_{j=1}^{n-1}\frac{a_{1j}a_{jn}}{a_{11}^2} + \frac{\sum\limits_{i=1}^{n-1}a_{1i}a_{in} - a_{11}^2\sum\limits_{j=2}^{n}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)}\cdot a_{nn},$$

if we focus on last equation, we can notice that it has the following form

$$a_{11}L + \frac{1}{a_{nn}^2}\sum_{j=2}^{n-1}a_{ij}a_{jn} = a_{nn}L + \frac{1}{a_{11}^2}\sum_{j=1}^{n-1}a_{ij}a_{jn}.$$

From which we obtain

$$\left(a_{11}-a_{nn}\right)L = \frac{1}{a_{11}^2}\sum_{j=1}^{n-1}a_{1j}a_{jn} - \frac{1}{a_{nn}^2}\sum_{j=1}^{n-1}a_{1j}a_{jn}.$$

Consequently,

$$\left(a_{11}-a_{nn}\right)L = \frac{\sum\limits_{j=1}^{n-1}a_{1j}a_{jn} - \sum^{n-1}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2},$$

hence,

$$L = \frac{\sum\limits_{j=1}^{n-1}a_{1j}a_{jn} - \sum^{n-1}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)}$$

which proves the third condition.

Proposition 2. *For any matrix $A \in \mathcal{A}$ and its Drazin's pseudo inverse X, $AXA = A$ holds.*

Proof. Let us take the multiplication of matrices

$$AXA = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \vdots & a_{1n-2} & a_{1n-1} & a_{1n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{2n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{3n} \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{n-2n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{n-1n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{nn} \end{pmatrix}.$$

$$\cdot \begin{pmatrix} \frac{1}{a_{11}} & \frac{a_{12}}{a_{11}^2} & \vdots & \frac{a_{nn}^2 \sum\limits_{i=1}^{n-1} a_{1i}a_{in} - a_{11}^2 \sum\limits_{j=2}^{n} a_{1j}a_{jn}}{a_{11}^2 a_{nn}^2 (a_{11}-a_{nn})} \\ 0 & 0 & \vdots & \frac{a_{2n}}{a_{nn}^2} \\ \dots & \dots & \vdots & \dots \\ 0 & 0 & \vdots & \frac{1}{a_{nn}} \end{pmatrix}.$$

$$\cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} & \vdots & a_{1n-2} & a_{1n-1} & a_{1n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{2n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{3n} \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{n-2n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{n-1n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{nn} \end{pmatrix}.$$

After the calculations, we obtain

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \vdots & a_{1n-2} & a_{1n-1} & L \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{2n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{3n} \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{n-2n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{n-1n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & a_{nn} \end{pmatrix}$$

where

$$L = a_{1n}+\frac{1}{a_{11}}\sum_{j=2}^{n-1}a_{1j}a_{jn}+\left(\frac{a_{nn}^2\sum\limits_{j=1}^{n-1}a_{1j}a_{jn}-a_{11}^2\sum\limits_{j=1}^{n-1}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)}\cdot a_{11}+\sum_{j=2}^{n-1}\frac{a_{1j}a_{jn}}{a_{nn}^2}\right)\cdot a_{nn}.$$

We need L to be a_{1n}. Therefore we should prove that

$$\frac{1}{a_{11}}\sum_{j=2}^{n-1}a_{1j}a_{jn}+\left(\frac{a_{nn}^2\sum\limits_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum\limits_{j=1}^{n-1}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)}\cdot a_{11}+\sum_{j=2}^{n-1}\frac{a_{1j}a_{jn}}{a_{nn}^2}\right)\cdot a_{nn}=0,$$

more precisely

$$\frac{1}{a_{11}}\sum_{j=2}^{n-1}a_{1j}a_{jn}+\frac{a_{nn}^2\sum\limits_{i=1}^{n-1}a_{1i}a_{in}-a_{11}^2\sum\limits_{j=1}^{n-1}a_{1j}a_{jn}}{a_{11}^2a_{nn}^2\left(a_{11}-a_{nn}\right)}+\sum_{j=2}^{n-1}\frac{a_{1j}a_{jn}}{a_{nn}^2}=0.$$

Multiplying both sides of the last equation with $a_{11}a_{nn}\left(a_{11}-a_{nn}\right)$we obtain

$$a_{nn}\left(a_{11}-a_{nn}\right)\sum_{j=2}^{n-1}a_{1j}a_{jn}+a_{11}\left(a_{11}-a_{nn}\right)\sum_{i=1}^{n-1}a_{1i}a_{in}.$$

After all possible cancellations, we obtain $a_{1n}=a_{1n}$. Hence, Proposition 4 holds. □

Let us demonstrate our construction with an example.

Example 1. *Let*

$$A=\begin{pmatrix}4 & 7 & 8\\ 0 & 0 & 6\\ 0 & 0 & 0\end{pmatrix}$$

be given. To determine the Drazin's pseudo inverse matrix, we denote as

$$X=\begin{pmatrix}x_{11} & x_{12} & x_{13}\\ 0 & 0 & x_{23}\\ 0 & 0 & x_{33}\end{pmatrix}$$

the pseudo-inverse matrix. According to (0.1), $x_{11}=\frac{1}{4}$, $x_{33}=\frac{1}{3}$, $x_{12}=\frac{7}{16}$, $x_{23}=\frac{2}{3}$, x_{13} *we determine from* (0.3) *relation, precisely from*

$$4x_{13}+7\cdot\frac{2}{3}+8\cdot\frac{1}{3}=8\cdot\frac{1}{4}+6\cdot\frac{7}{16}+3x_{13},$$

we find $x_{13} = \frac{-65}{24}$. *Therefore, the pseudo-inverse matrix is*

$$X = \begin{pmatrix} \frac{1}{4} & \frac{7}{16} & \frac{-65}{24} \\ 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Easily conditions $1), 2), 3)$ *of* $Definition\, 2$ *can be verified, so* X *is Drazin's pseudo-inverse for given matrix* A, *also the example satisfies the property of* $Proposition\, 4$.

References

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ДРАЗИНОВ ПСЕВДО-ИНВЕРЗНА НА ДЕСНОАГОЛНА СИНГУЛАРНА МАТРИЦА

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Р е з и м е

Низ теоретските испитувања кои се направени во овој труд за Дразинова псевдоинверзија во асоцијативен прстен, градиме инверзија во специјален прстен. Горно аголните матрици се сингуларни матрици, чија детерминанта е нула, што значи тие немаат инверзна матрица. Својствата на Дразиновата псевдо-инверзија овазможуваат да се најде таканаречена псевдо-инверзна на тие матрици. Во овој труд конструираме таква псевдо-инверзија на сингуларните матрици во прстенот од реални квадратни матрици. Конструкцијата ќе ја изведеме според дефиницијата на псевдо-инверзија дадена од Дразин. Во случајот кога елементите од секој прстен R се несингуларни псевдо-инверзната на Дразин станува инверзна која го исполнува условот $\forall a \in R$, $\exists a^{-1} \in R$, $a{\cdot}a^{-1} = e$, каде e е единичен елемент во R.

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