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FREE $(m+k,m)-$BANDS

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Abstract. A characterization of $(m+k,m)-$bands using the rectangular bands is given in [3]. This result is used to obtain a free $(m+k,m)-$band.

1. Introduction

First, we will introduce some notations which will be used further on:

1) The elements of Q^s, where Q^s denotes the s-th Cartesian power of Q, will be denoted by x_1^s.

2) The symbol x_i^j will denote the sequence $x_i, x_{i+1}, \dots, x_j$ for $i \leq j$, and the empty sequence for $i > j$.

3) If $x_1 = x_2 = \dots = x_s = x$, then x_1^s is denoted by the symbol $\overset{s}{x}$.

4) The set $\{1,2,\dots,s\}$ will be denoted by $\mathbb{N}_s$.

Let $Q \neq \emptyset$ and n,m are positive integers. If $[\ ]$ is a mapping from Q^n into Q^m, then $[\ ]$ is called an $(n,m)-$operation. A pair $(Q;[\ ])$ where $[\ ]$ is an $(n,m)-$operation is said to be an (n,m) groupoid. Every $(n,m)-$operation on Q induces a sequence $[\ ]_1, [\ ]_2, \dots, [\ ]_m$ of $n-$ary operations on the set Q, such that

$$((\forall i \in \mathbb{N}_m) \quad [x_1^n]_i = y_i) \Leftrightarrow [x_1^n] = y_1^m.$$

Let $m \geq 2$, $k \geq 1$. An $(m+k,m)-$groupoid $(Q;[\ ])$ is called an $(m+k,m)-$semigroup if for each $i \in \{0,1,2,\dots,k\}$

$$\left[x_1^i \left[x_{i+1}^{i+m+k}\right] x_{i+m+k+1}^{m+2k}\right] = \left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right]$$

An $(m+k,m)-$groupoid $(Q;[\ ])$ is said to be a projection $(m+k,m)-$groupoid if there are $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_m \leq m+k$, such that

$$[x_1^{m+k}] = x_{\alpha_1} x_{\alpha_2} \dots x_{\alpha_m},$$

for any $x_1^{m+k} \in Q^{m+k}$.

Let $0 \leq p \leq m$. An $(m+k,m)-$groupoid $(Q;[\ ])$ is said to be a $p-$zero $(m+k,m)-$groupoid if $[x_1^{m+k}] = x_1^p x_{p+k+1}^{m+k}$, for any $x_1^{m+k} \in Q^{m+k}$.

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Proposition 1.1. ([3, *Proposition* 1.3]) *Any* $p-$*zero* $(m+k,m)-$*groupoid* $(Q;[\,]^p)$ *is an* $(m+k,m)-$*semigroup.*

Proposition 1.2. ([3, *Proposition* 1.5]) *If* $(Q;[\,])$ *is a projection* $(m+k,m)-$*groupoid which is also an* $(m+k,m)-$*semigroup, then* $(Q;[\,])$ *is a* $p-$*zero* $(m+k,m)-$*semigroup, for some* $0 \leq p \leq m$.

Propositions 1.1 and 1.2 imply that there are exactly $m+1$ projection $(m+k,m)$-semigroups.

Let $(A_i;\ [\,]^i)$, $i = 1, 2, ..., t$ be $(m+k,m)-$semigroups. Their direct product is an $(m+k,m)-$semigroup, where the $(m+k,m)-$operation $[\,]$ is defined by

$$[x_1^{m+k}] = y_1^m \Leftrightarrow x_i = (x_{i,1},\ x_{i,2}, ..., x_{i,t}),\ y_j = (y_{j,1}, y_{j,2}, ..., y_{j,t}),$$
$$y_{j,r} = [x_{1,j}x_{2,j}...x_{m+k,j}]^r\ ,\ i \in \mathbb{N}_{m+k}, j \in \mathbb{N}_m, r \in \mathbb{N}_t.$$

Let $\mathbf{A}_p = (A_p;\ [\,]^p)$ be p-zero $(m+k,m)-$semigroups, $0 \leq p \leq m$. The direct product of A_m, A_{m-1}, ... , A_0 is called $(m+k,m)-$band.

If $(A_m \times A_{m-1} \times ... \times A_0;\ [\,])$ is an $(m+k,m)-$band then its $(m+k,m)-$operation $[\,]$ is of the form

$$[x_1^{m+k}] = y_1^m \quad \Leftrightarrow \quad x_i = (x_{i,1}, x_{i,2}, ..., x_{i,m+1}),$$
$$y_j = (x_{j,1}, x_{j,2}, ..., x_{j,m+1-j}, x_{j+k,m+2-j}, ..., x_{j+k,m+1}), i \in \mathbb{N}_{m+k}, j \in \mathbb{N}_m.$$

The next proposition gives a characterization of $(m+k,m)-$bands as $(m+k,m)$-semigroups in which five identities are satisfied.

Proposition 1.3. ([3, *Proposition* 2.2]) *An* $(m+k,m)-$*semigroup* $\mathbf{Q} = (Q,[\,])$ *is an* $(m+k,m)-$*band if and only if the following conditions are satisfied in* $\mathbf{Q}$*:*

(B I) $\left[x_1^{m+k}\right]_i = \left[y_1^{i-1} x_i y_{i+1}^{i+k-1} x_{i+k} y_{i+k+1}^{m+k}\right]_i$,

(B II) $\left[\overset{j-1}{a} \left[\overset{i-1}{a}\ x\ \overset{k-1}{a}\ y\ \overset{m-i}{a}\right]_i \overset{k-1}{a}\ z\ \overset{m-j}{a}\right]_j = \left[\overset{i-1}{a}\ x\ \overset{k-1}{a} \left[\overset{j-1}{a}\ y\ \overset{k-1}{a}\ z\ \overset{m-j}{a}\right]_j \overset{m-i}{a}\right]_i$,

(B III) $\left[\overset{i-1}{a} \left[\overset{j-1}{a}\ x\ \overset{k-1}{a}\ y\ \overset{m-j}{a}\right]_j \overset{k-1}{a}\ z\ \overset{m-i}{a}\right]_i = \left[\overset{i-1}{a}\ x\ \overset{k-1}{a}\ z\ \overset{m-i}{a}\right]_i$,

(B IV) $\left[\overset{j-1}{a}\ x\ \overset{k-1}{a} \left[\overset{i-1}{a}\ y\ \overset{k-1}{a}\ z\ \overset{m-i}{a}\right]_i \overset{m-j}{a}\right]_j = \left[\overset{j-1}{a}\ x\ \overset{k-1}{a}\ z\ \overset{m-j}{a}\right]_j$,

(B V) $\left[\overset{m+k}{x}\right] = \overset{m}{x}$,

for a fixed element $a \in Q$, $i, j \in \mathbb{N}_m$ *and* $j \leq i$.

The second characterization of $(m+k,m)-$bands, using the usual rectangular bands, where a rectangular band is a semigroup $(Q;\ *)$ that satisfies the identities $x*y*z = x*z$ and $x*x = x$, for each x, y, $z \in Q$ is given in [3], also.

Proposition 1.4. ([3, *Proposition* 3.1]) $\mathbf{Q} = (Q;\ [\,])$ *is an* $(m+k,m)-$*band if and only if there are rectangular bands* $(Q;\ *_i)$, $i \in \mathbb{N}_m$, *such that*

(i) $(x *_i y) *_j z = x *_i (y *_j z)$,

(ii) $(x *_j y) *_i z = x *_i z$,

(iii) $x *_j (y *_i z) = x *_j z$,
for $i, j \in \mathbb{N}_m, j \leq i$
and
$\left[x_1^{m+k}\right]_i = x_i *_i x_{i+k},\ x_1^{m+k} \in Q^{m+k},\ i \in \mathbb{N}_m$.

This result of Proposition 1.4. is used to obtain a free $(m+k,m)$-band.

2. Free $(m+k,m)-$bands

Let B be a nonempty set. We define a sequence of sets $B_0, B_1, \ldots, B_p, \ldots$ by induction.
Let $B_0 = B$, B_p alredy defined and let $C_p = \{xy|\ x, y \in B_p\}$. Then, let
$B_{p+1} = B_p \cup (\mathbb{N}_m \times C_p)$ and $\overline{B} = \bigcup_{p \geq 0} B_p$.

Define a *length* for elements of $\overline{B}$, i.e. a mapping $|\ | : \overline{B} \to \mathbb{N}$ as follows:
If $a \in B$ then $|a| = 1$. Suppose that for each $u \in B_p$, $|u|$ is defined, then for $(i, xy) \in B_{p+1}$ we take $|(i, xy)| = 1 + |x| + |x|\,|y|$.

By induction on the length we are going to define a mapping $\varphi : \overline{B} \to \overline{B}$.
For $a \in B$ let

$$(0) \quad \varphi(a) = a.$$

Let $u = (i, xy) \in \overline{B}$ and suppose that for each $v \in \overline{B}$ with $|v| < |u|$, $\varphi(v)$ be defined and

(i) if $\varphi(v) \neq v$ then $|\varphi(v)| < |v|$;
(ii) $\varphi(\varphi(v)) = \varphi(v)$.

Because $|x| < |u|$ and $|y| < |u|$, it follows that $\varphi(x)$ and $\varphi(y)$ are defined.

If $\varphi(x) \neq x$ or $\varphi(y) \neq y$ then let

$$(1) \quad \varphi(i, xy) = \varphi(i, \varphi(x)\varphi(y)).$$

If $\varphi(x) = x$ and $\varphi(y) = y$, we consider several cases:
For $u = (i, xx)$, let

$$(2) \quad \varphi(u) = \varphi(x);$$

For $u = (i, (j, zw)y)$, $j \leq i$, let

$$(3) \quad \varphi(u) = \varphi(i, zy);$$

For $u = (i, x(j, zw))$, $i \leq j$, let

$$(4) \quad \varphi(u) = \varphi(i, xw);$$

For $u = (i, (j, zw)y)$, $i < j$, let

$$(5) \quad \varphi(u) = \varphi(j, z(i, wy));$$

For $u = (i, x(j, xz))$, $j < i$, let

$$(6) \quad \varphi(u) = \varphi(j, xz).$$

If $\varphi(u)$ can not be defined by $(1), (2), (3), (4), (5)$ or (6) let

$$(7) \quad \varphi(u) = u.$$

We will give some properties of φ.

Lemma 2.1. *φ is a well defined mapping.*

Proof. The proof of this property is by induction on the length of the elements $u = (i, xy)$ of $\overline{B}$.

Let $\varphi(x) \neq x$ or $\varphi(y) \neq y$. Then $|x| < |u|$, $|y| < |u|$ and from (i) we have $|\varphi(x)| < |x|$ or $|\varphi(y)| < |y|$. Hence, $|(i, \varphi(x)\varphi(y)| = 1 + |\varphi(x)| + |\varphi(x)|\,|\varphi(y)| < 1 + |x| + |x|\,|y| = |(i, xy)| = |u|$.

Let $\varphi(x) = x$ and $\varphi(y) = y$.

If $u = (i, xx)$ then $|x| < |u|$.

If $u = (i, (j, zw)y)$, $j \leq i$ then $|(i, zy)| = 1 + |z| + |z|\,|y| < 1 + 1 + |z| + |z|\,|w| + |y| + |z|\,|y| + |z|\,|w|\,|y| = |(i, (j, zw)y)| = |u|$.

If $u = (i, x(j, zw))$, $i \leq j$ then $|(i, xw)| = 1 + |x| + |x|\,|w| < 1 + |x| + |x| + |x|\,|z| + |x|\,|z|\,|w| = |(i, x(j, zw))| = |u|$.

If $u = (i, (j, zw)y)$, $i < j$ then $|(j, z(i, wy))| = 1 + |z| + |z| + |z|\,|w| + |z|\,|w|\,|y| < 1 + 1 + |z| + |z|\,|w| + |y| + |z|\,|y| + |z|\,|w|\,|y| = |(i, (j, zw)y)| = |u|$.

If $u = (i, x(j, xz))$, $j < i$ then $|(j, xz)| < |u|$.

Concidering the fact that on the right hand side of (1), (2), (3), (4), (5) and (6) of the definition of φ, φ is applied on elements with length less then the length of u, we conclude that φ is a well defined mapping. □

Lemma 2.2. *Let $u \in \overline{B}$.*
a) $|\varphi(u)| \leq |u|$.
b) If $\varphi(u) \neq u$ then $|\varphi(u)| < |u|$.
c) $\varphi(\varphi(u)) = \varphi(u)$.

Proof. By induction on the length.
a) If $\varphi(u)$ is defined by (0) or (7), then $\varphi(u) = u$. So, $|\varphi(u)| = |u|$.
If $\varphi(u)$ is defined by (1), (2), (3), (4), (5) or (6), then on the right hand side of (1), (2), (3), (4), (5) and (6) of the definition of φ, φ is applied on element v with length less then the length of u and by the inductive hipothesis $|\varphi(v)| \leq |v|$. Hence, $|\varphi(u)| = |\varphi(v)| \leq |v| < |u|$.
b) It follows from *a*).
c) If $\varphi(u)$ is defined by (0) or (7), then $\varphi(u) = u$. So, $\varphi(\varphi(u)) = \varphi(u)$.
If $\varphi(u)$ is defined by (1), (2), (3), (4), (5) or (6), then on the right hand side of (1), (2), (3), (4), (5) and (6) of the definition of φ, φ is applied on element v, such that $|v| < |u|$. By the inductive hipothesis $\varphi(\varphi(v)) = \varphi(v)$, hence $\varphi(\varphi(u)) = \varphi(\varphi(v)) = \varphi(v) = \varphi(u)$. □

Lemma 2.3. *Let $u = (i, xy) \in \overline{B}$. Then:*
a) $\varphi(u) = \varphi(i, \varphi(x)\varphi(y))$.
b) $\varphi(u) = \varphi(i, \varphi(x)y) = \varphi(i, x\varphi(y))$.

Proof. *a*) If $\varphi(x) \neq x$ or $\varphi(y) \neq y$ then *a*) follows from (1) of the definition of φ. If $\varphi(x) = x$ and $\varphi(y) = y$, then *a*) is obvious.
b) By induction on the length of $u = (i, xy)$.

If $\varphi(x) \neq x$ or $\varphi(y) \neq y$ then $\varphi(u) = \varphi(i, \varphi(x)\varphi(y))$. Because $|(i, \varphi(x)\varphi(y)| < |u|$, by induction and using $a)$ we have $\varphi(i, \varphi(x)\varphi(y)) = \varphi(i, \varphi(\varphi(x))\varphi(y)) = \varphi(i, \varphi(x)y)$. Similary, $\varphi(u) = \varphi(i, x\varphi(y))$.

If $\varphi(x) = x$ and $\varphi(y) = y$, then the property is obvious. □

Lemma 2.4. *If $u = (i, xx) \in \overline{B}$ then $\varphi(u) = \varphi(x)$.*

Proof. By induction on the length of $u = (i, xy)$.

If $\varphi(x) \neq x$ then by (1) we have $\varphi(i, xx) = \varphi(i, \varphi(x)\varphi(x))$. Then, by induction, since $|(i, \varphi(x)\varphi(x))| = 1 + |\varphi(x)| + |\varphi(x)|\,|\varphi(x)| < 1 + |x| + |x|\,|x| = |u|$, and using Lemma 2.3 $a)$ we have $\varphi(i, \varphi(x)\varphi(x)) = \varphi(\varphi(x)) = \varphi(x)$.

If $\varphi(x) = x$, then the property follows from (2) of the definition of φ. □

Lemma 2.5. (I) *If $u = (i, (j, zw)y)$, $j \leq i$ then $\varphi(u) = \varphi(i, zy)$.*
(II) *If $u = (i, x(j, zw))$, $i \leq j$ then $\varphi(u) = \varphi(i, xw)$.*
(III) *If $u = (i, (j, zw)y)$, $i < j$ then $\varphi(u) = \varphi(j, z(i, wy))$.*
(IV) *If $u = (i, x(j, xz))$, $j < i$ then $\varphi(u) = \varphi(j, xz)$.*

Proof. By induction on the length of $u = (i, xy)$.
(I) A) Let $\varphi(j, zw) \neq (j, zw)$ or $\varphi(y) \neq y$.
A1) If $\varphi(y) \neq y$ then, by induction and using Lemma 2.3. $b)$ we have:
$\varphi(i, (j, zw)y) = \varphi(i, (j, zw)\varphi(y)) = \varphi(i, z\varphi(y)) = \varphi(i, zy)$.
We have applied (I) on $(i, (j, zw)\varphi(y))$ because $|(i, (j, zw)\varphi(y))| < |(i, (j, zw)y)|$.
A2) Let $\varphi(j, zw) \neq (j, zw)$.
A2.1) If $\varphi(z) \neq z$ or $\varphi(w) \neq w$ then, using Lemma 2.3. $b)$ and (1) of the definition of φ, we have:
$\varphi(i, (j, zw)y) = \varphi(i, \varphi(j, zw)y) = \varphi(i, \varphi(j, \varphi(z)\varphi(w))y) = \varphi(i, (j, \varphi(z)\varphi(w))y)$
$= \varphi(i, \varphi(z)y) = \varphi(i, zy)$.
In the above, we have applied (I) on $(i, (j, \varphi(z)\varphi(w))y)$ because $|(i, (j, \varphi(z)\varphi(w))y)| < |u|$.
A2.2) Let $\varphi(z) = z$ and $\varphi(w) = w$.
A2.2.1) Let $z = w$. Then, using Lemma 2.3. $b)$ and (2) of the definition of φ, we have:
$\varphi(i, (j, zz)y) = \varphi(i, \varphi(j, zz)y) = \varphi(i, \varphi(z)y) = \varphi(i, zy)$.
A2.2.2) Let $z = (l, z_1z_2)$, $l \leq j$. Then, by induction, using Lemma 2.3. $b)$ and (3) of the definition of φ, we have:
$\varphi(i, (j, (l, z_1z_2)w)y) = \varphi(i, \varphi(j, (l, z_1z_2)w)y) = \varphi(i, \varphi(j, z_1w)y) = \varphi(i, (j, z_1w)y)$
$= \varphi(i, z_1y) = \varphi(i, (l, z_1z_2)y) = \varphi(i, zy)$.
We have applied (I) on $(i, (j, z_1w)y)$ and $(i, (l, z_1z_2)y)$, since their lengths are less then the length of u.
A2.2.3) Let $w = (l, w_1w_2)$, $j \leq l$. Then, by induction, using Lemma 2.3. $b)$ and (4) of the definition of φ, we have:
$\varphi(i, (j, z(l, w_1w_2))y) = \varphi(i, \varphi(j, z(l, w_1w_2))y) = \varphi(i, \varphi(j, zw_2)y) = \varphi(i, (j, zw_2)y)$
$= \varphi(i, zy)$.
We have applied (I) on $(i, (j, zw_2)y)$, because $|(i, (j, zw_2)y)| < |u|$.
A2.2.4) Let $u = (i, (j, (l, z_1z_2)w)y)$, $j < l$.Then, by induction, using Lemma 2.3. $b)$ and (5) of the definition of φ, we have:

$\varphi(i,(j,(l,z_1z_2)w)y) = \varphi(i,\varphi(j,(l,z_1z_2)w)y) = \varphi(i,\varphi(l,z_1(j,z_2w))y)$
$= \varphi(i,(l,z_1(j,z_2w))y)$.
If $j < l \leq i$, then by induction, since $|(i,(l,z_1(j,z_2w))y)| < |u|$ and $|(i,(l,z_1z_2)y)| < |u|$, we have:
$\varphi(i,(l,z_1(j,z_2w))y) = \varphi(i,z_1y) = \varphi(i,(l,z_1z_2)y) = \varphi(i,zy)$.
If $j \leq i < l$, then by induction and using Lemma 2.3. b) we have:
$\varphi(i,(l,z_1(j,z_2w))y) = \varphi(l,z_1(i,(j,z_2w)y)) = \varphi(l,z_1\varphi(i,(j,z_2w)y))$
$= \varphi(l,z_1\varphi(i,z_2y)) = \varphi(l,z_1(i,z_2y)) = \varphi(i,(l,z_1z_2)y) = \varphi(i,zy)$.
We have applied (III) on $u' = (i,(l,z_1(j,z_2w))y)$ and $u'' = (i,(l,z_1z_2)y)$ and (I) on $u''' = (i,(j,z_2w)y)$, since $|u'| < |u|$, $|u''| < |u|$ and $|u'''| < |u|$.
A2.2.5) Let $u = (i,(j,z(l,zw_2))y)$, $l < j$. Then, by induction, using Lemma 2.3. b) and (6) of the definition of φ, we have:
$\varphi(i,(j,z(l,zw_2))y) = \varphi(i,\varphi(j,z(l,zw_2))y) = \varphi(i,\varphi(l,zw_2)y) = \varphi(i,(l,zw_2)y)$
$= \varphi(i,zy)$.
We have applied (I) on $(i,(l,zw_2)y)$, since $|(i,(l,zw_2)y)| < |u|$.
B) Let $\varphi(j,zw) = (j,zw)$ and $\varphi(y) = y$.
B1) Let $y = (j,zw)$. Then, by (2) we have:
$\varphi(i,(j,zw)(j,zw)) = \varphi(j,zw)$.
B1.1) If $j < i$ then, by induction we have:
$\varphi(j,zw) = \varphi(i,z(j,zw)) = \varphi(i,zy)$.
We have applied (IV) on $(i,z(j,zw))$. It is possible, since $|(i,z(j,zw))| < |u|$.
B1.2) If $j = i$, then, by induction we have:
$\varphi(i,zw) = \varphi(i,z(i,zw)) = \varphi(i,zy)$
In the above, we have applied (II) on $(i,z(i,zw))$, because its length is less then the length of u.
B2) If $y \neq (j,zw)$ then the property follows from (3) of the definition of φ.
The above discusion completes the inductive step for (I).

(II) A) Let $\varphi(x) \neq x$ or $\varphi(j,zw) \neq (j,zw)$.
A1) If $\varphi(x) \neq x$ then, by induction and using Lemma 2.3. b) we have:
$\varphi(i,x(j,zw)) = \varphi(i,\varphi(x)(j,zw)) = \varphi(i,\varphi(x)w) = \varphi(i,xw)$.
We have applied (II) on $(i,\varphi(x)(j,zw))$ because $|(i,\varphi(x)(j,zw))| < |u|$.
A2) Let $\varphi(j,zw) \neq (j,zw)$.
A2.1) If $\varphi(z) \neq z$ or $\varphi(w) \neq w$ then, by induction, using Lemma 2.3. b) and (1) of the definition of φ, we have:
$\varphi(i,x(j,zw)) = \varphi(i,x\varphi(j,zw)) = \varphi(i,x\varphi(j,\varphi(z)\varphi(w))) = \varphi(i,x(j,\varphi(z)\varphi(w)))$
$= \varphi(i,x\varphi(w)) = \varphi(i,xw)$.
We have applied (II) on $(i,x(j,\varphi(z)\varphi(w)))$ because $|(i,x(j,\varphi(z)\varphi(w)))| < |u|$.
A2.2) Let $\varphi(z) = z$ and $\varphi(w) = w$.
A2.2.1) Let $z = w$. Then, using Lemma 2.3. b) and (2), we have:
$\varphi(i,x(j,ww)) = \varphi(i,x\varphi(j,ww)) = \varphi(i,x\varphi(w)) = \varphi(i,xw) = \varphi(i,xw)$.
A2.2.2) Let $z = (l,z_1z_2)$, $l \leq j$. Then, by induction, using Lemma 2.3. b) and (3) of the definition of φ, we have:
$\varphi(i,x(j,(l,z_1z_2)w)) = \varphi(i,x\varphi(j,(l,z_1z_2)w)) = \varphi(i,x\varphi(j,z_1w)) = \varphi(i,x(j,z_1w))$
$= \varphi(i,xw)$.

We have applied (II) on $(i,x(j,z_1w))$ since $|(i,x(j,z_1w))|<|u|$.
A2.2.3) Let $w=(l,w_1w_2)$, $j\le l$. Then, by induction, using Lemma 2.3. b) and (4) of the definition of φ, we have:

$\varphi(i,x(j,z(l,w_1w_2)))=\varphi(i,x\varphi(j,z(l,w_1w_2)))=\varphi(i,x\varphi(j,zw_2))=\varphi(i,x(j,zw_2))$
$=\varphi(i,xw_2)=\varphi(i,x(l,w_1w_2))=\varphi(i,xw)$.

In the above, we have applied (II) on $(i,x(j,zw_2))$ and $(i,x(l,w_1w_2))$, since their lengths are less then the length of u.
A2.2.4) Let $z=(l,z_1z_2)$, $j<l$. Then, by induction, using Lemma 2.3. b) and (5) of the definition of φ, we have:

$\varphi(i,x(j,(l,z_1z_2)w))=\varphi(i,x\varphi(j,(l,z_1z_2)w))=\varphi(i,x\varphi(l,z_1(j,z_2w)))$
$=\varphi(i,x(l,z_1(j,z_2w)))=\varphi(i,x(j,z_2w))=\varphi(i,xw)$.

We have applied (II) on $u'=(i,x(l,z_1(j,z_2w)))$ and $u''=(i,x(j,z_2w))$, since $|u'|<|u|$ and $|u''|<|u|$.
A2.2.5) Let $w=(l,zw_2)$, $l<j$, then, using Lemma 2.3. b) and (6) of the definition of φ, we have:

$\varphi(i,x(j,z(l,zw_2)))=\varphi(i,x\varphi(j,z(l,zw_2)))=\varphi(i,x\varphi(l,zw_2))=\varphi(i,x(l,zw_2))$
$=\varphi(i,xw)$.

B) Let $\varphi(x)=x$ and $\varphi(j,zw)=(j,zw)$.
B1) If $x=(j,zw)$, then by (2) of the definition of φ, we have:

$\varphi(i,(j,zw)(j,zw))=\varphi(j,zw)$.

B1.1) Let $i<j$, then, by induction and using Lemma 2.4. we have:

$\varphi(j,zw)=\varphi(j,z\varphi(w))=\varphi(j,z\varphi(i,ww))=\varphi(j,z(i,ww))=\varphi(i,(j,zw)w)$
$=\varphi(i,xw)$.

We have applied (III) on $(i,(j,zw)w)$, because $|(i,(j,zw)w)|<|u|$.
B1.2) If $i=j$, then we apply (I) on $(i,(i,zw)w)$, because $|(i,(i,zw)w)|<|u|$. Thus:

$\varphi(i,zw)=\varphi(i,(i,zw)w)=\varphi(i,xw)$.

B2) Let $x=(l,x_1x_2)$, $l\le i$. Then, by induction and using (3) of the definition of φ, we have:

$\varphi(i,(l,x_1x_2)(j,zw))=\varphi(i,x_1(j,zw))=\varphi(i,x_1w)=\varphi(i,(l,x_1x_2)w)=\varphi(i,xw)$.

We have applied (II) on $u'=(i,x_1(j,zw))$ and (I) on $u''=(i,(l,x_1x_2)w)$, since $|u'|<|u|$ and $|u''|<|u|$.
B3) If $\varphi(u)$ is not defined by (2) and (3) of the definition of φ, then the property follows from (4) of the definition.

The above discusion completes the inductive step for (II).

(III) A) Let $\varphi(y)\ne y$ or $\varphi(j,zw)\ne(j,zw)$.
A1) If $\varphi(y)\ne y$ then, by induction and using Lemma 2.3. b) we have:

$\varphi(i,(j,zw)y)=\varphi(i,(j,zw)\varphi(y))=\varphi(j,z(i,w\varphi(y)))=\varphi(j,z\varphi(i,w\varphi(y)))$
$=\varphi(j,z\varphi(i,wy))=\varphi(j,z(i,wy))$.

We have applied (III) on $(i,(j,zw)\varphi(y))$ because $|(i,(j,zw)\varphi(y))|<|u|$.
A2) Let $\varphi(j,zw)\ne(j,zw)$.
A2.1) If $\varphi(z)\ne z$ or $\varphi(w)\ne w$ then, by induction, using Lemma 2.3. b) and (1) of the definition of φ, we have:

$\varphi(i,(j,zw)y)=\varphi(i,\varphi(j,zw)y)=\varphi(i,\varphi(j,\varphi(z)\varphi(w))y)=\varphi(i,(j,\varphi(z)\varphi(w))y)$

$= \varphi(j, \varphi(z)(i, \varphi(w)y)) = \varphi(j, z(i, \varphi(w)y)) = \varphi(j, z\varphi(i, \varphi(w)y)) = \varphi(j, z\varphi(i, wy))$
$= \varphi(j, z(i, wy))$.
In the above, we have applied (III) on $u' = (i, (j, \varphi(z)\varphi(w))y)$ because $|u'| < |u|$.
A2.2) Let $\varphi(z) = z$ and $\varphi(w) = w$.
A2.2.1) Let $z = w$. Then, by induction, using Lemma 2.3. b) and (2) of the definition of φ we have:

$\varphi(i, (j, zz)y) = \varphi(i, \varphi(j, zz)y) = \varphi(i, \varphi(z)y) = \varphi(i, zy) = \varphi(j, z(i, zy))$.

We have applied (IV) on $u' = (j, z(i, zy))$, since $|u'| < |u|$.
A2.2.2) Let $z = (l, z_1z_2)$, $l \leq j$. Then, by induction, using Lemma 2.3. b) and (3) of the definition of φ, we have:

$\varphi(i, (j, (l, z_1z_2)w)y) = \varphi(i, \varphi(j, (l, z_1z_2)w)y) = \varphi(i, \varphi(j, z_1w)y) = \varphi(i, (j, z_1w)y)$
$= \varphi(j, z_1(i, wy)) = \varphi(j, (l, z_1z_2)(i, wy)) = \varphi(j, z(i, wy))$

We have applied (III) on $u' = (i, (j, z_1w)y)$ and (I) on $u'' = (j, (l, z_1z_2)(i, wy))$, since $|u'| < |u|$ and $|u''| < |u|$.
A2.2.3) Let $w = (l, w_1w_2)$, $j \leq l$. Then, by induction, using Lemma 2.3. b) and (4) of the definition of φ, we have:

$\varphi(i, (j, z(l, w_1w_2))y) = \varphi(i, \varphi(j, z(l, w_1w_2))y) = \varphi(i, \varphi(j, zw_2)y) = \varphi(i, (j, zw_2)y)$
$= \varphi(j, z(i, w_2y)) = \varphi(j, z(l, w_1(i, w_2y))) = \varphi(j, z\varphi(l, w_1(i, w_2y)))$
$= \varphi(j, z\varphi(i, (l, w_1w_2)y)) = \varphi(j, z(i, (l, w_1w_2)y)) = \varphi(j, z(i, wy))$.

In the above, we have applied (III) on $u' = (i, (j, zw_2)y)$ and $u'' = (i, (l, w_1w_2)y)$ and (II) on $u''' = (j, z(l, w_1(i, w_2y)))$, since $|u'| < |u|$, $|u''| < |u|$ and $|u'''| < |u|$.
A2.2.4) Let $z = (l, z_1z_2)$, $j < l$. Then, by induction, using Lemma 2.3. b) and (5) of the definition of φ, we have:

$\varphi(i, (j, (l, z_1z_2)w)y) = \varphi(i, \varphi(j, (l, z_1z_2)w)y) = \varphi(i, \varphi(l, z_1(j, z_2w))y)$
$= \varphi(i, (l, z_1(j, z_2w))y) = \varphi(l, z_1(i, (j, z_2w)y)) = \varphi(l, z_1\varphi(i, (j, z_2w)y))$
$= \varphi(l, z_1\varphi(j, z_2(i, wy))) = \varphi(l, z_1(j, z_2(i, wy))) = \varphi(j, (l, z_1z_2)(i, wy)) = \varphi(j, z(i, wy))$.

We have applied (III) on $u' = (i, (l, z_1(j, z_2w))y)$, $u'' = (i, (j, z_2w)y))$ and $u''' = (j, (l, z_1z_2)(i, wy))$, since $|u'| < |u|$, $|u''| < |u|$ and $|u'''| < |u|$.
A2.2.5) Let $w = (l, zw_2)$, $l < j$, then using Lemma 2.3. b) and (6) of the definition of φ, we have:

$\varphi(i, (j, z(l, zw_2))y) = \varphi(i, \varphi(j, z(l, zw_2))y) = \varphi(i, \varphi(l, zw_2)y) = \varphi(i, (l, zw_2)y)$.

If $l \leq i < j$, then by induction and using Lemma 2.3. b) we have:

$\varphi(i, (l, zw_2)y) = \varphi(i, zy) = \varphi(j, z(i, zy)) = \varphi(j, z\varphi(i, zy))$
$= \varphi(j, z\varphi(i, (l, zw_2)y)) = \varphi(j, z(i, (l, zw_2)y)) = \varphi(j, z(i, wy))$.

We have applied (I) on $u' = (i, (l, zw_2)y)$ and (III) on $u'' = (j, z(i, zw))$, since $|u'| < |u|$ and $|u''| < |u|$.

If $i < l < j$ then by induction and using Lemma 2.3. b) we have:

$\varphi(i, (l, zw_2)y) = \varphi(l, z(i, w_2y)) = \varphi(j, z(l, z(i, w_2y))) = \varphi(j, z\varphi(l, z(i, w_2y)))$
$= \varphi(j, z\varphi(i, (l, zw_2)y)) = \varphi(j, z(i, (l, zw_2)y)) = \varphi(j, z(i, wy))$.

We have applied (III) on $u' = (i, (l, zw_2)y)$ and $u'' = (i, (l, zw_2)y)$ and (IV) on $u''' = (j, z(l, z(i, w_2y)))$, because $|u'| < |u|$, $|u''| < |u|$ and $|u'''| < |u|$.
B) Let $\varphi(y) = y$ and $\varphi(j, zw) = (j, zw)$.
B1) Let $y = (j, zw)$, then by induction, using Lemmas 2.3. b) and 2.4. and (2) of the definition of φ, we have:

$\varphi(i, (j, zw)(j, zw)) = \varphi(j, zw) = \varphi(j, z\varphi(w)) = \varphi(j, z\varphi(i, ww))$

$= \varphi(j, z\varphi(i, w(j, zw))) = \varphi(j, z(i, w(j, zw))) = \varphi(j, z(i, wy))$.
In the above, we have applied (II) on $u' = (i, w(j, zw))$, because $|u'| < |u|$.
B2) It is not possible $\varphi(u)$ to be defined by (3) of the definition of φ.
B3) Let $y = (l, y_1y_2)$, $i \leq l$. Then, by induction, using Lemma 2.3. b) and (4) of the definition of φ, we have:

$\varphi(i, (j, zw)(l, y_1y_2)) = \varphi(i, (j, zw)y_2) = \varphi(j, z(i, wy_2)) = \varphi(j, z\varphi(i, wy_2))$
$= \varphi(j, z\varphi(i, w(l, y_1y_2)) = \varphi(j, z(i, w(l, y_1y_2))) = \varphi(j, z(i, wy))$.

We have applied (III) on $u' = (i, (j, zw)y_2)$ and (II) on $u'' = (i, w(l, y_1y_2))$, since $|u'| < |u|$ and $|u''| < |u|$.
B4) If $\varphi(u)$ is not defined by (2), (3) and (4) of the definition of φ, then the property follows from (5) of the definition.

The above discusion completes the inductive step for (III).

(IV) A) Let $\varphi(x) \neq x$ or $\varphi(j, xz) \neq (j, xz)$.
A1) If $\varphi(x) \neq x$ then, by induction, using Lemma 2.3. b) and (1) of the definition of φ, we have:

$\varphi(i, x(j, xz)) = \varphi(i, \varphi(x)\varphi(j, xz)) = \varphi(i, \varphi(x)\varphi(j, \varphi(x)z)) = \varphi(i, \varphi(x)(j, \varphi(x)z))$
$= \varphi(j, \varphi(x)z) = \varphi(j, xz)$.

We have applied (IV) on $(i, \varphi(x)(j, \varphi(x)z))$ since $|(i, \varphi(x)(j, \varphi(x)z))| < |u|$.
A2) Let $\varphi(j, xz) \neq (j, xz)$.
A2.1) In A1) we have considered the case $\varphi(x) \neq x$. Next, we consider $\varphi(x) = x$. Let $\varphi(z) \neq z$. Then, by induction and using Lemma 2.3. b) we have:

$\varphi(i, x(j, xz)) = \varphi(i, x\varphi(j, xz)) = \varphi(i, x\varphi(j, x\varphi(z))) = \varphi(i, x(j, x\varphi(z)))$
$= \varphi(j, x\varphi(z)) = \varphi(j, xz)$.

We have applied (IV) on $(i, x(j, x\varphi(z)))$ since $|(i, x(j, x\varphi(z)))| < |u|$.
A2.2) Let $\varphi(z) = z$.
A2.2.1) If $x = z$ then, using Lemma 2.3. b) and (2), we have:

$\varphi(i, x(j, xx)) = \varphi(i, x\varphi(j, xx)) = \varphi(i, x\varphi(x)) = \varphi(i, xx) = \varphi(x) = \varphi(j, xx)$.

A2.2.2) Let $x = (l, x_1x_2)$, $l \leq j$. Then, by induction and using Lemma 2.3. b) and (3) of the definition of φ, we have:

$\varphi(i, (l, x_1x_2)(j, (l, x_1x_2)z)) = \varphi(i, (l, x_1x_2)\varphi(j, (l, x_1x_2)z)) = \varphi(i, (l, x_1x_2)\varphi(j, x_1z))$
$= \varphi(i, (l, x_1x_2)(j, x_1z)) = \varphi(i, x_1(j, x_1z)) = \varphi(j, x_1z) = \varphi(j, (l, x_1x_2)z) = \varphi(j, xz)$.

We have applied (I) on $u' = (i, (l, x_1x_2)(j, x_1z))$ and $u'' = (j, (l, x_1x_2)z)$ and (IV) on $u''' = (i, x_1(j, x_1z))$, because $|u'| < |u|$, $|u''| < |u|$ and $|u'''| < |u|$.
A2.2.3) Let $z = (l, z_1z_2)$, $j \leq l$. Then, by induction, using Lemma 2.3. b) and (4) of the definition of φ, we have:

$\varphi(i, x(j, x(l, z_1z_2))) = \varphi(i, x\varphi(j, x(l, z_1z_2))) = \varphi(i, x\varphi(j, xz_2))$
$= \varphi(i, x(j, xz_2)) = \varphi(j, xz_2) = \varphi(j, x(l, z_1z_2)) = \varphi(j, xz)$.

In the above, we have applied (IV) on $u' = (i, x(j, xz_2))$ and (II) on $u'' = (j, x(l, z_1z_2))$ since $|u'| < |u|$ and $|u''| < |u|$.
A2.2.4) Let $x = (l, x_1x_2)$, $j < l$. Then, using Lemma 2.3. b) and (5) of the definition of φ, we have:

$\varphi(i, (l, x_1x_2)(j, (l, x_1x_2)z)) = \varphi(i, (l, x_1x_2)\varphi(j, (l, x_1x_2)z))$
$= \varphi(i, (l, x_1x_2)\varphi(l, x_1(j, x_2z))) = \varphi(i, (l, x_1x_2)(l, x_1(j, x_2z)))$.

We will consider three cases.

A2.2.4.1) If $j < i < l$ then, by induction and using Lemma 2.3. b) we have:
$\varphi(i,(l,x_1x_2)(l,x_1(j,x_2z))) = \varphi(i,(l,x_1x_2)((j,x_2z)) = \varphi(l,x_1(i,x_2(j,x_2z)))$
$= \varphi(l,x_1\varphi(i,x_2(j,x_2z))) = \varphi(l,x_1\varphi(j,x_2z)) = \varphi(l,x_1(j,x_2z)) = \varphi(j,(l,x_1x_2)z)$
$= \varphi(j,xz)$.
We have applied (II) on $u_1 = (i,(l,x_1x_2)(l,x_1(j,x_2z)))$, (III) on $u_2 = (i,(l,x_1x_2)((j,x_2z)$ and $u_3 = (j,(l,x_1x_2)z)$ and (IV) on $u_4 = (i,x_2(j,x_2z))$, because $|u_\lambda| < |u|$ for $\lambda = 1,2,3,4$.
A2.2.4.2) If $j < l < i$ then, by induction we have:
$\varphi(i,(l,x_1x_2)(l,x_1(j,x_2z))) = \varphi(i,x_1(l,x_1(j,x_2z))) = \varphi(l,x_1(j,x_2z))$
$= \varphi(j,(l,x_1x_2)z) = \varphi(j,xz)$
We have applied (I) on $u' = (i,(l,x_1x_2)(l,x_1(j,x_2z)))$, (IV) on $u'' = (i,x_1(l,x_1(j,x_2z)))$ and (III) on $u''' = (j,(l,x_1x_2)z)$, because $|u'| < |u|$, $|u''| < |u|$ and $|u'''| < |u|$.
A2.2.4.3) If $j < i = l$ then, by induction, we have:
$\varphi(i,(i,x_1x_2)(i,x_1(j,x_2z))) = \varphi(i,x_1(i,x_1(j,x_2z)) = \varphi(i,x_1(j,x_2z)) = \varphi(j,(i,x_1x_2)z)$
$= \varphi(j,xz)$.
We have applied (I) on $u' = (i,(i,x_1x_2)(i,x_1(j,x_2z)))$, (II) on $u'' = (i,x_1(i,x_1(j,x_2z))$ and (III) on $u''' = (j,(i,x_1x_2)z)$, since $|u'| < |u|$, $|u''| < |u|$ and $|u'''| < |u|$.
A2.2.5) Let $z = (l,xz_2)$, $l < j$, then, by induction, using Lemma 2.3. b) and (6) of the definition of φ, we have:
$\varphi(i,x(j,x(l,xz_2))) = \varphi(i,x\varphi(j,x(l,xz_2))) = \varphi(i,x\varphi(l,xz_2)) = \varphi(i,x(l,xz_2))$
$= \varphi(l,xz_2) = \varphi(j,x(l,xz_2)) = \varphi(j,xz)$.
We have applied (IV) on $u' = (i,x(l,xz_2))$ and $u'' = (j,x(l,xz_2))$, since $|u'| < |u|$ and $|u''| < |u|$.
B) Let $\varphi(x) = x$ and $\varphi(j,xz) = (j,xz)$.
B1) It is not possible $\varphi(u)$ to be defined by (2).
B2) Let $x = (l,x_1x_2)$, $l \le i$. Then, by induction, using Lemma 2.3. b) and (3) of the definition of φ, we have:
$\varphi(i,(l,x_1x_2)(j,(l,x_1x_2)z)) = \varphi(i,x_1(j,(l,x_1x_2)z)) = \varphi(j,(i,x_1(l,x_1x_2))z)$
$= \varphi(j,\varphi(i,x_1(l,x_1x_2))z)$.
We have applied (III) on $(j,(i,x_1(l,x_1x_2))z)$ since $|(j,(i,x_1(l,x_1x_2))z)| < |u|$.
B2.1) Let $i = l$. Then, by induction and using Lemma 2.3., we have:
$\varphi(j,\varphi(i,x_1(i,x_1x_2))z) = \varphi(j,\varphi(i,x_1x_2)z) = \varphi(j,(i,x_1x_2)z) = \varphi(j,xz)$.
We have applied (II) on $u' = (i,x_1(i,x_1x_2))$, since $|u'| < |u|$.
B2.2) Let $l < i$. Then, by induction and using Lemma 2.3. b), we have:
$\varphi(j,\varphi(i,x_1(l,x_1x_2))z) = \varphi(j,\varphi(l,x_1x_2)z) = \varphi(j,(l,x_1x_2)z)\varphi(j,xz)$.
We have applied (IV) on $(i,x_1(l,x_1x_2))$ because $|(i,x_1(l,x_1x_2))| < |u|$.
B3) It is not possible $\varphi(u)$ to be defined by (4).
B4) Let t $x = (l,x_1x_2)$, $i < l$. Then, by induction, using Lemmas 2.3. b) and 2.4. and (5) of the definition of φ, we have:
$\varphi(i,(l,x_1x_2)(j,(l,x_1x_2)z)) = \varphi(l,x_1(i,x_2(j,(l,x_1x_2)z)))$
$= \varphi(l,x_1\varphi(i,x_2(j,(l,x_1x_2)z))) = \varphi(l,x_1\varphi(j,(i,x_2(l,x_1x_2))z))$
$= \varphi(l,x_1\varphi(i,x_2(l,x_1x_2))z)) = \varphi(l,x_1\varphi(j,\varphi(i,x_1x_2)z)) = \varphi(l,x_1\varphi(j,\varphi(x_2)z))$
$= \varphi(l,x_1\varphi(j,x_2z)) = \varphi(l,x_1(j,x_2z)) = \varphi(j,(l,x_1x_2)z) = \varphi(j,xz)$.
In the above, we have applied (III) on $u' = (j,(i,x_2(l,x_1x_2))z)$ and $u'' = (j,(l,x_1x_2)z)$ and (II) on $u''' = (i,x_2(l,x_1x_2))$, since $|u'| < |u|$, $|u''| < |u|$ and $|u'''| < |u|$.

B5) If $\varphi(u)$ is not defined by (2), (3), (4) and (5) of the definition of φ, then the property follows from (6) of the definition.

The above discusion completes the inductive step for (IV). □

Let $Q=\varphi(\overline{B})$. If $u\in Q$ then there is $v\in\overline{B}$ such that $\varphi(v)=u$ and, by Proposition 2.2 c), we have:

$\varphi(u)=\varphi(\varphi(v))=\varphi(v)=u$.

It is clear that if $\varphi(u)=u$ then $u\in\varphi(\overline{B})=Q$. Hence, $Q=\{u|\ u\in\overline{B},\varphi(u)=u\}$.

We define mappings $*_i:Q\times Q\to Q$, $i\in\mathbb{N}_m$ by $x*_i y=\varphi(i,xy)$.

Lemma 2.6. *For each $i\in\mathbb{N}_m$, $(Q;*_i)$ are rectangular bands that satisfy* (i)*,* (ii) *and* (iii) *from Proposition* 1.4..

Proof. If $x,y\in Q$ then $(i,xy)\in\overline{B}$ and consequently $\varphi(i,xy)\in Q$. Hence, $*_i$ are well defined mappings i.e. $(Q;*_i)$ are groupoids for each $i\in\mathbb{N}_m$.

Let $x,y,z\in Q$, $i\in\mathbb{N}_m$. Then, using Lemmas 2.3. b) and 2.5. (I) and (II), we have:

$(x*_i y)*_i z=\varphi(i,\varphi(i,xy)z)=\varphi(i,(i,xy)z)=\varphi(i,xz)=x*_i z$,

and

$x*_i(y*_i z)=\varphi(i,x\varphi(i,yz))=\varphi(i,x(i,yz))=\varphi(i,xz)=x*_i z$.

Let $x\in Q$, $i\in\mathbb{N}_m$. Using Lemma 2.4. we have:

$x*_i x=\varphi(i,xx)=\varphi(x)=x$.

Hence, $(Q;*_i)$ are semigroups for each $i\in\mathbb{N}_m$. Moreover, $x*_i y*_i z=x*_i z$ and $x*_i x=x$, for each $i\in\mathbb{N}_m$. So, $(Q;*_i)$ are rectangular bands.

A) Let $j\le i$. Then, $(x*_i y)*_j z=\varphi(j,\varphi(i,xy)z)\overset{2.3.b)}{=}\varphi(j,(i,xy)z)$.

If $j=i$ then:

$\varphi(i,(i,xy)z)\overset{2.5.(\mathrm{I})}{=}\varphi(i,xz)=x*_i z=x*_i y*_i z=x*_i(y*_i z)=x*_i(y*_j z)$.

If $j<i$, then:

$\varphi(j,(i,xy)z)\overset{2.5.(\mathrm{III})}{=}\varphi(i,x(j,yz))\overset{2.3.b)}{=}\varphi(i,x\varphi(j,yz))=x*_i(y*_j z)$.

Hense, $(Q;*_i)$ are rectangular bands that satisfy (i) from Proposition 1.4.

B) Let $j\le i$. Then:

$(x*_j y)*_i z=\varphi(i,\varphi(j,xy)z)\overset{2.3.b)}{=}\varphi(i,(j,xy)z)\overset{2.5.(\mathrm{I})}{=}\varphi(i,xz)=x*_i z$.

Hence, $(Q;*_i)$ are rectangular bands that satisfy (ii) from Proposition 1.4.

C) Let $j\le i$. Then:

$x*_j(y*_i z)=\varphi(j,x\varphi(i,yz))\overset{2.3.b)}{=}\varphi(j,x(i,yz))\overset{2.5.(\mathrm{II})}{=}\varphi(j,xz)=x*_j z$.

Hense, $(Q;*_i)$ are rectangular bands that satisfy (iii) from Proposition 1.4. □

Let $[\,]:Q^{m+k}\to Q^m$ be the mapping defined by:

$(\forall x_1^{m+k}\in Q^{m+k})\left[x_1^{m+k}\right]_i=x_i*_i x_{i+k}$,

for each $i\in\mathbb{N}_m$. According to Proposition 1.4. and Lemma 2.6., $(Q;[\,])$ is an $(m+k,m)$-band.

Theorem 2.7. $(Q;[\,])$ *is a free* $(m+k,m)-$*band with a basis* B.

Proof. It is clear that $B\subseteq Q$. Let $\langle B\rangle$ be the $(m+k,m)-$subsemigroup of Q generated by B. Let $u=(i,xy)\in Q$ where $x,y\in\langle B\rangle$ and a be a fixed element of

B. Then, $\left[\overset{i-1}{a}\, x \overset{k-1}{a}\, y \overset{m-i}{a}\right]_i \in \langle B\rangle$, for each $i \in \mathbb{N}_m$, i.e.

$$u = \varphi(u) = \varphi(i, xy) = x *_i y = \left[\overset{i-1}{a}\, x \overset{k-1}{a}\, y \overset{m-i}{a}\right]_i \in \langle B\rangle.$$

Hence, $Q \subseteq \langle B\rangle$. Because $\langle B\rangle \subseteq Q$, it follows that $Q = \langle B\rangle$ and so $(Q; [\,])$ is a $(m + k, m)-$band generated by B.

Let $(Q'; [\,]')$ be another $(m + k, m)-$band generated by B and let $\lambda : B \to Q'$ be a mapping. By induction on the length we are going to define a mapping $g : Q \to Q'$ as follows:

$g(b) = \lambda(b)$, for $b \in B$

and

$$g(i, xy) = \left[\overset{i-1}{g(a)}\, g(x)\, \overset{k-1}{g(a)}\, g(y)\, \overset{m-i}{g(a)}\right]'_i.$$

Concidering the fact that on the right hand side of the definition of g, g is applied on elements with length less then the length of $u = (i, xy)$, it is obvious that g is a well defined mapping.

Let $x, y \in Q$. We will prove, by induction, that $g(\varphi(i, xy)) = g(i, xy)$. If $u = (i, xy) \in Q$ then $\varphi(u) = u$ and $g(\varphi(u)) = g(u)$. If $u = (i, xy) \notin Q$ then, since $x, y \in Q$, $\varphi(i, xy)$ is not defined by (1).

A) Let $u = (i, xx)$ i.e. $\varphi(u)$ is defined by (2). Then, by induction and using the identity (B V), we have:

$$g(\varphi((i, xx)) = g(\varphi((x)) = g(x) = \left[\overset{i-1}{g(a)}\, g(x)\, \overset{k-1}{g(a)}\, g(x)\, \overset{m-i}{g(a)}\right]'_i = g(i, xx).$$

We have applied the inductive hipothesis on x, since $|x| < |u|$.

B) Let $u = (i, (j, zw)y)$, $j \le i$, i.e. $\varphi(u)$ is defined by (3). Then, by induction and using the identity (B III), we have:

$$g(\varphi(i, (j, zw)y)) = g(\varphi(i, zy)) = g(i, zy) = \left[\overset{i-1}{g(a)}\, g(z)\, \overset{k-1}{g(a)}\, g(y)\, \overset{m-i}{g(a)}\right]'_i$$

$$= \left[\overset{i-1}{g(a)} \left[\overset{j-1}{g(a)}\, g(z)\, \overset{k-1}{g(a)}\, g(w)\, \overset{m-j}{g(a)}\right]'_j \overset{k-1}{g(a)}\, g(y)\, \overset{m-i}{g(a)}\right]'_i = \left[\overset{i-1}{g(a)}\, g(j, zw)\, \overset{k-1}{g(a)}\, g(y)\, \overset{m-i}{g(a)}\right]'_i$$

$= g(i, (j, zw)y)$.

We have applied the inductive hipothesis on (i, zy), since $|(i, zy)| < |u|$.

C) Let $\varphi(u)$ is defined by (4), i.e. $u = (i, x(j, zw))$, $i \le j$. Then, by induction and using the identity (B IV), we have:

$$g(\varphi(i, x(j, zw))) = g(\varphi(i, xw)) = g(i, xw) = \left[\overset{i-1}{g(a)}\, g(x)\, \overset{k-1}{g(a)}\, g(w)\, \overset{m-i}{g(a)}\right]'_i$$

$$\left[\overset{i-1}{g(a)}\, g(x)\, \overset{k-1}{g(a)} \left[\overset{j-1}{g(a)}\, g(z)\, \overset{k-1}{g(a)}\, g(w)\, \overset{m-j}{g(a)}\right]'_j \overset{m-i}{g(a)}\right]'_i = \left[\overset{i-1}{g(a)}\, g(x)\, \overset{k-1}{g(a)}\, g(j, zw)\, \overset{m-i}{g(a)}\right]'_i$$

$= g(i, x(j, zw))$.

We have applied the inductive hipothesis on (i, xw), since $|(i, xw)| < |u|$.

D) Let $\varphi(u)$ is defined by (5), i.e. $u = (i, (j, zw)y)$, $i < j$. Then, by induction and using the identity (B II), we have:

$$g(\varphi(i,(j,zw)y)) = g(\varphi(j,z(i,wy))) = g(j,z(i,wy)) = \left[\overset{j-1}{g(a)}\, g(z)\, \overset{k-1}{g(a)}\, g(i,wy)\, \overset{m-j}{g(a)}\right]'_j$$

$$= \left[\overset{j-1}{g(a)}\, g(z)\, \overset{k-1}{g(a)} \left[\overset{i-1}{g(a)}\, g(w)\, \overset{k-1}{g(a)}\, g(y)\, \overset{m-i}{g(a)}\right]'_i \overset{m-j}{g(a)}\right]'_j$$

$$= \left[\overset{i-1}{g(a)} \left[\overset{j-1}{g(a)}\, g(z)\, \overset{k-1}{g(a)}\, g(w)\, \overset{m-j}{g(a)}\right]'_j \overset{k-1}{g(a)}\, g(y)\, \overset{m-i}{g(a)}\right]'_i = \left[\overset{i-1}{g(a)}\, g(j,zw)\, \overset{k-1}{g(a)}\, g(y)\, \overset{m-i}{g(a)}\right]'_i$$

$$= g(i,(j,zw)y).$$

We have applied the inductive hipothesis on $(j,z(i,wy))$, since $|(j,z(i,wy))| < |u|$.

E) Let $u = (i,x(j,xz))$, $j < i$, i.e. $\varphi(u)$ is defined by (6). Then, by induction and using the identities (B V) and (B II) we have:

$$g(\varphi(i,x(j,xz))) = g(\varphi(j,xz)) = g(j,xz) = \left[\overset{j-1}{g(a)}\, g(x)\, \overset{k-1}{g(a)}\, g(z)\, \overset{m-j}{g(a)}\right]'_j$$

$$= \left[\overset{j-1}{g(a)} \left[\overset{i-1}{g(a)}\, g(x)\, \overset{k-1}{g(a)}\, g(x)\, \overset{m-i}{g(a)}\right]'_i \overset{k-1}{g(a)}\, g(z)\, \overset{m-j}{g(a)}\right]'_j$$

$$= \left[\overset{i-1}{g(a)}\, g(x)\, \overset{k-1}{g(a)} \left[\overset{j-1}{g(a)}\, g(x)\, \overset{k-1}{g(a)}\, g(z)\, \overset{m-j}{g(a)}\right]'_j \overset{m-i}{g(a)}\right]'_i$$

$$= \left[\overset{i-1}{g(a)}\, g(x)\, \overset{k-1}{g(a)}\, g(j,xz)\, \overset{m-i}{g(a)}\right]'_i = g(i,x(j,xz)).$$

We have applied the inductive hipothesis on (j,xz), since $|(j,xz)| < |u|$.

Let $x_j \in Q$, $j \in \mathbb{N}_{m+k}$. Then:

$$g(\left[x_1^{m+k}\right]_i = g(x_i *_i x_{i+k}) = g(\varphi(i,x_ix_{i+k})) = g(i,x_ix_{i+k})$$

$$= \left[\overset{i-1}{g(a)}\, g(x_i)\, \overset{k-1}{g(a)}\, g(x_{i+k})\, \overset{m-i}{g(a)}\right]'_i$$

$$\overset{(BI)}{=} [g(x_1)\dots g(x_{i-1})g(x_i)g(x_{i+1})\dots g(x_{i+k-1})g(x_{i+k})g(x_{i+k+1})\dots g(x_{m+k})]'_i,$$

for each $i \in \mathbb{N}_m$.

Hence, g is an $(m+k,m)-$homomorphism which is an extention for λ. So, $(Q;[\ ])$ is a free $(m+k,m)-$band with a basis B. □

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