EQUIVALENCE OF DIFFERENT DEFFINITIONS OF SPACE OF ENDS

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Abstract. In this paper it is proved that the previous description of the space of ends of a connected, locally conected, localy compact, metric space which uses admissible sequences, concides with the description which uses the inverse limit of the components of complements of compacta.

key words: Metric space, locally connected, locally compact, component, quasicomponent, admissible sequence, set of ends, space of ends

The notion of an end of a space appeared in the papers of Freudenthal at the end of the first half of twentieth century. This notion is a very important tool in study of non-compact spaces.

In [2] is given a description of the space of ends of a topological space by use of admissible sequences. This description is equivalent to the general notion in the case when the topological space is locally compact separable metric spaceand its space of quasicomponents is compact ([2]).

If X is connected, the set and the space of ends can be defined by use of inverse limits,. This approach has advantages based on the wide application of inverse limits in topology.

In the literature there is no proof of the equivalence of these two descriptions of the space of ends. In this paper we give strict proof that these two definitions coincide.

Let X be a connected locally connected locally compact metric space. Then X is separable (theorem 7.3, p.241 [4]). If C is a compact set in X, we denote by $S(X\backslash C)$ the set of components of $X\backslash C$. The set of ends of X is

$$\varprojlim_{C} S(X\backslash C)$$

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where the inverse limit is taken over all compact subsets of X. There is a cofinal sequence of compacta in X, $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_n \subseteq \ldots$, i.e. for any compact C, $C \subseteq X$, there exists compact C_n, such that $C \subseteq C_n$ [8]. It is a known fact from the theory of inverse limits that in this case:

$$\varprojlim_{C} S(X \backslash C) = \varprojlim_{n} S(X \backslash C_n).$$

So, the set of ends of X consists of all sequences of components

$$(S_1, S_2, \ldots S_n, \ldots) \in \prod_{n=1}^{\infty} S(X \backslash C_n)$$

such that $S_1 \supseteq S_2 \supseteq \ldots \supseteq S_n \supseteq \ldots$.

By [8], the sequence $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_n \subseteq \ldots$ can be chosen so that $S(X \backslash C_n)$ is finite.

As in [8], we put discrete topology on each $S(X \backslash C_n)$ and conclude that the space of ends

$$\varprojlim_{C} S(X \backslash C).$$

is compact metric space.

Now we repeat the definition of the set of ends EX of X given in [2] for the case when X is locally connected separable metric space and its space of quasi components is compact.

Definition: Let X be a topological space and $Y \subseteq X$. Y **separates** X if and only if $X \backslash Y$ is not connected.

Definition: A sequence (a_n) of points of X is **admissible in** X provided that:

(1) no subsequence of (a_n) converges to a point of X;

(2) no compact subset of X separates (in X) two subsequences of (a_n).

Clearly, the condition (2) is equivalent to the condition: there is no compact set C in X, such that $X \backslash C = A \cup B$, A, B are open-closed disjoint sets in $X \backslash C$ and there exist subsequences (a'_n) $(a"_n)$ of (a_n), such that (a'_n) is in A, and $(a"_n)$ in B. Let $\mathcal{A}_X$ be the set of all admissible sequences in X. We define a relation "$\sim$" in $\mathcal{A}_X$ on this way: Let (a_n), $(b_n) \in \mathcal{A}_X$. $(a_n) \sim (b_n)$ if and only if no compact subset of X, separates an infinite subsequence of (a_n) from an infinite sequence of (b_n). Clearly, $(a_n) \sim (b_n)$ holds if and only if there is an admissible

sequence (c_n), such that each of (a_n) and (b_n) is a subsequence of (c_n).
Proposition 1 The relation "$\sim$" is an equivalence relation on $\mathcal{A}_X$.

Proof: Reflexivity and symetry are clear. We prove transitivity: Let $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$ and suppose that $(a_n) \not\sim (c_n)$. It means that there exists a compact set C in X, such that $X \backslash C = A \cup B$, where A, B are open-closed disjoint sets in $X \backslash C$ and there exists a subsequence (a'_n) of (a_n) in A and a subsequence (c'_n) of (c_n) in B. Because of $(a_n) \sim (b_n)$, it follows that at most a finite number of members of a sequence (b_n) are in B, which is not true because of the condition $(b_n) \sim (c_n)$. So, $(a_n) \sim (c_n)$.

For each sequence $(a_n) \in \mathcal{A}_X$, $[(a_n)]$ will denote the equivalence class containing (a_n).

Defintion: A sequence (a_m) of points of X is **eventually in** $A \subseteq X$, if and only if a finite number of members of the sequence α are outside of A, i.e. if there exists $m_0 \in \mathbf{N}$, such that for all $m \geq m_0$, $a_m \in A$.

We define the set of ends of the space X, to be the set $EX = \{[(a_n)] \,|(a_n) \in \mathcal{A}_X\}$.

Let $\mathcal{B} = \{G|G \subseteq X$ open in X and ∂G is compact $\}.ForeachG$ $\in \mathcal{B}$, let $G^* = \{[(a_m)] \in EX\,|(a_m)$ is eventually in $G\}$ and let $\mathcal{B}^* = \{G^*|G \in \mathcal{B}\}$.

If G_1, $G_2 \in \mathcal{B}$, then

$$(G_1 \cap G_2)^* = \{[(a_m)] \in EX\,|(a_m) \text{ eventually in } G_1 \cap G_2\} =$$

$$= \{[(a_m)] \in EX\,|(a_m) \text{ eventually in } G_1\} \cap \{[(a_m)] \in EX\,|(a_m)$$

$$\text{eventually } in\ G_2\} = G_1^* \cap G_2^*$$

and hence $\mathcal{B}^*$ is a basis for a topology on the set EX.
Proposition 2 EX is T_2-space.

Proof: Let $[(a'_n)]\,,\, [(a"_n)] \in EX$ and $[(a'_n)] \neq [(a"_n)]$. It means that $(a'_n) \not\sim (a"_n)$, i.e. there exists compact set C, such that $X \backslash C = A \cup B$, where A, B are open-closed disjoint set in $X \backslash C$ and there exist a subsequece (b'_n) of (a'_n) in A and a subsequence $(b"_n)$ of $(a"_n)$ in B. Since $X = X^o \cup \partial X \cup ExtX$ and the union is disjoint, and A, B are disjoint open sets, it follows that $\partial A \subseteq C$. So ∂A is compact. By analogy ∂B is compact, so A, $B \in \mathcal{B}$, and $(a'_n) \in A^*, (a"_n) \in B^*$.

Now we will prove that $A^* \cap B^* = \emptyset$. Suppose that there exists $[(a_n)] \in A^* \cap B^*$. Then (a_n) is eventually in A and in B, which is not possible. Hence EX is a T_2 - space.

Theorem 1 Let X be a connected, locally connected, locally compact metric space. There exists a homeomorphism between the topological space $\varprojlim_{n} S\left(X \backslash C_n\right)$ and EX, i.e. the above two definitions of the space of ends are coincide.

Proof: We define a function

$$f : \varprojlim_{n} S\left(X \backslash C_n\right) \rightarrow \{[(a_n)] \,|(a_n) \in \mathcal{A}_X\}$$

in this way: $f\left(S_1, S_2, \ldots\right) = [(a_n)]$, if (a_n) is eventually in S_i, for every $i \in \mathbf{N}$. We will show that f is homeomorphism. Because the space $\varprojlim_{n} S\left(X \backslash C_n\right)$ is compact [7], and $\{[(a_n)] \,|(a_n) \in \mathcal{A}_X\}$ is T_2 - space it is sufficient to show that f is continuous biection.

1) The function is well defined: Let $f\left(S_1, S_2, \ldots\right) = [(a_n)]$ and let $f\left(S_1, S_2, \ldots\right) = [(b_n)]$. Then (a_n) is eventually in S_i, for every $i \in \mathbf{N}$ and (b_n) is eventually in S_i, for every $i \in \mathbf{N}$), i.e. $a_n)$, (b_n) are eventually in S_i, for every $i \in \mathbf{N}$.

Suppose that $(a_n) \not\sim (b_n)$. It means that there exists a compact subset C of X, that separates a subsequence of (a_n) from a subsequence of (b_n). Because of the cofinality of $(C_n | n \in \mathbf{N})$, it follows that there exists $n_0 \in \mathbf{N}$, such that $C \subseteq C_{n_0}$. Hence C_{n_0} separate a subsequence of (a_n) from a subsequence of (b_n), which is contradiction with the condition: (a_n), (b_n) are eventually in the component S_{n_0} of $X \backslash C_{n_0}$. It follows that $(a_n) \sim (b_n)$, i.e. $[(a_n)] = [(b_n)]$.

2) f is injective: Let $f\left(S_1, S_2, \ldots\right) = f\left(L_1, L_2, \ldots\right) = [(a_n)]$. Then (a_n) is eventually in S_i, for every $i \in \mathbf{N}$ and (a_n) is eventually in L_i, for every $i \in \mathbf{N}$, i.e. (a_n) is eventually in $S_i \cap L_i$ for every $i \in \mathbf{N}$. Because S_i and L_i are components of $X \backslash C_i$, for every $i \in \mathbf{N}$,and $S_i \cap L_i \neq 0$ it follows that $S_i = L_i$, for every $i \in \mathbf{N}$. Hence $(S_1, S_2, \ldots) = (L_1, L_2, \ldots)$.

3) f is surjective: Let $[(a_n)] \in \{[\alpha] \,|(a_n) \in \mathcal{A}_X\}$ and let $n \in \mathbf{N}$. Since the compact set C_n contains at most finite members of the sequence (a_n), it follows that (a_n) is eventually in $X \backslash C_n$.

We will show that (a_n) is eventually in some component S^j of $X \backslash C_n$. Suppose this is not true. Let the components of $X \backslash C_n$ be $\{S_1, S_2, \ldots, S_k\}$. There exists a component S^j of $X \backslash C_n$ which contains a subsequence (a'_n) of (a_n) and there is another subsequence $(a"_n)$ of (a_n) contained in $\cup_{i=1, i \neq j}^{k} S^j$.Since X is locally connected, then $X \backslash C_n$ is locally connected and it follows that each component of $X \backslash C_n$ is open. Specially,S^j is open and $\cup_{i=1, i \neq j}^{k} S^j$ is open. We obtain that C_n separates X in two open subsets S^j and $\cup_{i=1,i\neq j}^{k} S^j$ and the sequence (a_n) contains subsequences in S^j and in $\cup_{i=1,i\neq j}^{k} S^j$. It follows that (a_n) is not admissible, which is a contradiction.

4) f is continuous: Let $f(S_1, S_2, \ldots) = [(a_n)]$. If G is a open set such that ∂G is compact and such that (a_n) is eventually in G, then the set $W = \{[(b_n)] \in EX \,|\, (b_n) \text{ eventually in } G\}$ is a basic neighbourhood of $[(a_n)]$. Moreover all basic neighbourhoods of $[(a_n)]$ are of this type.

We will show that there exists $n_0 \in \mathbf{N}$, such that $S_{n_0} \subseteq G$.

Really, because the sequence of compacta (C_n) is cofinal, it follows that there exists $n_0 \in \mathbf{N}$ such that $C_{n_0} \supseteq \partial G$, i.e. $X \backslash C_{n_0} \subseteq X \backslash \partial G = G \cup extG$. Then for the component S_{n_0} of $X \backslash C_{n_0}$ one has $S_{n_0} \subseteq G \cup extG$, i.e. $S_{n_0} = (G \cap S_{n_0}) \cup (extG \cap S_{n_0})$. Since S_{n_0} is connected, from the last equation and $G \cap S_{n_0} \neq \emptyset$, it follows that $S_{n_0} \subseteq G \cap S_{n_0}$. We obtain $S_{n_0} \subseteq G$, which proves the above assertion.

In this way we get that $V = \{[(c_m)] \in EX \,|\, (c_m) \text{ eventually in } S_{n_0}\} \subseteq W$. We choose a neighbourhood U of $(S_1, S_2, \ldots)$, $U = \prod_{n \in \mathbf{N}} U_n$, where

$$U_n = \begin{cases} S(X \backslash C_n), & \text{for } n > n_0 \\ S_n, & \text{for } n \leq n_0 \end{cases}$$

We claim that $f(U) \subseteq V \subseteq W$. Really, if

$$\left(S_1, S_2, \ldots, S'_{n_0}, S'_{n_0+1}, \ldots\right) \in U,$$

and

$$\left(S_1, S_2, \ldots, S'_{n_0}, S'_{n_0+1}, \ldots\right) = [(d_n)],$$

then (d_n) is eventually in S_{n_0} and it follows that $(d_n) \in V$.

If the space is not locally connected, in general we cannot choose a compact C_n such that $X \backslash C_n$ to have only a finite number of components. Can we use quasicomponents instead of components ? More precisely, we state the following problem: Let X be a locally compact metric space whoose space of quasicomponents is compact, can the space of ends of X be described as the inverse limit of the space of quasicomponents of $X \backslash C$, C compact in X?

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ЕКВИВАЛЕНТНОСТ НА РАЗЛИЧНИТЕ ДЕФИНИЦИИ НА ПРОСТОРОТ НА КРАЕВИ

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Апстракт: Во трудот е покажано дека претходната дефиниција на просторот на краеви на сврзан, локално конечен, локално компактен метрички простор која е определена со помош на низи кои немаат точки на акумулација се совпаѓа со дефиницијата која користи инверзен лимес на комплементи на компактни множества.

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