UNIFORM SEPARATION

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Abstract. The main properties of weakly connected spaces, the componnents of weak connectedness and uniformly locally connected spaces are stated and proved.

In the most recent papers of Berarducci, Dikranjan and Pelant concerning a class of functions which is between continuous and uniform continuous an important role is played by notions of weak connectedness (under name uniform connectedness in [1]) and uniform local connectedness [2] (also [3]). In this paper we consider some of the properties related to these notions.

Let X be a metric space, A and B are non-empty subsets of X.

Definition: A and B are *uniformly separated* in X if there exists a uniformly continuous function $f: X \to [0,1]$ such that $f(A)=0$ and $f(B)=1$.

Theorem 1: Let Z be a metric space, A and B are non-empty subsets of Z and $X = A \cup B$. Then, A and B are uniformly separated in X if and only if A and B are are uniformly separated in Z.

Proof: Let $X = A \cup B$, $A \neq 0$, $B \neq 0$ and $X \subseteq Z$. Let A and B be uniformly separated in X. Then there exists a uniformly continuous function $f: X \to [0,1]$ such that $f(A)=0$ and $f(B)=1$. Then there is a (unique) extension of f to a uniformly continuous function $\overline{f}: \overline{X} \to [0,1]$. Now, from Katetov theorem, there exists a uniformly continuous function $\tilde{f}: Z \to [0,1]$, an extension of $\overline{f}: \overline{X} \to [0,1]$. The function $\tilde{f}: Z \to [0,1]$ satisfies $\tilde{f}(A) = f(A) = 0$, $\tilde{f}(B) = f(B) = 1$ and $\tilde{f}(X) = f(X)$. It follows A and B be uniformly separated in Z.

Now, let A and B be uniformly separated in Z. Then there exists a uniformly continuous function $f: Z \to [0,1]$, which satisfies $f(A)=0$ and $f(B)=1$. Then the restriction $f: X \to [0,1]$ satisfies the same conditions i.e. A and B be uniformly separated in X.

Definition: Let X be a metric space and $Y \subseteq X$. Y is *weakly connected*, if Y is not a union of two uniformly separated subsets.

Theorem 2: a) If A and B are uniformly separated subsets in X, then A and B are separated in X.

b) If X is connected X then is weakly connected.

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Proof: a) Let A and B be uniformly separated in X , then there exists a uniformly continuous function $f: X \to [0,1]$ such that $f(A)=0$ and $f(B)=1$. Then A and B are closed in X as inverse images of one point sets. It follows that A and B are separated in X.

b) Suppose X is weakly connected. Then $X = A \cup B$, and A and B are uniformly separated subsets. From a), A and B are separated in X, and it follows X is a union of two separated subsets.

Theorem 3: Let X be a metric space and $Y \subseteq Z \subseteq X$. A subset Y is weakly connected in X if and only if Y is weakly connected in Z.

Proof: Suposse Y is weakly connected in Z and that Y is not weakly connected in X. There exists non empty subsets A and B of X, such that $Y = A \cup B$, and A and B are uniformly separated in X. From the previous theorem, A and B are uniformly separated in Z. It follows Y is not weakly connected in Z.

In the same way we can prove the other direction of the theorem.
$C_1 = f^{-1}(0) \cap A$

Theorem 4: Let A and B be uniformly separated nonempty subsets of X, $X = A \cup B$, and let C be weakly connected in X. Then $C \subseteq A$ or $C \subseteq B$.

Proof: Let A and B be uniformly separated subsets of X and $X = A \cup B$. Then there exists a uniformly continuous function $f: X \to [0,1]$ such that $f(A)=0$ and $f(B)=1$.

Suppose that the conclusion is not true i.e. that $C \cap A \neq \varnothing$ and $C \cap B \neq \varnothing$. Than if we put $C_1 = C \cap A$ and $C_2 = C \cap B$, then $C = C_1 \cup C_2$, $C_1 \neq 0$, $C_2 \neq 0$

The restriction of f, $f|_C : C \to [0,1]$ is uniformly continuous and

$$f|_C(C_1) = f(C_1) \subseteq f(A) = 0$$

$$f|_C(C_2) = f(C_2) \subseteq f(B) = 1$$

It follows C_1 and C_2 uniformly separated, and C is not weakly connected. It follows that the proposition that $C \cap A \neq \varnothing$ and $C \cap B \neq \varnothing$ is not true. Then $C \subseteq A$ or $C \subseteq B$.

Corrolary 5: Let A and B be uniformly separated and $X = A \cup B$. and let C be connected in X. Then $C \subseteq A$ or $C \subseteq B$.

Proof: If C is connected , then C is weakly connected. From Theorem 4, it follows that $C \subseteq A$ or $C \subseteq B$.

Theorem 6: Let C be weakly connected in X and $C \subseteq D \subseteq \overline{C}$. Then D is weakly connected.

Proof: Suppose that D is not weakly connected. Then there exist uniformly separated sets A and B such that $D = A \cup B$. There exists a uniformly continuous function $f : D \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. Since $C \subseteq D$, from the previous theorem it follows that $C \subseteq A$ or $C \subseteq B$.

Let $C \subseteq A$. Then $D \subseteq \overline{C} \subseteq \overline{A}$. From $f(A) = 0$ it follows $f(\overline{A}) = 0$. Then, $f(D) = 0$. This is in contradiction with $f(B) = 1$.

Theorem 7: Let X be weakly connected and $f : X \to Y$ be exists uniformly continuous function. Then $f(X)$ is weakly connected.

Proof: Suppose that $f(X)$ is not weakly connected. Then $f(X) = C \cup D$, $C \neq \varnothing$ $D \neq \varnothing$ and there exists a uniformly continuous function $g : f(X) \to [0,1]$ such that $g(C) = 0$, $g(D) = 1$. Let $f^{-1}(C) = A, f^{-1}(D) = B$. Then $X = A \cup B$.

We consider the function $h = g \circ f : X \to \left[0,1\right]$. The function h is uniformly continuous and

$$h(A) = g(f(A)) = g(f(f^{-1}(C))) \subseteq g(C) = 0$$

$$h(B) = g(f(B)) = g(f(f^{-1}(D))) \subseteq g(D) = 1.$$

This is a contradiction with the proposition that X is weakly connected.

Theorem 8: Let $C_a, a \in A$ be weakly connected spaces. If, for all $a, a' \in A$, $C_a \cap C_{a'} \neq \varnothing$, then the union $\bigcup_{a \in A} C_a$ is weakly connected.

Proof: Suppose that $\bigcup_{a \in A} C_a$ is not weakly connected. Then there exist uniformly separated sets U and V, and $\bigcup_{a \in A} C_a = U \cup V$. By Theorem 3, $C_a \subseteq U$ or $C_a \subseteq V$. From $U \neq \varnothing$ and $V \neq \varnothing$, there exist indices $a, a' \in A$ such that $C_a \subseteq U$ and $C_{a'} \subseteq V$. Since U and V are uniformly separated, it follows that $C_a \cap C_{a'} = \varnothing$. This is in contradiction with $C_a \cap C_{a'} \neq \varnothing$.

Definition: Let X be a metric space. A *component of weak connectedness* $C_w(x)$ of a point x is the biggest weakly connected set containing x.

In the metric space X we define a relation $\sim$ in the following way: $x \sim y$ if there exists weakly connected subset of X containing x and y.

Refleksivity and symetry of $\sim$ are obvious, while transitivity follows from the previous theorem

Really, from $x \sim y$, there exists a weakly connected subset C_1 containing x and y, and from $y \sim z$ there exists a weakly connected subset C_2 containing y and z. Since $C_1 \cap C_2$, from the previous theorem, the set $C = C_1 \cup C_2$ is a weakly connected subset containing x and z. It follows $x \sim z$.

The components of weak connectedness of X are exactly the equivalence classes of the relation $\sim$.

Theorem 9: The components of weak connectedness are closed.

Proof: Let $C_w(x)$ be the component of weak connectedness of the point x. From Theorem 6, $\overline{C_w(x)}$ is closed. From the definition, $C_w(x)$ is the biggest weakly connected subset C_1 containing the point x, and it follows that $C_w(x) = \overline{C_w(x)}$ i.e. $C_w(x)$ is closed.

Theorem 10: A component of weak connectedness is a union of components of connectedness.

Proof: Let $C(x)$ be component of connectedness of x. Then $C(x)$ is weakly connected. It follows, there exists a component of weak connectedness D such that $C \subseteq D$.

Theorem 11: If X is locally connected, then the components of weak connectedness are open sets.

Proof: The components of connectedness are open sets. Now, each component of weak connectedness is an open set as a union of some components of connectedness.

Definition: The metric space X is *uniformly locally connected*, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any two poins x, y satisfying $d(x,y) < \delta$ are contained in a connected set with diameter less than ε.

Theorem 12: If X is uniformly locally connected, then X is locally connected.

Proof: Let X be uniformly locally connected metric space. For a given $\varepsilon > 0$, there exists $\delta > 0$ such that: if $d(x_0, x) < \delta$, then there exists a connected set C_x containing x and x_0 such that $\operatorname{diam} C_x < \varepsilon$.

Then $C_x \subseteq B_\varepsilon(x_0)$ and $\bigcup_{x \in B_\delta(x_0)} C_x \subseteq B_\varepsilon(x_0)$.

Since $x_0 \in C_x$, from Theorem 8 we have that $\bigcup_{x \in B_\delta(x_0)} C_x$ is connected.

Suppose that $B_\delta(x_0)$ is not connected. Then there exist two nonempty open sets A and B such that $B_\delta(x_0) = A \cup B$ and $A \cap B = \varnothing$.

Let $z \in B$. Then $x_0, z \in C_z$. Let $C = \bigcup_{z \in B} C_z$.

From theorem 8, C is connected.

If $y \in A$, then $x_0, y \in C_y$. Let $D = \bigcup_{y \in A} C_y$.

D is a connected set. We have

$$B \subseteq C \cap B_\delta(x_0),$$
$$A \subseteq D \cap B_\delta(x_0)$$

and $x_0 \in A$, $x_0 \in B$.

It follows, $x_0 \in A \cap B$ - a contradiction with $A \cap B \neq \varnothing$. The conclusion is that the set $B_{Gd}(x_0)$ is connected i.e X is locally connected.

Corrolary 13: If X is a uniformly locally connected metric space, than the components of weak connectedness are open sets.

Theorem 14: Product of a finite number of weakly connected metric spaces is weakly connected.

Proof: Let X_1 and X_2 be weakly connected metric spaces.

Let $(x_1, x_2) \in X_1 \times X_2$. Let $(y_1, y_2) \in X_1 \times X_2$ be an arbitrary point. Then $(x_1, x_2), (x_1, y_2) \in \{x_1\} \times X_2$. The metric space $\{x_1\} \times X_2$ is weakly connected. It follows that $(x_1, y_2) \in C_w(x_1, x_2)$ i.e. $C_w(x_1, y_2) = C_w(x_1, x_2)$.

In the same way, from $(x_1, y_2), (y_1, y_2) \in X_1 \times \{y_2\}$ it follows that $C_w(y_1, y_2) = C_w(x_1, y_2)$. We conclude that $X_1 \times X_2$ is weakly connected.

References:

1. A. Berarducci, D.Dikranjan, Uniformly approachable functions and spaces, Rend. Ist. Matematica Univ. Di Trieste **25**, (1993), 23 -56
2. A. Berarducci, D.Dikranjan, J. Pelant, Functions with distant fibers and uniform continuity
3. J.G.Hocking, G.S.Young, Topology, Addison - Wesley 1961
4. K. Kuratowski, Topology, (Russian translation) Mir, Moskva 1969 (originally published: Academic Press, New York and London, Panstwowe Wydawnictwo Naukowe, Warszawa, 1968)

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РАМНОМЕРНА РАЗДЕЛЕНОСТ

Никита Шекутковски, Татјана Атанасова Пачемска

Апстракт. Основните својства на слабо сврзаните простори, компонентите на слаба сврзаност и рамномерно локално сврзаните простори се формулирани и докажани.

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