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(3,1,ρ)-METRIZABLE TOPOLOGICAL SPACES

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Abstract. In this paper we show that a topological space is pseudo-o-metrizable if and only if it is a $(3,1,\rho)$-metrizable. As a corollary we obtain that a topological space is o-metrizable if and only if it is a $(3,1,\triangle)$-metrizable. At the end it is shown that a topological space is simmetrizable iff it is a $(3,1,\triangle)$-metrizable, with a metric satisfying $d(x,x,y) = d(x,y,y)$.

1. Generalized metrics

The geometric problems in metric spaces, and their axiomatic classification have been considered in [Me]. Later, several notion of generalized metrics have been introduced, like the notion of symmetric, o-metric and pseudo-o-metric in [S] and [M], 2-metric in [G1], m-metric in [G2], and generalized metric in [U]. In [D] we have introduced the notion of (n,m,ρ)-metrics. The goal of this paper is to examine the connection between some of those notions and their induced topological structures

First we will define some of these notions.

Let M be a given nonempty set, and let $d : M \times M \mapsto \mathbb{R}_0^+$ be a given map, where $\mathbb{R}_0^+$ is the set of the nonnegative real numbers. We consider the following axioms;

(d_1) For each $x \in M, d(x,x) = 0$;

(d_2) For each $x, y \in M$, $d(x,y) = 0$ if and only if $x = y$;

(d_3) For each $x, y \in M$, $d(x,y) = d(y,x)$;

(d_4) For each $x, y, z \in M$, $d(x,y) \leq d(y,z) + d(z,x)$.

A map d satisfying (d_1) is called **real distance** in [M], and **pseudo-o-metric** in [N].

A map d satisfying (d_2) is called **o-metric** in [N].

A map d satisfying (d_2) and (d_3) is called **symmetric** in [N].

The axioms (d_2), (d_3) and (d_4) are the axioms for the usual notion of a metric.

The notion of **2-metric** is defined in [G1] as a map $d : M \times M \times M \mapsto \mathbb{R}_0^+$ satisfying:

(m_1) For each $x, y \in M$, there is $z \in M$, such that $d(x,y,z) \neq 0$;

(m_2) For each $x, y, z \in M$, $d(x,y,z) = 0$ iff at least two of the three points x, y, z are equal;

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(m_3) For each $x, y, z \in M$, $d(x, y, z) = d(x, z, y) = d(y, z, x)$; and

(m_4) For each $x, y, z, t \in M$, $d(x, y, z) \leq d(x, y, t) + d(x, t, z) + d(t, y, z)$.

The generalization to m-metric is given in [G2].

In [D] we have generalized the notion of equivalence to the notion of an (n,m)-equivalence, and used it to define the notion of (n, m, ρ)-metric. We recall their definitions.

Let n, m be two positive integers, such that $n - m = k > 1$, and let M be a nonempty set.

Let M^n denote the n^{th} Cartesian power of M. We will use the notation $\mathbf{x} = a_1 a_2 \dots a_n$ or just $\mathbf{x} = a_1^n$ instead of $x = (a_1, a_2, \dots, a_n)$ for the elements $\mathbf{x} \in M^n$. For $x \in M$, we denote the element $(x, x, \dots, x)$ by x^n.

Definition 1. *The* **n-fold permutation product** *of M, i.e. the* $\mathbf{n}^{th}$**-symmetric power** *of M, is the set $M^{(n)} = M^n / \sim$, where $\sim$ is the equivalence relation defined on M^n by*

$$x_1^n \sim y_1^n \text{ if and only if } (x_1, \dots, x_n) \text{ is a permutation of } (y_1, \dots, y_n). \quad (1)$$

We will use the same notation $\mathbf{x} = a_1^n$ for the elements in $M^{(n)}$ keeping in mind that $a_1^n = b_1^n$ in $M^{(n)}$, for $a_i, b_i \in M$, iff $(b_1, b_2, \dots, b_n)$ is a permutation of $(a_1, a_2, \dots, a_n)$.

Definition 2. *A subset ρ of $M^{(n)}$ is called* **symmetric n-relation** *on M. A symmetric n-relation on M is called* **reflexive n-relation** *on M if for each a in M, a^n is in ρ. A symmetric n-relation on M is called* **transitive (n,m)-relation** *on M, i.e.* **(n,m)-transitive**, *if for each $\mathbf{x}$ in $M^{(n)}$ and each $\mathbf{b}$ in $M^{(m)}$,*

$$(\mathbf{ub} \in \rho \text{ for each } \mathbf{u} \in M^{(k)}, n = m + k, \text{ with } \mathbf{uv} = \mathbf{x}) \text{ implies } \mathbf{x} \in \rho \quad (2)$$

A reflexive n-relation on M which is (n,m)-transitive is called **(n,m)-equivalence** *on M. Instead of saying transitive (n,1)-relation and (n,1)-equivalence, we say only transitive n-relation on M, and n-equivalence on M.*

With these notions, a 2-equivalence is the usual notion of an equivalence relation.

Example 1. (1) The set $\Delta = \{x^n \mid x \text{ in } M\}$ is an (n,m)- equivalence on M for each $1 < m < n$.

(2) The set $\nabla = \{(x, x, y) \mid (x, x, y) \in M^{(3)}\}$ is a (3,1)- equivalence on M.

(3) The set Col= $\{(A, B, C) \mid A, B, C$ are colinear points in $E^2\}$ is a (3,t)-equivalence on E^2, for $t = 1, 2$, where E^2 is the euclidean plane.

(4) The set Com= $\{(A, B, C, D) \mid A, B, C, D$ are complanar in $E^3\}$ is a (4,t)-equivalence on E^3, for $t = 1, 2, 3$, where E^3 is the euclidean 3-dimensional space.

Definition 3. *Let ρ be an (n,m)-equivalence on M. A map $d : M^{(n)} \mapsto \mathbb{R}_0^+$, satisfiying the following axioms:*

(i) $d(\mathbf{x}) = 0$ iff $\mathbf{x} \in \rho$; and

(ii) For each $\mathbf{a} \in M^{(m)}$ and each $\mathbf{x} \in M^{(n)}$, $d(\mathbf{x}) \leq \sum d(\mathbf{ua})$, where the sum is over all the $\mathbf{u} \in M^{(k)}$ such that there is a $\mathbf{v} \in M^{(m)}$ with $\mathbf{uv} = \mathbf{x}$ in $M^{(n)}$;

is said to be an **(n,m,ρ)-metric** *on M, and the pair (M,d) is said to be* **(n,m,ρ)-metric space**.

The sum in (ii) is over all the parts $\mathbf{u} \in M^{(k)}$ *of* $\mathbf{x}$, *where* $n = m + k$. *In the case* $m = 1$, *instead of saying* $(n, 1, \rho)$*-metric we say only* (n, ρ)*-metric. When there is no ambiguity about the* (n, m)*-equivalence* ρ, *we omit it and write only* (n, m)*-metric instead of* (n, m, ρ)*-metric and n-metric instead* (n, ρ)*-metric.*

With the above notions, the notion of a $(2, \triangle)$-metric is the same with the usual notion of metric, while the notion of $(3, 1, \nabla)$-metric is the notion of 2-metric as defined in [G1].

Example 2. Let $\triangle$ be the (n,m)-equivalence defined in Example 1, (1), and let $d : M^{(n)} \mapsto \mathbb{R}_0^+$ be defined by $d(\mathbf{x}) = 0$ iff $\mathbf{x} \in \triangle$, and $d(x) = 1$ otherwise. Then it is easy to check that d is an $(n, m, \triangle)$-metric, and so, (M, d) is an $(n, m, \triangle)$-metric space. We call this $(n, m, \triangle)$-metric and $(n, m, \triangle)$-metric space, **discrete (n,m)-metric** and **discrete (n,m)- metric space.**

Example 3. Let $P : (E^2)^{(3)} \mapsto \mathbb{R}_0^+$ and $V : (E^3)^{(4)}\mathbb{R}_0^+$ be defined by:

P(A,B,C)=the area of the triangle whose vertices are the points A, B, and C; and

V(A,B,C,D)=the volume of the tetrahedron whose vertices are the points A, B, C, D.

In the case when A,B,C are collinear, P(A,B,C)=0, and when A,B,C,D are coplanar, V(A,B,C,D)=0.

Then, it can be checked that P is a (3,Col)-metric on E^2 and V is a (4,Com)-metric on E^3, i.e. (E^2, P) is a (3,Col)-metric space, and (E^4, V) is a (4,Com)-metric space.

2. Topological spaces metrizable by generalized metrics

Next, we will consider only $(3, 1, \rho)$-metrics, and the topologies induced by them. Let (M,d) be a $(3, 1, \rho)$-metrics space. It is possible to define open balls as follows. Let $x, y \in M$, $r > 0$ and let:

(OB1) $B(x, y, r) = \{z \mid d(x, y, z) < r\}$, the open ball in M with center (x,y) and radius r;

(OB2) $B(x, x, r) = \{z \mid d(x, x, z) < r\}$, the open ball in M with center x and radius r;

(OB3) $B(x, r) = \{(u, v) \mid d(x, u, v) < r\}$, the open ball in $M^{(2)}$ with center x and radius r.

For all the three definition of open ball, the collection of all open bals is not a base for a topology. Using these collection of open balls as generating sets we define three topologies:

(T1) The topology on M with a base the set of all finite intersections of OB1;

(T2) The topology on M with a base the set of all finite intersections of OB2;

(T3) The topology on $M^{(2)}$ with a base the set of all finite intersections of OB3.

But following the approach as in [N] and [M], we say that a topological space (M, τ) is:

(3M) $(\mathbf{3},\mathbf{1},\rho)$**-metrizable**, if there is a $(3,1,\rho)$-metric on M such that U is open iff for each x in U, there is $r>0$, such that $B(x,x,r)\subseteq U$;

(3WM)) **weak** $(\mathbf{3},\mathbf{1},\rho)$**-metrizable** , if there is a $(3,1,\rho)$-metric on M such that U is open iff for each x in U, there is $r>0$, such that $B(x,r)\subseteq U^{(2)}$;

(3SM)) **strong** $(\mathbf{3},\mathbf{1},\rho)$**-metrizable**, if there is a $(3,1,\rho)$-metric on M such that τ has a base the following collection: $\{U \mid U\subseteq M$, for each $(x,y)\in U^{(2)}$, there is $r>0$, such that $B(x,y,r)\subseteq U\}$.

The proof of the following proposition is straightforward.

Proposition 1. *Let M be a given $(3,1,\rho)$-metric space. The collection of all the sets defined to be open in (3M) together with the empty set is a topology on M. The collection of all the sets defined to be open in (3WM) together with the empty set is a topology on M. The collection $\{U \mid U\subseteq M$, for each $(x,y)\in U^{(2)}$, there is $r>0$, such that $B(x,y,r)\subseteq U\}$ defined in (3SM) is a base for a topology on M.*

Now we recall the definitions for o-metrizable and pseudo-o-metrizable spaces [N].

A topological space (M,τ) is called:

o-metrizable, if there is a o-metric D on M such that U is open iff for each x in U, there is $r>0$, such that $T(x,r)=\{y \mid D(x,y)<r\}\subseteq U$;

pseudo-o-metrizable, if there is a pseudo-o-metric D on M such that U is open iff for each x in U, there is $r>0$, such that $T(x,r)=\{y \mid D(x,y)<r\}\subseteq U$.

Theorem 1. *A topological space (M,τ) is pseudo-o-metrizable iff it is a $(3,1,\rho)$-metrizable.*

Proof. Let (M,τ) be pseudo-o-metrizable via a pseudo-o-metric D. Define a map $d: M^{(3)}\mapsto \mathbb{R}_0^+$ by:

$d(x,y,z)=D(x,y)+D(y,x)+D(x,z)+D(z,x)+D(y,z)+D(z,y)$, for $x\neq y\neq z\neq x$, and

$d(x,y,z)=D(a,b)$, for $(x,y,z)=(a,a,b)$ in $M^{(3)}$.

Let ρ be the 3-relation on M defined by:

$$\rho=\{(a,a,b)\in M^{(3)} \mid D(a,b)=0\}.$$

The definition of ρ and the fact that D is a pseudo-o-metric imply that $\triangle\subseteq\rho$. To show that ρ is a (3,1)-equivalence, let $(x,y,z)\in M^{(3)}$, let $b\in M$ and let $(x,y,b),(x,b,z),(b,y,z)\in\rho$. The definition of ρ implies that $b=x$, $b=y$, $b=z$ or $x=y=z$. W.l.o.g. let $b=x$. Then $(x,y,z)=(x,y,z)\in\rho$. Hence, ρ is a $(3,1)$-equivalence on M. The definition of d and the fact that D is a pseudo-o-metric imply that $d(x,y,z)=0$ iff $x=y=z$, or $(x,y,z)=(a,a,b)$ in $M^{(3)}$ such that $D(a,b)=0$, i.e. iff $(x,y,z)\in\rho$.

Next, for $x\neq y\neq z\neq x$, and $x\neq a$, $y\neq a$, $z\neq a$:

$$\begin{aligned}
&d(x,y,a)+d(x,a,z)+d(a,y,z)=\\
&=D(x,y)+D(y,x)+D(x,a)+D(a,x)+D(y,a)+D(a,y)\\
&+D(x,a)+D(a,x)+D(x,z)+D(z,x)+D(a,z)+D(z,a)\\
&+D(a,y)+D(y,a)+D(a,z)+D(z,a)+D(y,z)+D(z,y)
\end{aligned}$$

$\geq D(x,y) + D(y,x) + D(x,z) + D(z,x) + D(y,z) + D(z,y) = d(x,y,z)$.

For $x \neq y \neq z \neq x$, and w.l.o.g., $x = a$:
$d(x,y,a) + d(x,a,z) + d(a,y,z) = D(x,y) + D(x,z) + d(x,y,z) \geq d(x,y,z)$.

For $x = y \neq z$, and $x \neq a$, $z \neq a$:

$$d(x,y,a) + d(x,a,z) + d(a,y,z)$$
$$= d(x,y,a) + D(x,a) + D(a,x) + D(x,z) + D(z,x) + D(a,z) + D(z,a)$$
$$+ d(a,y,z) \geq D(x,z) = d(x,x,z) = d(x,y,z).$$

For $x = y \neq z$, and $x = a$:
$d(x,y,a)+d(x,a,z)+d(a,y,z) = 0+D(x,z)+d(a,y,z) \geq D(x,z) = d(x,x,z) = d(x,y,z)$.

For $x = y \neq z$, and $z = a$:
$d(x,y,a) + d(x,a,z) + d(a,y,z) = D(x,a) + D(z,x) + d(a,y,z) \geq D(x,z) = d(x,x,z) = d(x,y,z)$.

For $x = y = z$:
$d(x,y,a)+d(x,a,z)+d(a,y,z) \geq 0 = D(x,x) = d(x,x,x) = d(x,y,z)$.

Hence, d is a $(3,1,\rho)$-metric.

The conclusion follows from the fact that:
$B(x,x,r) = \{y \mid d(x,x,y) < r\} = \{y \mid D(x,y) < r\} = T(x,r)$.

Conversely, let (M,τ) be $(3,1,\rho)$-metrizable space via a $(3,1,\rho)$-metric d.

Define $D : M \times M \mapsto \mathbb{R}_0^+$ by $D(x,y) = d(x,x,y)$.

Since d is $(3,1,\rho)$-metric and $\triangle \subseteq \rho$, it follows that $D(x,x) = d(x,x,x) = 0$, i.e. D is a pseudo-o-metric on M.

The conclusion follows from the definition of D and the fact that:
$T(x,r) = \{y \mid D(x,y) < r\} = \{y \mid d(x,x,y) < r\} = B(x,x,r)$.

□

Corrolary 1. *A topological space (M,τ) is o-metrizable iff it is a $(3,1,\triangle)$-metrizable space.*

Proof. Since any o-metric is also a pseudo-o-metric, any o-metrizable space is also $(3,1,\rho)$-metrizable space, by Theorem 2, where $\rho = \{(a,a,b) \in M^{(3)} \mid D(a,b) = 0\} = \triangle$. □

Theorem 2. *A topological space (M,τ) is symmetrizable iff it is a $(3,1,\triangle)$-metrizable, via a $(3,1,\triangle)$-metric d, satisfying $d(x,x,y) = d(x,y,y)$ for any $x,y \in M$.*

Proof. Let (M,τ) be symmetrizable via a symmetric D. Define a map $d : M^{(3)} \mapsto \mathbb{R}_0^+$ by:

$$d(x,y,z) = D(x,y) + D(x,z) + D(y,z), \quad \text{for any } x,y,z.$$

Then $d(x,y,z) = 0$ iff $D(x,y) = D(x,z) = D(y,z) = 0$ iff $x = y = z$ iff $(x,y,z) \in \triangle$. Moreover,

$$d(x,y,a) + d(x,a,z) + d(a,y,z)$$
$$= D(x,y)+D(x,a) + D(y,a)+D(x,a)+D(x,z)+D(a,z)+D(a,y)+D(a,z)+D(y,z)$$

$\geq D(x,y) + D(x,z) + D(y,z) = d(x,y,z)$; and

$d(x,x,y) = D(x,x) + D(x,y) + D(x,y) = 2D(x,y)$

$= D(x,y) + D(x,y) + D(y,y) = d(x,y,y)$.

Hence, d is a $(3,1,\triangle)$-metric d, satisfying $d(x,x,y) = d(x,y,y)$.
The conclusion follows from the fact that:

$$B(x,x,2r) = \{y \mid d(x,x,y) < 2r\} = \{y \mid D(x,y) + D(y,x) < 2r\} =$$
$$= \{y \mid D(x,y) < r\} = T(x,r).$$

□

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