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ON β-CONNECTED SPACE BETWEEN SUBSETS

ERDAL EKICI AND BILJANA KRSTESKA

Abstract. Two new concepts β-connected space between subsets and β-set-connected functions are introduced and some of their basic properties are studied.

1. Introduction

Noiri [6] introduced slightly β-continuous functions. This is a new weak form of both slightly continuity [4] and β-continuity [1]. In this paper we introduce the notions of β-connected spaces between subsets and β-set-connected functions. We show that within the class of surjective functions the classes of β-set-connected functions and slightly β-continuous functions coincide.

2. Preliminaries

Throughout this paper, by X we denote a topological space. Let A be a subset of X. We denote the interior, the closure and the complement of a subset A of X by $int(A)$, $cl(A)$ and A^c respectively.

A subset A of a space X is called β-open if $A \subset cl(int(cl(A)))$ [1]. The complement of a β-open set is called β-closed [1]. The intersection of all β-closed sets containing A is called β-closure [1] of A and is denoted by β-$cl(A)$. The β-interior of A is defined by the union of all β-open sets contained in A and is denoted by β-$int(A)$.

The family of all β-open (resp. β-closed, β-clopen, clopen) sets of X is denoted by $\beta O(X)$ (resp. $\beta C(X)$, $\beta CO(X)$, $CO(X)$).

Let $f : X \to Y$ be a function. A subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called a graph of f and is denoted by $G(f)$.

Definition 1. *A space X is β-connected if X can not be expressed as an union of two nonempty β-open sets [7].*

Definition 2. *A function $f : X \to Y$ is slightly β-continuous [6] if $f^{-1}(V) \in \beta O(X)$ for any clopen set V of Y.*

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3. β-CONNECTED SPACE BETWEEN SUBSETS

Definition 3. *A space X is β-connected between subsets A and B if there exists no β-clopen set K for which $A \subset K$ and $K \cap B = \varnothing$.*

Definition 4. *A function $f : X \to Y$ is β-set-connected if whenever X is β-connected between A and B, then $f(X)$ is connected between $f(A)$ and $f(B)$ with respect to relative topology on $f(X)$.*

Theorem 5. *A function $f : X \to Y$ is β-set-connected if and only if $f^{-1}(K)$ is β-clopen for every clopen subset K of $f(X)$ with respect to relative topology on $f(X)$.*

Proof. ($\Rightarrow$): Let K be any clopen subset of $f(X)$ with respect to the relative topology on $f(X)$. Suppose that $f^{-1}(K)$ is not β-closed in X. Then there exists $x \in X \backslash f^{-1}(K)$ such that for every β-open set U with $x \in U$, $U \cap f^{-1}(K) \neq \varnothing$.

Suppose that there exists a β-clopen set A such that $f^{-1}(K) \subset A$ and $x \notin A$. Then $x \in X \backslash A \subset X \backslash f^{-1}(K)$ and $X \backslash A$ is a β-open set containing x and disjoint from $f^{-1}(K)$. This contradiction implies that X is set β-connected between x and $f^{-1}(K)$.

Since f is β-set-connected, $f(X)$ is connected between $f(x)$ and $f(f^{-1}(K))$. So $f(f^{-1}(K)) \subset K$ and $f(x) \notin K$, is a contradiction. Hence, $f^{-1}(K)$ is β-closed in X. By using the complements, we obtain that $f^{-1}(K)$ is β-open.

($\Leftarrow$): Suppose that there exist subsets A and B of X for which $f(X)$ is not connected between $f(A)$ and $f(B)$ in relative topology on $f(X)$. Hence, there exists a set $K \subset f(X)$ that is clopen in the relative topology on $f(X)$ such that $f(A) \subset K$ and $K \cap f(B) = \varnothing$. Then $A \subset f^{-1}(K)$, $B \cap f^{-1}(K) = \varnothing$ and $f^{-1}(K)$ is β-clopen, which implies that X is not β-connected between A and B. We obtain that f is β-set-connected. □

Theorem 6. *Let $f : X \to Y$ be a function. If f is β-set-connected, then it is slightly β-continuous.*

Proof. Let F be a clopen subset of Y. Then $F \cap f(X)$ is clopen in the relative topology on $f(X)$. Since f is β-set-connected, it follows that
$f^{-1}(F) = f^{-1}(F \cap f(X))$ is β-clopen in X. □

Theorem 7. *Let $f : X \to Y$ be a function. If f is slightly β-continuous surjection, then it is β-set-connected.*

Proof. It follows from Theorem 5. □

Remark 8. *The following example shows that in general slightly β-continuity is not equivalent to β-set-connectedness.*

Example 9. *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a, b, c\}\}$ and $\upsilon = \{X, \varnothing, \{a\}, \{b, c\}, \{a, b, c\}\}$. We define a function $f : (X, \tau) \to (X, \upsilon)$ as follows*

$$f(a) = b,\ f(b) = b,\ f(c) = c,\ f(d) = a.$$

Then f is slightly β-continuous but it is not β-set-connected.

Theorem 10. *If X is a β-connected space and $f : X \to Y$ is slightly β-continuous surjection, then Y is connected [6].*

Definition 11. *A topological space X is called hyperconnected [8] if every open set in X is dense in X.*

Remark 12. *The following example shows that slightly β-continuous surjection does not preserve hyperconnectedness.*

Example 13. *Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$ and $\sigma = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is slightly β-continuous surjective. The space (X, τ) is hyperconnected. But (X, σ) is not hyperconnected.*

Definition 14. *A filter base Λ is β-convergent to a point x in X if for any $U \in \beta O(X)$ containing x, there exists $B \in \Lambda$ such that $B \subset U$.*

Definition 15. *A filter base Λ is co-convergent to a point x in X if for any $U \in CO(X)$ containing x, there exists $B \in \Lambda$ such that $B \subset U$.*

Theorem 16. *If $f : X \to Y$ is slightly β-continuous, then for each point $x \in X$ and each filter base Λ in X β-converging to x, the filter base $f(\Lambda)$ is co-convergent to $f(x)$.*

Proof. Let $x \in X$ and Λ be any filter base in X β-converging to x. Since f is slightly β-continuous, then for any $V \in CO(Y)$ containing $f(x)$, there exists $U \in \beta O(X)$ containing x such that $f(U) \subset V$. Since Λ is β-converging to x, there exists $B \in \Lambda$ such that $B \subset U$. This means that $f(B) \subset V$ and therefore the filter base $f(\Lambda)$ is co-convergent to $f(x)$. □

Definition 17. *A space X is β-T_1 [5] (resp. clopen T_1 [2]) if for each pair of distinct points x and y of X, there exist β-open (resp. clopen) sets U and V containing x and y respectively such that $y \notin U$ and $x \notin V$.*

Definition 18. *A space X is β-T_2 [5] (resp. clopen T_2 [2]) if for each pair of distinct points x and y in X, there exist disjoint β-open (resp.clopen) sets U and V in X such that $x \in U$ and $y \in V$.*

Definition 19. *A graph $G(f)$ of a function $f : X \to Y$ is strongly β-co-closed if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in \beta CO(X)$ containing x and $V \in CO(Y)$ containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 20. *A graph $G(f)$ of a function $f : X \to Y$ is strongly β-co-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in \beta CO(X)$ containing x and $V \in CO(Y)$ containing y such that $f(U) \cap V = \emptyset$.*

Theorem 21. *If $f : X \to Y$ is slightly β-continuous and Y is clopen T_1, then $G(f)$ is strongly β-co-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) \backslash G(f)$. Then $f(x) \neq y$ and there exists a clopen set V of Y such that $f(x) \in V$ and $y \notin V$. According to the assumption that f is slightly β-continuous, it follows that $f^{-1}(V) \in \beta CO(X)$ containing x. Put $U = f^{-1}(V)$. So $f(U) \subset V$. Therefore, $f(U) \cap (Y \backslash V) = \emptyset$ and $Y \backslash V \in CO(Y)$ containing y. Hence $G(f)$ is strongly β-co-closed in $X \times Y$. □

Theorem 22. *If $f : X \to Y$ be an injective function with a strongly β-co-closed graph $G(f)$, then X is β-T_1 space.*

Proof. Let x and y be any two distinct points of X. Then $(x, f(y)) \in (X \times Y) \backslash G(f)$. According to the definition of strongly β-co-closed graph, there exist a β-clopen set U of X and $V \in CO(Y)$ such that $(x, f(y)) \in U \times V$ and $f(U) \cap V = \emptyset$. Hence $U \cap f^{-1}(V) = \emptyset$. Therefore, $y \notin U$. Thus it follows that X is β-T_1. □

Definition 23. *A function $f : X \to Y$ is always β-open [3] if $f(V) \in \beta O(Y)$ for each $V \in \beta O(X)$.*

Theorem 24. *If $f : X \to Y$ is a surjective always β-open function with a strongly β-co-closed graph $G(f)$, then Y is β-T_2 space. .*

Proof. Let y_1 and y_2 be any distinct points of Y. Since f is surjective $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) \backslash G(f)$. According to the assumption that $G(f)$ is a strongly β-co-closed graph, there exist a β-clopen set U of X and $V \in CO(Y)$ such that $(x, y_2) \in U \times V$ and $(U \times V) \cap G(f) = \emptyset$. Then, $f(U) \cap V = \emptyset$. Since, f is always β-open, it follows that $f(U)$ is β-open such that $f(x) = y_1 \in f(U)$. Hence Y is β-T_2. □

Definition 25. *A β-frontier of a subset A of X is β-$fr(A) = \beta$-$cl(A) \cap \beta$-$cl(X \backslash A)$.*

Theorem 26. *The set of all points $x \in X$ in which a function $f : (X, \tau) \to (Y, \sigma)$ is not slightly β-continuous is the union of β-frontier of the inverse images of clopen sets containing $f(x)$.*

Proof. Suppose that f is not slightly β-continuous at $x \in X$. Then there exists a clopen set V of Y containing $f(x)$ such that $f(U)$ is not contained in V for each $U \in \beta O(X)$ containing x. Then $U \cap (X \backslash f^{-1}(V)) \neq \varnothing$ for every $U \in \beta O(X)$ containing x and hence $x \in \beta$-$cl(X \backslash f^{-1}(V))$. On the other hand $x \in f^{-1}(V) \subset \beta$-$cl(f^{-1}(V))$ and hence $x \in \beta$-$fr(f^{-1}(V))$.

Conversely, suppose that f is slightly β-continuous at $x \in X$ and let V be a clopen set of Y containing $f(x)$. Then there exists $U \in \beta O(X)$ containing x such that $U \subset f^{-1}(V)$. Hence $x \in \beta$-$int(f^{-1}(V))$. Therefore, $x \notin \beta$-$fr(f^{-1}(V))$ for each clopen set V of Y containing $f(x)$. □

Theorem 27. *If f is a slightly β-continuous function from a β-connected space X onto space Y, then Y is not a discrete space.*

Proof. Suppose that Y is discrete. Let A be a proper nonempty open subset of Y. Then $f^{-1}(A)$ is any proper nonempty β-clopen subset of X, which is a contradiction to the assumption that X is β-connected. □

Theorem 28. *A space X is β-connected if every slightly β-continuous function from a space X into any T_0-space Y is constant.*

Proof. Suppose that X is not β-connected and every slightly β-continuous function from X into Y is constant. Since X is not β-connected, there exists a proper nonempty β-clopen subset A of X. Let $Y = \{a, b\}$ and $\tau = \{Y, \varnothing, \{a\}, \{b\}\}$ be a topology of Y. Let $f : X \to Y$ be a function such that $f(A) = \{a\}$ and $f(X \backslash A) = \{b\}$. Then f is non-constant and slightly β-continuous such that Y is T_0, which is a contradiction. Hence, X is β-connected. □

Theorem 29. *If $f : (X, \tau) \to (Y, \sigma)$ is a slightly β-continuous injection and (Y, σ) is clopen T_2, then (X, τ) is β-T_2.*

Proof. Let x, y be any distinct points of X. Then $f(x) \neq f(y)$. Since (Y, σ) is clopen T_2, there exist clopen sets U, V in Y containing $f(x)$, $f(y)$, respectively, such that $U \cap V = \emptyset$. Since f is slightly β-continuous, there exist G, $H \in \beta O(X)$ containing x, y, respectively, such that $f(G) \subset U$ and $f(H) \subset V$. This implies that $G \cap H = \emptyset$. Hence (X, τ) is β-T_2. □

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Department of Mathematics, Canakkale Onsekiz Mart University,Terzioglu Campus, 17020 Canakkale/TURKEY

Faculty of natural Sciences and Mathematics, Institute of Mathematics, P.O. Box 162, 1000 Skopje, MACEDONIA

E-mail address: eekici@comu.edu.tr

E-mail address: biljanak@iunona.pmf.ukim.edu.mk