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DIRECTIONALLY NONLINEAR CONNECTION IN SPACETIME MANIFOLD

EMILIJA G. CELAKOSKA [1)]

Dedicated to Academician Blagoj Popov on the Occasion of His 85^{th} Birthday

Abstract. We introduce a covariant derivative in direction of specific vectors making subset H of the fiber. The corresponding connection is directionally nonlinear since the subset H does not have subspace structure, so it does not make a distribution. It follows that the directional vector can not be separated from the connection coefficients. Thus we make a connection tensor instead of standard $\Gamma^k_{ij}V^i$. The connection is also nonholonomic because it allows directionally limited parallel transport only along the vectors from H. The elements of the subset H of the fiber correspond to the 4-velocities on the spacetime manifold.

1. Introduction

In the standard formal definition, a fiber bundle (E, B, π, F) is given by C^∞ base space B, total space E, a continuous surjective map $\pi : E \to B$ and an algebraic structure of $\pi^{-1}(x)$ for every $x \in B$ so that the compatibility condition holds: For every $x \in B$ there is an open neighborhood U, a fiber space F and a homeomorphism $\varphi : U \times F \to \pi^{-1}(U)$ such that for every $x \in U$ it is $\pi\varphi(x, v) = x$ for all $v \in F$ and the map $v \mapsto \varphi(x, v)$ gives an isomorphism of F and $\pi^{-1}(x)$. The structure group G is a Lie group of symmetries which provide the matching conditions between overlapping charts, that is, for any two overlapping charts (U_i, φ_i) and (U_j, φ_j),

$$\varphi_i^{-1}\varphi_j : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F, \quad \varphi_i^{-1}\varphi_j(x, \xi) = (x, t_{ij}(x)\xi)$$

where the continuous map $t_{ij} : U_i \cap U_j \to G$ is transition function satisfying $t_{ii}(x) = 1$, $t_{ij}(x) = t_{ji}(x)^{-1}$ and the cocycle condition $t_{ik}(x) = t_{ij}(x)t_{jk}(x)$ on a triple overlap.

In the special case of tangent bundle, the fiber is a vector space with dimension n, at a point. In local coordinates $(x_1, ..., x_n)$ the vector fields $\frac{\partial}{\partial x_i}$ span this vector space. Transition function from these coordinates to another set of coordinates is given by the Jacobian of the coordinate change, which makes an equivalence class of tangent vector representations at a point i.e. a tangent vector.

The next natural step is to make a parallel transport connecting the fibers which means introducing a connection that will characterize the covariant derivative. Every connection is determined with the relation

$$\nabla_{\partial_j}\partial_k = \Gamma^i_{jk}\partial_i \tag{1}$$

where ∂_i stands for the basis vectors $\frac{\partial}{\partial x_i}$ of the vector space. So, in the local chart it is obviously $\nabla_{\partial_j} Y = \partial_j Y^i + \Gamma^i_{jk} Y^k$ which is the classical expression for covariant

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differentiation and $\nabla_X \partial_k = \omega_k^i(X)\partial_i$, which is (1,1) tensor field where ω_k^i are 1-forms. So,

$$\Gamma_{jk}^i \equiv \omega_k^i(\partial_j) \tag{2}$$

and it is not necessarily a tensor. In the classical definition of connection the linearity property is consisted, meaning that the map $F \times F \to F$ satisfy the linearity conditions on the both arguments,

$$(i) \quad \nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z,$$
$$(ii) \quad \nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z.$$

The most well-known example in Riemannian geometry and used in General Relativity is the Levi-Civita connection, which is metric-compatibile. Given a metric g_{ij} on the manifold B, it is $\nabla_{\partial_k} g_{ij} = 0$ or, equivalently,

$$\Gamma_{jk}^i = \frac{1}{2} g_{im} \left(\partial_k g_{mj} + \partial_j g_{mk} + \partial_m g_{jk}\right)$$

So, constructing a connection directly from the metric, the curvature and the geodesics are easy to express. Moreover, from the metric as a field tensor and its derivatives in the form of the Ricci scalar the Lagrangian is constructed, so the action is the well known Einstein-Hilbert action delivering the Einstein field equations. However, it is not the case that Levi-Civita connection and metric provide unique description of spacetime with gravitation. A connection may contain not only curvature, but also torsion and/or nonmetricity and there are many such theories describing gravitation. In more general theories [1] than General Relativity (GR), like for example Einstein-Cartan and gauge theories for the Poincare group the gravitation is carried by curvature and torsion. The most general linear connections used in metric-affine theories introduce nonmetricity to carry partially or completely the gravitation field [2], [3].

According to this point of view, torsion and nonmetricity represent additional degrees of freedom, and consequently new physics phenomena might be associated with them [4]. Note that curvature, torsion and nonmetricity are properties of a connection not of a manifold. This is implied by their resident appearance in the bundle and therefore the possibility to define many different connections on the same manifold.

2. Tangent Vectors on Spacetime Manifold

Among the roles played by the tangent vectors, the most fundamental is the role of the velocity vector along a curve. For a given vector field X on B, a curve $l : (a, b) \to B,\ a, b \in \mathbb{R}$ is the integral curve of X if the vector $\dot{l}(t) \in T_{l(t)}B$ at each point coincides with the value of X at that point. There is a system of differential equations that should be satisfied by the integral curve through an arbitrary point $p \in B$. Choosing for simplicity $l(0) = p$ in the local chart $(U; x_1, ..., x_n)$ with $X = X^i(p)\partial_i$ we have $l(t) = (x_1(t), ..., x_n(t))$, so $\dot{l}(t) = \frac{dx^i(t)}{dt}\partial_i$ and the system of differential equations that should be satisfied by the curve is $\frac{dx^i(t)}{dt} = X^i(x_1(t), ..., x_n(t))$. This is a system of ordinary differential equations of first order, so a solution passing through p exists and it is unique. The vector

$$\dot{l}(t) = \frac{dx^i(t)}{dt}\partial_i \tag{3}$$

is velocity vector and the differentiation is by time parameter.

For the spacetime manifold, we will refer spacetime velocities as 4-velocities. In this case, differentiation by a time parameter and the presence of time coordinate axis in a local chart like in (3) make 4-velocity vector a notion with a compound structure. Treating t as a coordinate, $\frac{d}{dt}$ is the basis vector ∂_0. The tangent bundle can be considered as a fiber bundle associated to the principal vierbein bundle $L(B)(B, O(1,3))$ with structure group $O(1,3)$ constructed in such a way that a vierbein u at $x \in B$ can be uniquely represented as $u = (X_0, X_1, X_2, X_3)$ where $X_k = X_k^i \partial_i$, $X_k^i \in O(1,3)$. Since

$$\partial_0(x^i(t)) \equiv \frac{dx^i(t)}{dt} = X^i(x^0(t), x^1(t), x^2(t), x^3(t))$$

i.e. we can denote it with X_0^i it follows that (3) is the vector X_0 in the vierbein field.

Morphisms among 4-velocities are orthogonal inhomogeneous Lorentz coordinate transformations, which include rotations, reflections and pure hyperbolic rotations, usually called boosts.

Now, we can make the following conclusions.

(a) On the set H of 4-velocities only boosts make practical sense as transition functions, since tensorial $O(3)$ rotations are not possible for different 4-velocities, and reflections are not in the connected component $SO(1,3)$. However, all transformations from $O(1,3)$ are admissible.

(b) 4-velocities are determined with the 3 dimensional local spatial basis. The time component turns into a normalizator coefficient determined by coefficients in front of the local coordinates that in practice may be given.

(c) Addition operation inherited from the vector space $\mathbb{R}^4$ is not closed.

Considering (c), it follows that the velocity vectors make only a subset H, not a vector subspace, and combining this with (b) it follows that it is not possible to construct a distribution. Namely, the fourth component is non-constant since it depends on the other three and moreover, it never acquires the value 0. While all standard structures on the bundle remain well defined, this situation obviously affects linearity properties (i) and (ii) of connections given in the introduction. There is a way [5], [6], to define another addition operation on H, so that H is closed [7] and moreover, to become a loop space. However, since in two consecutive applications of the operation, additional terms describing a rotation must appear, as the set of boosts is not closed, the connection still remains void of linearity properties due to such operation. It is an operation marked with $\oplus$ defined for spatial velocity vectors

$$\vec{u} \oplus \vec{v} = \frac{1}{1 + \frac{\vec{u}\cdot\vec{v}}{c^2}} \left(\vec{u} + \sqrt{1 - \frac{u^2}{c^2}}\vec{v} + \frac{1}{c^2} \frac{1}{1 + \sqrt{1 - \frac{u^2}{c^2}}} (\vec{u} \cdot \vec{v})\vec{u} \right)$$

and the multiplication with scalar is

$$\alpha \circ \vec{v} = \frac{c}{v} \frac{\left(1 + \frac{v}{c}\right)^{\alpha} - \left(1 - \frac{v}{c}\right)^{\alpha}}{\left(1 + \frac{v}{c}\right)^{\alpha} + \left(1 - \frac{v}{c}\right)^{\alpha}} \vec{v}$$

where c is the limit velocity magnitude implied by the pseudo signature of the Lorentz group elements and physically is considered as the velocity of electromagnetic wave, or simply, speed of light. The addition is non-commutative and non-associative while multiplication by a scalar is non-distributive i.e. $\alpha \circ (\vec{u} \oplus \vec{v}) \neq \alpha\vec{u} \oplus \alpha\vec{v}$. Now, if we apply these operations on the covariant derivative of the standard linear connection used in GR to the corresponding tangent vectors of spacetime velocities U, V we obtain

$$\nabla_{U \oplus V} = (U^i \oplus V^i)\partial_j + \Gamma^i_{jk}(U^k \oplus V^k),$$

$$\nabla_{\alpha\circ V} = (\alpha \circ V^i)\partial_j + \Gamma^i_{jk}(\alpha \circ V^k).$$

It is obvious that linearity in these covariant derivatives does not hold. However, the bundle structure is not affected by this, since we use the inherited addition and multiplication by scalar from the vector space $\mathbb{R}^4$. Thus, in order to respect the nonlinear nature of the operations on velocity vectors, we accept that the connection should be constructed as nonlinear.

3. Connection Tensor

We assume space-time to be a 4-dimensional differentiable manifold B endowed with a Lorentzian metric g of signature $(+---)$. Our connection requires a globally Lorentzian spacetime that is a strong restriction imposed to B, but compatible with observations [12]. We suppose that on B there is a congruence of timelike geodesics being the integral curves of the tangent vector field U. We shall restrict tangent bundle to the normalized vector fields $U \equiv g(U,U)^{-1/2}U$ which represent tangent vectors parameterized by their proper time, i.e. tangent vectors of 4-velocity. At each point of space-time, there is a tangent space attached to it, given by a Minkowski space, which will be the fiber of the corresponding tangent bundle. The indices in tangent space are raised and lowered by the Minkowski metric $\eta = diag(1,-1,-1,-1)$ or its inverse. Thus, the vector field U has a constant magnitude implied by $\eta(U,U) = U^iU_i = c^2$, where c is the speed of light.

The 4-dimensional spacetime manifold B can be considered as a configuration space of a nonholonomic structure, where nonholonomity is expressed by the reduction of the tangent bundle TB on the space of 4-velocities. It means that geodesics are restricted to the space of 4-velocities, i.e. the geodesic with initial 4-velocity evolve so that all subsequent velocities are 4-velocities. However, since the space of 4-velocities is hyperbolic (nonlinear) [8] and it is only a subset (not subspace) of tangent space at each point, we are not able to straightforwardly apply the apparatus based on linear vector distributions, since a true distribution can not be defined. Thus, we shall build a nonholonomic connection step by step. In a tangent vector basis

$$\{\mathbf{e_0} = \frac{1}{c}\partial t, \mathbf{e_1} = \partial x, \mathbf{e_2} = \partial y, \mathbf{e_3} = \partial z\}$$

of the spacetime coordinates (ct, x, y, z) the 4-vector of velocity has a constant magnitude c and takes the form

$$W = (W^0, W^1, W^2, W^3) = \gamma_w(c, w_x\frac{\partial x}{\partial t}, w_y\frac{\partial y}{\partial t}, w_z\frac{\partial z}{\partial t}),$$

$\gamma_w = \frac{1}{\sqrt{1-\frac{w^2}{c^2}}}$ and $\vec{w} = (w_x, w_y, w_z)$ is the corresponding 3-vector of velocity with magnitude w.

To achieve a suitable nonholonomic connection we shall start with an antisymmetric connection form ϕ_{ij} obtained as a exterior derivative of the normalized 4-velocity covector

$$\frac{1}{c}W_{\#} = \frac{1}{c}\eta_{ki}W^i$$

i.e.

$$\frac{1}{c}W_{\#} = \frac{1}{c}(W_0dt + W_1dx + W_2dy + W_3dz).$$

The corresponding exterior derivative of $W_{\#}$ gives

$$dW_{\#} = \left(\frac{1}{c}\frac{\partial W_1}{\partial t} - \frac{\partial W_0}{\partial x}\right) dt \wedge dx + \left(\frac{1}{c}\frac{\partial W_2}{\partial t} - \frac{\partial W_0}{\partial y}\right) dt \wedge dy +$$

$$+\left(\frac{1}{c}\frac{\partial W_3}{\partial t}-\frac{\partial W_0}{\partial z}\right)dt\wedge dz+\left(\frac{\partial W_2}{\partial x}-\frac{\partial W_1}{\partial y}\right)dx\wedge dy+$$
$$+\left(\frac{\partial W_3}{\partial x}-\frac{\partial W_1}{\partial z}\right)dx\wedge dz+\left(\frac{\partial W_3}{\partial y}-\frac{\partial W_2}{\partial z}\right)dy\wedge dz$$

that results in the following connection form

$$\phi_{ij}(W_\#)=\frac{1}{c}\begin{bmatrix}0 & a_x-\sigma_x & a_y-\sigma_y & a_z-\sigma_z\\ \sigma_x-a_x & 0 & \omega_z & -\omega_y\\ \sigma_y-a_y & -\omega_z & 0 & \omega_x\\ \sigma_z-a_z & \omega_y & -\omega_x & 0\end{bmatrix}=\frac{1}{c}\begin{bmatrix}0 & \vec{a}^T-\vec{\sigma}^T\\ \vec{\sigma}-\vec{a} & \Omega\end{bmatrix}$$

where

$$\vec{\sigma}-\vec{a}=(\sigma_x-a_x,\sigma_y-a_y,\sigma_z-a_z)=$$
$$=\left(\frac{\partial W_0}{\partial x}-\frac{1}{c}\frac{\partial W_1}{\partial t},\frac{\partial W_0}{\partial y}-\frac{1}{c}\frac{\partial W_2}{\partial t},\frac{\partial W_0}{\partial z}-\frac{1}{c}\frac{\partial W_3}{\partial t}\right)=\operatorname{grad}W_0-\frac{1}{c}\frac{\partial\vec{w}}{\partial t}$$

is a 3-vector of difference between gradient of W_0 given by $\vec{\sigma}$ and acceleration $\vec{a}$, while

$$\vec{\omega}=(\omega_x,\omega_y,\omega_z)=\left(\frac{\partial W_2}{\partial z}-\frac{\partial W_3}{\partial y},\frac{\partial W_3}{\partial x}-\frac{\partial W_1}{\partial z},\frac{\partial W_1}{\partial y}-\frac{\partial W_2}{\partial x}\right)=$$
$$=\left(\frac{\partial W^3}{\partial y}-\frac{\partial W^2}{\partial z},\frac{\partial W^3}{\partial x}-\frac{\partial W^1}{\partial z},\frac{\partial W^2}{\partial x}-\frac{\partial W^1}{\partial y}\right)=\operatorname{rot}\vec{w}$$

is the 3-vector of angular velocity.

We define our connection tensor by

$$\Phi=\phi^i_j=\eta^{ik}\phi_{kj}=\frac{1}{c}\begin{bmatrix}0 & \vec{a}^T-\vec{\sigma}^T\\ \vec{\sigma}-\vec{a} & \Omega\end{bmatrix}$$

which is antisymmetric in space-space entries and symmetric in space-time entries. It is obvious that the tensor Φ can be written uniquely as

$$\Phi=\frac{1}{c}\begin{bmatrix}0 & \vec{a}^T-\vec{\sigma}^T\\ \vec{\sigma}-\vec{a} & \Omega\end{bmatrix}=$$
$$=\frac{1}{c}\begin{bmatrix}0 & 0\\ 0 & \Omega\end{bmatrix}+\frac{1}{c}\begin{bmatrix}0 & \vec{a}^T-\vec{\sigma}^T\\ \vec{\sigma}-\vec{a} & 0\end{bmatrix}=T+N$$

where the first tensor T is antisymmetric and the second tensor N is symmetric. The antisymetric part T can be interpreted as a tensor of torsion and the symmetric part N is interpreted as a tensor of nonmetricity. Physically, the 4-velocity W can be interpreted as an "internal" velocity of the gravitational source, e.g. spin + moving through time ($\vec{\sigma}$). The gradient of W_0 represented by $\vec{\sigma}$ carries gravitation of the gravitational source in rest.

4. Introduction of the Nonlinear Connection

The tensor Φ is antisymmetric in space-space entries and symmetric in space-time entries and so, it takes values in the Lie algebra of the Lorentz group. To complete our connection we shall modify the action of the tensor Φ by a tensor P from the Lorentz group in a way $P^{-1}\Phi P$ to obtain a most general connection based on Φ that remains in the Lie algebra of the Lorentz group. Physically interpreted, the goal of the tensor P is to "synchronize" 4-velocities of the gravitational source and the test body. Thus, we introduce P as an isometry link in the space of 4-velocities. This isometry is a Lorentz link by boost that links given 4-velocity U and V [9], [10], [11] given by

$$P(U,V) = \delta - \frac{(U+V)\otimes g(U+V)}{c^2 + U\cdot V} + 2\frac{V\otimes gU}{c^2}, \quad (U+V)^2 \neq 0 \tag{4}$$

Important property of the isometry link (4) is related to the relationship between Lorentz boost and orthonormal basis. Namely, in Minkowski space, if $\{U, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is an orthonormal basis in a frame of the tangent vector U, then $\{V, P(U,V)\mathbf{e_1}, P(U,V)\mathbf{e_2}, P(U,V)\mathbf{e_3}\}$ is the corresponding orthonormal basis in the frame of the tangent vector V which is parallel to those of U. Thus, by introducing Lorentz boost P, we achieve the connection S given in tensorial form by

$$S = P^{-1}\Phi P = \eta P^T \eta \Phi P \tag{5}$$

or rewritten in the local chart

$$S_i^j = (\eta P^T \eta)_i^k \Phi_m^j P_k^m.$$

The connection defined by (5) is nonmetric because Φ is nonmetric. Additionally, the connection S is directionally nonlinear and should be considered as nonholonomic because it can not be separated from the tangent vector V, i.e. V cannot be extracted from S. The directional nonlinearity of the S is expected and even desirable because the directions of parallel transport are tangent vectors of 4-velocity. It can be checked that

$$\nabla_{\alpha X+Y} \neq \alpha\nabla X + \nabla Y$$

where X and Y are tangent vectors of 4-velocity and α is a constant. This directional nonlinearity is justified by the nonlinearity of relativistic velocity addition and multiplication by a scalar.

In the forthcoming paper we shall give the geodesics based on the introduced nonlinear connection and show that the standard relativistic effects get the same outcome as in GR although the corresponding results are obtained in a rather different way.

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Department of Mathematics an Informatics
Faculty of Mechanical Engineering
St. Cyril and Methodius University
E-mail address: `cemil@mf.edu.mk`