PARTITIONINGS INDUCED BY A MAP

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Abstract. By a partitioning of a topological space we understand a covering consisting of disjoint open sets. For a locally compact space Y and a map $f: X \to Y$ any finite partition $\mathcal{R}$. induces a finite partition $\mathcal{R}_f$ and a unique map $f_*: \mathcal{R} \to \mathcal{R}_f$.

For compact spaces X, Y which are not locally connected, an approximative map $\underline{f}$ from X towards Y induces a finite portioning $\mathcal{R}_{\underline{f}}$ and a map $\underline{f}_*: \mathcal{R} \to \mathcal{R}_{\underline{f}}$.

An application to dynamical systems willbe given.

In this paper first we define a partitioning induced by a map. Based on this, we introduce a partitioning induced by an approximative map. Approximative maps are used for defining a notion of shape [2].

At last, it is presented a result about behavior of open closed subsets of a uniform attractors in a dynamical system. This result is motivated by a result of Sanjurjo [3] about components of attractors.

For a topological space X, by $C(X)$ we denote the set of components of X. If $f: X \to Y$ is a map, then this map induces a map $C(f): C(X) \to C(Y)$.

Definition 1. A ***partitioning*** of a topological space X is a family $\{X_a \mid a \in A\}$ of disjoint, non-empty, open and closed subsets of X whose union is X.

In [4] it is shown the following:

Theorem 1. If the space of quasicomponents $Q(X)$ of the topological space X is compact, then every partitioning $\{X_a \mid a \in A\}$ of X is finite.

It is not difficult to show that *any partitioning of a compact space is finite*.

From now on in this paper, we will consider only spaces which have only partionings that are finite. In this way , anytime we consider a partitioning, we will think of a finite partitioning.

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Let X and Y be locally compact spaces. Let X be a space with compact space of quasicomponents, let Y be locally connected and let $f: X \to Y$ be a continuous function.

Let $\mathcal{R} = \{X_1, X_2, \ldots, X_n\}$ be a partitioning of X,

i.e., $X = X_1 \cup X_2 \cup \ldots \cup X_n$, where $X_1, X_2, \ldots, X_n$ are open and closed disjoint subsets of X.

We are going to construct a partitioning $\mathcal{R}_f$ of a subspace $\bigcup_{R \in \mathcal{R}_f} R$ of Y, and an induced surjective map $f_*: \mathcal{R} \to \mathcal{R}_f$ having the property $(gf)_* = g_* f_*$.

First we mention that *any open and closed set is a union of components of X.*

Since for each $i, i = 1,2,\ldots,n$, X_i is open and closed, then $X_i = \bigcup_{\alpha \in A} C_\alpha$, where C_α are components of X. There is a component $D_{\beta(\alpha)}$ of Y, so that $f(C_\alpha) \subseteq D_{\beta(\alpha)}$. Then

$$f(X_i) = f(\bigcup_{\alpha \in A} C_\alpha) = \bigcup_{\alpha \in A} f(C_\alpha) \subseteq \bigcup_{\alpha \in A} D_{\beta(\alpha)} = Y_i'.$$

Because of the space Y being locally connected, each Y_i' is open.

Now we define a relation $\sim$ in the following way:

$Y_i' \sim Y_j'$, if there exists a finite sequence Y_{i_1}', Y_{i_2}', ..., Y_{i_p}', so that $Y_i' \cap Y_{i_1}' \neq \varnothing$, $Y_{i_1}' \cap Y_{i_2}' \neq \varnothing$, ..., $Y_{i_p}' \cap Y_j' \neq \varnothing$. It is not difficult to show that $\sim$ is an equivalence relation. Let us denote by $\tilde{Y}_i'$ the equivalence class of Y_i'.

Let

$$Y_l = \bigcup_{Y_j' \in \tilde{Y}_i'} Y_j' .$$

Since Y is locally connected, Y_i is an open set in Y as an union of connected components. Let $Y_1, \ldots, Y_k$ be all the sets that are obtained in this way.

Let

$$Y_0 = Y \setminus (Y_1 \cup \ldots \cup Y_k).$$

Then Y_0 is also a union of components, and consequently is open. It follows that $Y_0, Y_1, \ldots, Y_k$ are open and closed disjoint sets.

As a result of that, a partitioning $\mathcal{R}_f = \{Y_1, \ldots, Y_k\}$ is obtained.

The following step is to define an induced map $f_* : \mathcal{R} \to \mathcal{R}_f$. It is done as follows: For $X_m \in \mathcal{R}$, there exists Y_i', so that $f(X_m) \subseteq Y_i'$ and there exists Y_j, so that $Y_i' \subseteq Y_j$. Then $f(X_m) \subseteq Y_j$. We define

$$f_*(X_m) = Y_j.$$

Remark. If the induced map of components $C(f) : C(X) \to C(Y)$ **is surjective** i.e., when $f(X)$ meets all components of Y, then $Y = Y_1 \cup \ldots \cup Y_k$, and there is no such a member of the partitioning as Y_0.

Theorem 2. Let X, Y, Z be topological spaces and Y, Z be locally connected. Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps and let $\mathcal{R}$ be a finite partitioning of X. Let $\mathcal{R}_f$ be the partitioning of Y induced by f and let $f_* : \mathcal{R} \to \mathcal{R}_f$ be the map induced by f. Let $(\mathcal{R}_f)_g$ be the partitioning of Z induced by g and $\mathcal{R}_f$, and $g_* : \mathcal{R}_f \to (\mathcal{R}_f)_g$ be the map induced by g and $\mathcal{R}_f$. And let $\mathcal{R}_{gf}$ be the partitioning induced by $gf : X \to Z$, and $(gf)_* : \mathcal{R} \to \mathcal{R}_{gf}$ be the map induced by gf.

Then, $(\mathcal{R}_f)_g = \mathcal{R}_{gf}$ and $(gf)_* = g_* f_*$.

Proof. Let $X_1 \in \mathcal{R}$ and let $f_*(X_1) = Y_1 \in \mathcal{R}_f$. These sets are union of components

$$X_1 = \bigcup_{a \in A} C_a, \quad Y_1 = \bigcup_{b \in B} D_b.$$

Let $g_*(Y_1) = Z_1$, and Z_1 is an union of components

$$Z_1 = \bigcup_{c \in C} E_c.$$

Let $f(C_a) \subseteq D_b$ and let $g(D_b) \subseteq E_c$. It must be that $E_c \in Z_1$, so it follows that

$$gf(C_a) \subseteq Z_1. \tag{1}$$

Now, let $(gf)_*(X_1) = Z_2$. It follows that

$$gf(C_a) \subseteq Z_2. \tag{2}$$

From (1) and (2) follows $Z_1 = Z_2$.

Theorem 3. Let X, Y be subsets of M - a locally compact ANR and let $\underline{f} = (f_n)$ be an approximative map from X towards Y. Let $\mathcal{R}$ be a partitioning of X. Then there exists a finite partitioning $\mathcal{R}_{\underline{f}}$ and a map $\underline{f}_*: \mathcal{R} \to \mathcal{R}_{\underline{f}}$, unique in the sense that for each $X_1 \in \mathcal{R}$, the sequence $f_n|_{X_1}: X_1 \to M$ is an approximative map from X_1 towards $\underline{f}_*(X_1) = Y_1$.

Proof. First we define $\underline{f}_*: \mathcal{R} \to \mathcal{R}_{\underline{f}}$. (f_n) is an approximative map from X towards Y, $f_n: X \to M$. Since M is an ANR, it follows that M is locally connected, and we can apply previous theorems. There exists a cofinal sequence of closed, locally connected neighbourhoods

$$V_1 \supset \ldots \supset V_k \supset V_{k+1} \ldots \supset Y,\ Y = \bigcap_{k=1}^{\infty} V_k.$$

For V_k, there is n_k, so that for $n \geq n_k$, f_n is homotopic to f_{n+1} in V_k. It follows $f_n(X) \subseteq V_k$.

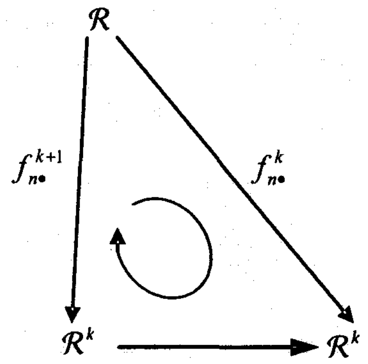
The map $f_n: X \to V_k$ induces a map $f_{n*}^k: \mathcal{R} \to \mathcal{R}_{f_n}$.

Let $X_1 \in \mathcal{R}$. Let by the map $f_n: X \to V_k$, $f_n(X_1) \subseteq Y_1$, i.e., $f_{n*}^k(X_1) = Y_1$, where $X_1 = \bigcup_{a \in A} C_a$. $f_n(C_a)$ is connected, and consequently $f_n(C_a) \subseteq D$, where D is a component in Y_1.

Since, for $n \geq n_k$, f_n is homotopic to f_{n+1} in V_k. For the map $f_{n+1}: X \to V_k$, we have $f_{n+1}(C_a) \subseteq D$, i.e. $f_{n+1}(X_1) \subseteq Y_1$, and so $f_{n+1*}^k(X_1) = Y_1$, for each $n \geq n_k$.

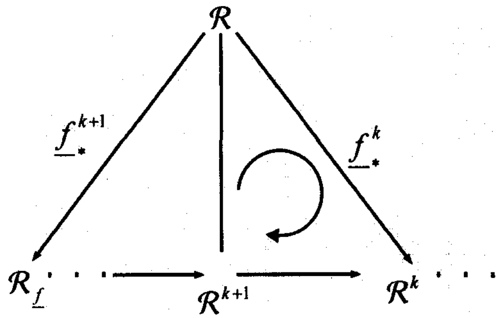
We obtain $\mathcal{R}_{f_n} = \mathcal{R}_{f_{n+1}}$ and also $f_{n*}^k = f_{n+1*}^k$ for each $n \geq n_k$. We denote all these partitions by $\mathcal{R}^k$ and all these maps by $\underline{f_*^k}: \mathcal{R} \to \mathcal{R}^k$.

Because of theorem 2, the diagram on picture 1 commutes for all $n \geq n_{k+1}$:



Picture 1

The diagram also commutes if f_{n*}^{k} is replaced by $\underline{f}_*^k$ (picture 2).



Picture 2

The inverse sequence

$$\ldots\ldots \to \mathcal{R}^{k+1} \to \mathcal{R}^k \to \ldots\ldots$$

has an iverse limit which we denote by $\mathcal{R}_{\underline{f}}$.

Since $|\mathcal{R}^k| \leq |\mathcal{R}|$, for each k, it follows that the cardinal number $|\mathcal{R}_{\underline{f}}|$ is finite. Let $Y_1 \in \mathcal{R}_{\underline{f}}$. Then $Y_1 = \bigcap_{k=1}^{\infty} W_k$, where $W_k \in \mathcal{R}^k$ is open and closed set in V_k. It follows W_k is closed in M.

Then Y_1 is closed. Since $\mathcal{R}_{\underline{f}}$ is a finite, it is also true that Y_1 is open. Consequently $\mathcal{R}_{\underline{f}}$ is a (finite) partitioning.

Since the diagram on picture 2 commutes, the maps $\underline{f}_*^k : \mathcal{R} \to \mathcal{R}^k$ induce a map $\underline{f}_* : \mathcal{R} \to \mathcal{R}_{\underline{f}}$.

(f_n) is an approximative map, i.e., for each neighborhood V of Y, f_n is homotopic to f_{n+1}, for almost all n, in V, with a homotopy F_n.

Let $h_n = f_n |_{X_1}$.

Let V_1 be a neighborhood of Y_1, so that $V_1 \cap Y_j = \varnothing$, for $j \neq 1$, $Y_j \in \mathcal{R}_{\underline{f}}$. There exists a neighborhood V of Y, so that V_1 is the component of connectedness of V that contains Y_1.

Let $X_1 = \bigcup_{\alpha\in A} C_\alpha$ where $C_\alpha, \alpha \in A$ are components of connectedness.

Let $f_n(C_\alpha) \subseteq D_\alpha \subseteq Y_1$, where D_α is a component of Y_1. Then it also holds $F_n(C_\alpha \times I) \subseteq D_\alpha \subseteq Y_1 \subseteq V_1$. It follows $F_n\left(\bigcup_{\alpha\in A}(C_\alpha \times I)\right) \subseteq Y_1$.

We define a homotopy

$$H_n : X_1 \times I \to Y_1 \text{ by } H_n = F_n |_{X_1 \times I}.$$

Then h_n is homotopic to h_{n+1}, with the homotopy H_n in V_1.

Let $\pi : M \times \mathbb{R} \to M$ be a flow in M. Let the compact set K be a uniform attractor of π. Let U be a compact neighborhood of K in $A(K)$ and such that each of the components of U meets K.

The approximative map (f_n) from U towards K induced by the flow π is defined in a natural way by putting $f_n : U \to M$ to be defined by $f_n(x) = \pi(x,n)$, for all $x \in U$.

Let the partitioning $\mathcal{R}$ of K be given. The inclusion $i : K \to U$ induces the partitioning $\mathcal{R}_i$ and the surjective map $i_* : \mathcal{R} \to \mathcal{R}_i$. The approximative map $\underline{f} : U \to K$ induces the partitioning $(\mathcal{R}_i)_{\underline{f}}$ and the map $\underline{f}_* : \mathcal{R}_i \to (\mathcal{R}_i)_{\underline{f}}$, as in theorem 3.

Let us denote the composition $\underline{f}_* \circ i_*$ by $F : \mathcal{R} \to (\mathcal{R}_i)_{\underline{f}}$.

Now we can prove the following

Theorem 4. Let K be a uniform attractor of π and let $\mathcal{R}$ be a partitioning of K. If $K_1 \in \mathcal{R}$ is a uniform attractor, then $F(K_1) \subseteq K_1$.

Proof. Let K_1 be a uniform attractor and let $K_1 \in \mathcal{R}$. Then $K_1 = \bigcup_{a\in A} C_a$, C_a is a component of K.

There exists $U_1 \in \mathcal{R}_i$, so that $K_1 \subseteq U_1$, i.e., $i_*(K_1) = U_1$.

Let us assume that $F(K_1) \not\subset K_1$. Then $\underline{f}_*(U_1) \not\subset K_1$, and it follows that there exists a component C of $\underline{f}_*(U_1)$, i.e., a component of K, so that $C \cap K_1 = \varnothing$.

K is a uniform attractor, U is a compact neighborhood of K, $\underline{f}: U \to K$ is the approximative map induced by π, so it follows that there exists a unique induced continuous map $\Lambda_{\underline{f}}: C(U) \to C(K)$, such that for every component $D' \in C(U)$, the sequence of maps $f_n|_{D'}: D' \to M$ defines an approximative map from D' towards $C' = \Lambda_{\underline{f}}(D')$.

Let D be a component of U_1, such that $\Lambda_{\underline{f}}(D) = C$.

It follows that maps $f_k|_D, D \to C$ define an approximative map.

There exists $k_0 > 0$ and a closed neighborhood W of C in M, $W \cap K_1 \neq \varnothing$, so that $\pi(x,k) = f_k(x) \in W$, for $x \in D, k \geq k_0$.

Because of $D \cap K_1 \neq \varnothing$, it follows that there exists $y \in D \cap K_1$, so that $\pi(y,k) \in W$, and $W \cap K_1 = \varnothing$, which is a contradiction to the fact that K_1 is uniform attractor.

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РАЗБИВАЊА ИНДУЦИРАНИ ОД НЕПРЕКИНАТО ПРЕСЛИКУВАЊЕ

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Апстракт. Разбивање на тополошки простор е покривач од отворени дисјунктни множества. За локално компактен тополошки простор Y и пресликување $f: X \to Y$ секое конечно разбивање индуцира конечно разбивање $\mathcal{R}_f$ и единствено пресликување $f_*: \mathcal{R} \to \mathcal{R}_f$.

За компактни простори X, Y кои не се локално сврзани, секое апроксимативно пресликување $\underline{f}$ од X кон Y индуцира разбивање $\mathcal{R}_{\underline{f}}$ и пресликување $\underline{f}_*: \mathcal{R} \to \mathcal{R}_{\underline{f}}$. Ќе биде дадена и една примена во теоријата на динамички системи.

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