Math. Maced.
Vol. 5 (2007)
25-36

CUBICAL WEAK (4,2)-CHAIN COMPLEXES OF TOPOLOGICAL SPACES

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Abstract. In this paper we give construction cubic weak $(4,2)$-chain complex of topological space X, based on maps from the cubes I^n into the space X, denoted by $wC(X)$. This gives a functor C from the category topological spaces to the category of weak $(4,2)$-chain complexes. Then, by applying the functors described in [13], we obtain three homology groups for a space X, denoted by $HC_{n,2}, HC_{n,+}, HC_{n,*}$.

1. Preliminaries

The (n,m)-groups and free (n,m)-groups were introduced in [1], [12]. Some results about (n,m)-groups, free (n,m)-groups and free commutative $(4,2)$-groups can be found for example in: [1], [2],[3], [4], [5], [6], [7], [8], [9], [10], [11], [12].

Let G be a nonempty set. A $(4,2)$**-semigroup** is a pair $(G,[\ ])$, where $[\ ]$: $G^4 \to G^2$ is a map satisfying the following associativity conditions:
$[[xyzt]uv] = [x[yztu]v] = [xy[ztuv]]$, for each $x,y,z,t,u,v \in G$.

A $(4,2)$-semigroup $(G,[\ ])$ induces a semigroup $(G^2,\circ)$, where "$\circ$" is the binary operation on G^2 defined by $x \circ y = [xy]$, for $x,y \in G^2$. Moreover:

A $(4,2)$-semigroup $(G,[\ ])$ is called commutative $(4,2)$-group if $(G^2,\circ)$ is commutative group.

The existence of a (4,2)-operation $[\ ]$ on G is equivalent to the existence of operations $[\ ]_1, [\ ]_2 : G^4 \to G$ defined by $[xyzt] = ([xyzt]_1, [xyzt]_2)$.

Let $(G,[\ ])$ and $(K,[\ ])$ be $(4,2)$-groups. A map $f : G \to K$ is called **(4,2)-homomorphism** if $f^2([xyzt]) = [f(x)f(y)f(z)f(t)]$, where $f^{(2)} : G^2 \to K^2$ is the product map of f defined by $f^2(u,v) = (f(u),f(v))$. It is obvious that f is a $(4,2)$-homomorphism if and only if $f^2 : (G^2,\circ) \to (K^2,\circ)$ is a group homomorphism.

The commutative $(4,2)$-group as objects and the $(4,2)$-homomorpisms as morfisms form a category, denoted by $(4,2)$**-Ab**.

We say that a subset H of a given $(4,2)$-group $(G,[\ ])$ is a $(4,2)$**-subgroup** if H^2 is a subgroup of $(G^2,\circ)$. A $(4,2)$-subgroup H of $(G,[\ ])$ is called *normal* $(4,2)$-*subgroup* if H^2 is a normal subgroup of $(G^2,\circ)$ and satisfies the condition

$$[x_1x_2H^2] = [y_1y_2H^2] \Leftrightarrow [x_ix_iH^2] = [y_iy_iH^2].$$

Above, $[xyH^2]$ denotes the set $\{[xyuv] |\ u,v \in H\}$, $[xH^2y]$ denotes the set

$\{[xuvy]|u,v \in H\}$ and $[H^2xy]$ denotes the set $\{[uvxy]|u,v \in H\}$.

Let $f : (G,[\,]) \to: (K,[\,])$ be a $(4,2)$-homomorpisms, 0 the zero in $(G,[\,])$, and 0' the zero in $(K,[\,])$ and $H = ker\{x|\ x \in G, f(x) = 0'\}$. Then H is a normal subgroup of $(G,[\,])$. Conversely, if H is a normal $(4,2)$-subgroup of $(G,[\,])$, then $H = ker\pi$, where $\pi : G \to G/H$ is the natural $(4,2)$-homomorpism from G to the factor $(4,2)$-group G/H.

We say that a $(4,2)$-group is *commutative* if the associated group $(G^2,\circ)$ is commutative ([10]).

Proposition 1.1. *([10]) A $(4,2)$-semigroup $(G,[\,])$ is a commutative $(4,2)$-group if and only if there exist $e \in G$ and a map $g : G \to G$ such that:*

(a) $[eexy] = (x,y)$ for each $x,y \in G$;
(b) $[xxg(x)g(x)] = (e,e)$ for each $x \in G$;
(c) $[xyzt] = [xtzy]$ for each $x,y,z,t \in G$. □

In a commutative $(4,2)$-group $(G,[\,])$:
(1) the neutral element e will be called *zero*,
(2) for each $x \in G$ the element $g(x)$ will be denoted by x'.

Remark. A $(4,2)$-subgroup of a commutative $(4,2)$-group does not have to be a normal $(4,2)$-subgroup, which is the case for the abelian groups, but a factor group of a commutative $(4,2)$-group by a normal $(4,2)$-subgroup is a commutative $(4,2)$-group.

Proposition 1.2. *([10]) Let $(G,[\,])$ be a commutative $(4,2)$-group. Then, for each $x,y,z,t \in G$:*

(a) $[eexy] = [xeey] = [xyee] = (x,y)$;
(b) $[xyzt] = [xtzy] = [zyxt] = [ztxy]$;
(c) $[xyzt] = (u,v) \Leftrightarrow [yxtz] = (v,u)$;
(d) $[exye] = (y,x)$;
(e) $[xxyy] = (u,v) \Rightarrow u = v$;
(f) $[xxx'x'] = [xx'x'x] = [x'xxx'] = (e,e)$;
(g) $D = \{(xx)|x \in G\} \subseteq G^2$ is a subgroup of the group $(G^2,\circ)$;
(h) $[xx'yy'] = (u,v) \Rightarrow v = u'$;
(i) $K = \{(x,x')|x \in G\} \subseteq G^2$ is a subgroup of the group $(G^2,\circ)$;
(j) $[xx'y'y]'_1 = [xx'y'y]'_2$. □

The commutative $(4,2)$-groups together with the $(4,2)$-homomorphism, form a category, denoted by $(4,2)$-Ab.

We will define three functors denoted by Φ_2, Φ_+ and Φ_* from the category $(4,2)$-Ab to the category Ab of abelian (commutative) groups. Let $\underline{G} = (G,[\,])$ be a given commutative $(4,2)$-group.

Let $\Phi_2(\underline{G})$ be the group $(G^2,\circ)$, defined above.

Let $\Phi_+(\underline{G}) = (G,+)$ where $x + y = a \Leftrightarrow [xxyy] = (a,a)$.

Let $\Phi_*(\underline{G}) = (G,_*)$, where $x * y = a \Leftrightarrow [x(-x)y(-y)] = (a,(-a))$.

A $(4,2)$-homomorphisms is mapped to itself by Φ_2, Φ_+ and Φ_*. It is easy to check that Φ_2, Φ_+ and Φ_* are covariant functors from $(4,2)$- Ab to Ab.

2. Weak $(4,2)$-chain complexes

A *weak* $(4,2)$*-chain complex*, denoted by wK, is a sequence

$$0 \xleftarrow{\partial_0} (K_0,[\,]) \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_{n-1}} (K_{n-1},[\,]) \xleftarrow{\partial_n} (K_n,[\,]) \xleftarrow{\partial_{n+1}} (K_{n+1},[\,]) \longleftarrow \dots \quad (2.1)$$

where $(K_n,[\ ])$ are commutative $(4,2)$-groups, $n \geq 0$, and $\partial_n : (K_n,[\ ]) \to (K_{n-1},[\ ])$, are $(4,2)$-homomorphisms such that $\partial_n\partial_{n+1} = 0$, i.e. such that for each $x \in K_{n+1}$, $\partial_n\partial_{n+1}(x) = 0$.

If wK and wK' are two weak $(4,2)$-chain complexes, we define a $(4,2)$-*chain map* $f : wK \to wK'$ as a sequence of $(4,2)$-homomorphisms $f_n : (K_n,[\ ]) \to (K'_{n-1},[\,])$ such that $\partial'_n f_n = f_{n-1}\partial_n$

If $f : wK \to wK'$ and $f' : wK' \to wK''$ are two $(4,2)$-chain maps, the composition $f' \circ f : wK \to wK''$ is defined by

$$(f' \circ f)_n = f'_n \circ f_n, n \geq 0$$

The weak $(4,2)$-chain complexes as objects and the $(4,2)$-chain maps as morphisms form a category $(4,2) - w\partial K$, called the *category of weak* $(4,2)$*-chain complexes*.

Next, we will define three functors (covariant) from the category $(4,2) - w\partial K$ to the usually category ∂K of chain complexes.

The first functor $F_2 : (4,2) - w\partial K \to \partial K$, is defined as follows :
For a given weak $(4,2)$-chain complexes $(4,2) - wK$ given by (2.1) , $F_2(wK)$ is the chain complexes

$$0 \xleftarrow{\partial^2_0} (K^2_0,\circ) \xleftarrow{\partial^2_1} \dots \xleftarrow{\partial^2_{n-1}} (K^2_{n-1},\circ) \xleftarrow{\partial^2_n} (K^2_n,\circ) \xleftarrow{\partial^2_{n+1}} \dots$$

where $(K^2_n,\circ) = \Phi_2(K_n,[\,])$ for the functor $\Phi_2 : (4,2) - Ab \to Ab$ defined in 1. and the homomorphism $\partial^2_n : (K^2_n,\circ) \to (K^2_{n-1},\circ)$ are defined by $\partial^2(x_1,x_2) = (\partial_n(x_1),\partial_n(x_2))$. Since $(K^2_n,\circ)$ is commutative group and $\partial^2_n \circ \partial^2_{n+1}(x_1,x_2) = (\partial_n\partial_{n+1}(x_1),\partial_n\partial_{n+1}(x_2)) = (e,e)$ it follows that $F_2(wK)$ is a chain complex.

Let $f : wK \to wK'$ be a $(4,2)$-chain map. We define $F_2(f)$, denoted also by f^2, to be the sequence of homomorphisms $f^2_n : F_2(wK) \to F_2(wK')$ defined by $f^2_n(x_1,x_2) = (f_n(x_1),f_n(x_2))$. Since,

$$\partial'^2_n \circ f^2_n(x_1,x_2) = (\partial' f_n(x_1),\partial' f_n(x_2)) = (f_{n-1}\partial_n(x_1),f_{n-1}\partial_n(x_2)) = \\ = f^2_{n-1} \circ \partial^2_n(x_1,x_2)$$

it follows that $F_2(f) = f^2 : F_2(wK) \to F_2(wK')$ is chain map.

Let $f : wK \to wK'$ and $f' : wK' \to wK''$ be two $(4,2)$-chain maps. Then $(F_2(f' \circ f))_n(x_1,x_2) = (f' \circ f)^2_n(x_1,x_2) = (f'_n \circ f_n(x_1), f'_n \circ f_n(x_2)) =$
$= (f'_n(f_n(x_1),f'_n(f_n(x_2)) = f'^2_n(f_n(x_1),f_n(x_2)) =$
$= f'^2_n(f^2_n(x_1,x_2)) = F_2(f')_n(F_2(f)_n(x_1,x_2)) = (F_2(f')_n \circ F_2(f)_n)(x_1,x_2)$
implies that $F_2(f' \circ f) = F_2(f') \circ F_2(f)$
It is obvious that $F_2(id) = id^2$.
Hence F_2 is a covariant functor.

The second functor $F_+ : (4,2) - w\partial K \to \partial K$, is defined as follows: For a weak $(4,2)$-chain complexes wK, given by (2.1) , $F_+(wK)$ is the chain complexes

$$0 \xleftarrow{\partial^+_0} (K^+_0,+) \xleftarrow{\partial^+_1} \dots \xleftarrow{\partial^+_{n-1}} (K^+_{n-1},+) \xleftarrow{\partial^+_n} (K^+_n,+) \xleftarrow{\partial^+_{n+1}} \dots$$

where $(K_+, +) = \Phi_+(K_n, [\,])$ for the functor $\Phi_+ : (4,2) - Ab \to Ab$ defined in 1. and the homomorphisms $\partial_n^+ : (K_n, +) \to (K_{n-1}, +)$ are defined as $\Phi_+(\partial_n)$. Since $\Phi_+ : (4,2) - Ab \to Ab$ is covariant functor and $\partial_n \partial_{n+1} = 0$, $(K_n^+, +)$ are commutative groups and a $\partial_n^+ \partial_{n+1}^+ = 0^+$. Hence K^+ is a chain complex.

Let $f : wK \to wK'$ be a $(4,2)$-chain map. We define $F_+(f)$, denoted also by f_n^+ to be the sequence of homomorphisms $f_n^+ : F_+(wK) \to F_+(wK')$ defined by $f_n^+(x) = f_n(x)$. Since ,

$$(\partial_n^{+'} \circ f_n^+)(x) = \phi^+(\partial^{'} \circ f_n)(x) = \phi(f_{n-1} \circ \partial_n(x)) = (f_{n-1}^+ \circ \partial_n^+)(x)$$

it follows that $f^+ : F_+(wK) \to F + (wK')$ is chain map.

Let $f : wK \to wK'$ and $f' : wK' \to K''$ be two $(4,2)$-chain maps. Then

$$F_+(f^{'} \circ f)_n(x) = (f^{'} \circ f)_n^+(x) = (f^{'} \circ f)_n(x) = f_n^{'} \circ f_n(x) = f_n^{+'} \circ f_n^+(x) =$$

$$= f_n^{+'}(f_n^+(x)) = F_+(f_n^{'})(F_+(f_n(x))) = F_+(f^{'})_n \circ F_+(f)_n(x)$$

implies that $F_+(f^{'} \circ f) = F_+(f^{'}) \circ F_+(f)$.
It is obvious that $F_+(id) = id^+$. Hence F_+ is covariant functor.

The third functor $F_* : (4,2) - w\partial K \to \partial K$, is defined as follows : For a weak $(4,2)$-chain complexes wK , given by (2.1), $F_*(wK)$ is the chain complexes

$$0 \xleftarrow{\partial_0^*} (K_0^*, *) \xleftarrow{\partial_1^1} ... \xleftarrow{\partial_{n-1}^*} (K_{n-1}^*, *) \xleftarrow{\partial_n^*} (K_n^*, *) \xleftarrow{\partial_{n+1}^*} ...$$

where $(K_n^*, *) = \Phi_*(K_n, [\,])$ for the functor $\Phi_* : (4,2) - Ab \to Ab$ defined in 1. and the homomorphisms $\partial_n^* : (K_n, *) \to (K_{n-1}, *)$ are $\Phi_+(\partial n)$. Since

$$\Phi_* : (4,2) - Ab \to Ab$$

is a covariant functor and $\partial_n \partial_{n+1} = 0$. It follows that $(K_n^*, *)$ are commutative group and $\partial_n^* \partial_{n+1}^* = 0^*$. Hence $F_*(wK)$ is a chain complex.

Let $f : wK \to wK'$ be a $(4,2)$-chain map. We define $F_*(f)$ denoted also by f_n^*, to be the sequence of homomorphisms $F_*(wK) \to F_*(wK')$ defined by $f_n^*(x) = f_n(x)$. Since,

$$(\partial_n^{'*} \circ f_n^*)(x) = \phi^*(\partial_n^{'} \circ f_n)(x) = \phi(f_{n-1} \circ \partial_n(x)) = (f_{n-1}^* \circ \partial_n^*)(x)$$

it follows that $F_*(f)$ is chain map.

Let $f : wK \to wK'$ and $f' : wK' \to wK''$ be two $(4,2)$-chain maps. Then

$$F_*(f^{'} \circ f)_n(x) = (f^{'} \circ f)_n^*(x) = (f^{'} \circ f)_n(x) = f_n^{'} \circ f_n(x) = f_n^{'*} \circ f_n^*(x) = f_n^{'*}(f_n^*(x)) =$$

$$= F_*(f_n^{'})(F_*(f_n(x))) = F_*(f^{'})_n \circ F_*(f)_n(x)$$

is that $F_*(f^{'} \circ f) = F_*(f^{'}) \circ F_*(f)$.
It is obvious that $F_*(id) = id^*$
Hence F_* is a covariant functor.

3. Cubical weak $(4,2)$-chain complexes of topological spaces

Let X be a topological space, $I^n = \{(t_1,\dots,t_n)|0 \le t_i \le 1, i = 1,2,\dots,n\}$ be the cube in $\mathbb{R}^n$, and $\dot{I}^n = \{(t_1,\dots,t_n)|0 \le t_i \le 1, t_i = 0 \text{ or } t_i = 1 \text{ for any } i = 1,2,\dots,n\}$ the *boundary* of I^n. Let Σ_n be the set of all continuous functions $f : I^n \to X$. For each $f \in \Sigma_n$ let $-f : I^n \to X$ be the map defined by $-f(x_1^n) = f((1-x_i)_{i=1}^n)$.

Let $(Cn(X), [\,])$ be the free commutative $(4,2)$-group with a basis Σ_n.

For a given n and each $i, 1 \le i \le n$, we define two maps $\delta_i, \delta^i : \Sigma_{n+1} \to \Sigma_n$ as follows. For a given $f \in \Sigma_n$, $\delta_i(f)$ denoted by f_{i0} and $\delta^i(f)$ denoted by f_{i1} are defined by:

$\delta_i(f)(x_1^n) = f_{i0}(x_1^n) = f(x_1^{i-1}, 0, x_i^n)$
$\delta^i(f)(x_1^n) = f_{i1}(x_1^n) = f((1-x_k)_{k=1}^{i-1}, 1, (1-x_k)_{k=i}^n))$

Proposition 3.1. *For the maps δ_i, δ^i and some of their compositions we have:*

$$\begin{array}{ll} \delta_\ell\left(\delta_k(f)\right) = \delta_{k-1}\left(\delta_\ell(f)\right), & \ell < k, \\ \delta_\ell\left(\delta^k(f)\right) = \delta_{k-1}\left(\delta^\ell(f)\right), & \ell < k \\ \delta^\ell\left(\delta_k(f)\right) = \delta^{k-1}\left(\delta^\ell(f)\right), & \ell < k \\ \delta^\ell\left(\delta_k(f)\right) = \delta^k\left(\delta^{\ell+1}(f)\right), & \ell \ge k \end{array} \tag{3.1}$$

□

Indeed,

$$\delta_\ell\left(\delta_k(f)\right)(x_1^{n-1}) = \delta_\ell\left(f_{ko}(x_1^{n-1})\right) = (f_{ko})_{l0}(x_1^{n-1}) = f_{ko}(x_1^{\ell-1}, 0, x_\ell^{n-1}) =$$
$$= \begin{cases} f(x_1^{k-1}, 0, x_k^{\ell-1}, 0, x_\ell^{n-1}), & k < \ell \\ f(x_1^{k-1}, 0, 0, x_\ell^{n-1}), & k = \ell \\ f(x_1^{\ell-1}, 0, x_l^{k-1}, 0, x_k^{n-1}), & \ell < k. \end{cases}$$

$$\delta^\ell(\delta_k f))(x_1^{n-1}) = \delta^\ell(f_{ko})(x_1^{n-1}) = (f_{ko})_{\ell 1}(x_1^{n-1}) =$$
$$= f_{ko}((1-x_i)_{i=1}^{\ell-1}, 1, (1-x_i)_{i=\ell}^{n-1}) =$$
$$= \begin{cases} f((1-x_i)_{i=1}^{k-1}, 0, (1-x_i)_{i=k}^{\ell-1}, 1, (1-x_i)_{i=\ell}^{n-1}), & k < \ell \\ f((1-x_i)_{i=1}^{k-1}, 0, 1, (1-x_i)_{i=k}^{n-1}), & k = \ell \\ f((1-x_i)_{i=1}^{\ell-1}, 1, (1-x_i)_{i=\ell}^{k-1}, 0, (1-x_i)_{i=k}^{n-1}), & k > \ell \end{cases}$$

$$\delta_\ell(\delta^k(f))((1-x_i)_{i=1}^{n-1}) = \delta_\ell(f_{k1}(1-x_i)_{i=1}^{n-1}) = (f_{k1})_{\ell 0}((1-x_i)_{i=1}^{n-1}) =$$
$$= f_{k1}((1-x_i)_{i=1}^{\ell-1}, 0, (1-x_i)_{i=\ell}^{n-1}) =$$
$$= \begin{cases} f(x_1^{k-1}, 1, x_k^{\ell-1}, 1, x_\ell^{n-1}), & k < \ell \\ f(x_1^{k-1}, 1, 1, x_\ell^{n-1}), & k = \ell \\ f(x_1^{\ell-1}, 1, x_\ell^{k-1}, 1, x_k^{n-1}), & \ell < k. \end{cases}$$

$$\delta^\ell(\delta^k(f))((1-x_i)_{i=1}^{n-1}) = \delta^\ell(f_{k1}(1-x_i)_{i=1}^{n-1}) = (f_{k1})_{\ell 1}((1-x_i)_{i=1}^{n-1}) =$$
$$= f_{k1}(x_1^{\ell-1}, 1, x_\ell^{n-1}) =$$

$$= \begin{cases} f((1-x_i)_{i=1}^{k-1}, 1, (1-x_i)_{i=k}^{\ell-1}, 0, (1-x_i)_{i=\ell}^{n-1}), & k < \ell \\ f((1-x_i)_{i=1}^{k-1}, 1, 0, (1-x_i)_{i=k}^{n-1}), & k = \ell \\ f((1-x_i)_{i=1}^{\ell-1}, 0, (1-x_i)_{i=\ell}^{k-1}, 1, (1-x_i)_{i=k}^{n-1}), & k > \ell \end{cases}$$

Theorem 1. *The following sequence*

$$0 \xleftarrow{\partial_0} (C_0(X), [\,]) \xleftarrow{\partial_1} (C_1(X), [\,]) \dots \xleftarrow{\partial_{n-1}} (C_{n-1}(X), [\,]) \xleftarrow{\partial_n} (C_n(X), [\,]) \longleftarrow \dots$$

is a (4,2)-weak chain complex, i.e. $\partial_n \partial_{n+1} = 0$, denote by $wC(X)$ and called cubic weak (4,2)-chain complex of X, where the (4,2)-homomorphisms

$$\partial_n : (C_n(X), [\,]) \to C_{n-1}(X), [\,]),\ n \geq 1$$

is defined by:

$$\partial_{2k-1}(f) = [\delta_1(f)(\delta_1)'(\delta_2(f))'\delta_2(f) \dots (\delta_{2k-2}(f))'\delta_{2k-2}(f)\delta_{2k-1}(f)(\delta_{2k-1}(f))' (\delta^1(f))'\delta^1(f)(\delta^2(f)(\delta^2(f))' \dots \delta^{2k-2}(f)(\delta^{2k-2}(f))'\delta^{2k-1}(f)]_1$$

$$\partial_{2k}(f) = [\delta_1(f)(\delta_1)'(\delta_2(f))'\delta_2(f) \dots \delta_{2k-1}(f)(\delta_{2k-1}(f))'(\delta_{2k}(f))'\delta_{2k}(f) (\delta^1(f))'\delta^1(f)\delta^2(f)(\delta^2(f))' \dots (\delta^{2k-1}(f))'\delta^{2k-1}(f)\delta^{2k}(f)(\delta^{2k}(f))']_1$$

For example for $n = 1$, $\partial_1 : (S_1(X), [\,]) \to (S_0(X), [\,])$
Let $f : [0,1] \to X$ be given. Then $-f(x) = f(1-x)$, and

$\partial_1(f) = [\delta_1(f)(\delta_1(f))'(\delta^1(f))'\delta^1(f)]_1$
$\partial_1(f) = [f(0)f'(0)f'(1)f(1)]_1$
$\partial_1(-f) = [(-f)(0)((-f)(0))'(-f(1))'(-f)(1)]_1 = [f(1)(f(1))'(f(0))'f(0)]_1$
$(\partial_1(f))' = [f(0)f'(0)f'(1)f(1)]_1' = [f(0)f'(0)f'(1)f(1)]_2 =$
$= [(f(0))'f(0)f(1)(f(1))']_1 = [f(1)(f(1))'(f(0))'f(0)]_1$
i.e. $\partial_1(-f) = (\partial_1(f))'$;
For $n = 2$, $\partial_2 : (C_2(X), [\,]) \to (C_1(X), [\,])$;
Let $f : I^2 \to X$ be given and $-f^2(t_1, t_2) = f^2(1-t_1, 1-t_2)$. Then

$\delta_1(f)(t) = f^2(0,t)$, $\delta_1(-f)(t) = f^2(1,1-t)$
$\delta_2(f)(t) = f^2(t,0)$, $\delta_2(-f)(t) = f^2(1-t,1)$
$\delta^1(f)(t) = f^2(1,1-t)$, $\delta^1(-f)(t) = f^2(0,t)$
$\delta^2(f)(t) = f^2(1-t,1)$, $\delta^2(-f)(t) = f^2(t,0)$

i.e. $\delta_1(-f) = \delta^1(f), \delta_2(-f) = \delta^2(f), \delta^1(-f) = \delta_1(f), \delta^2(-f) = \delta_2(f)$. Then
$\partial_2(f) = [\delta_1(f)(\delta_1(f))'(\delta_2(f))'\delta_2(f)(\delta^1(f))'\delta^1(f)\delta^2(f)(\delta^2(f))']_1$
$\partial_2(-f) = [\delta_1(-f)(\delta_1(-f))'(\delta_2(-f))'\delta_2(-f)(\delta^1(-f))'\delta^1(-f)\delta^2(-f)(\delta^2(-f))']_1$
$\partial_2(-f) = [\delta^1(f)(\delta^1(f))'(\delta^2(f))'\delta^2(f)(\delta_1(f))'\delta_1(f)\delta_2(f)(\delta_2(f))']_1$
$\partial_2(-f) = [(\delta_1(f))'\delta_1(f)\delta_2(f)(\delta_2(f))'\delta^1(f)(\delta^1(f))'(\delta^2(f))'\delta^2(f)]_1$
$(\partial_2(f))' = [\delta_1(f)(\delta_1(f))'(\delta_2(f))'\delta_2(f)(\delta^1(f))'\delta^1(f)\delta^2(f)(\delta^2(f))']_1'$
$(\partial_2(f))' = [\delta_1(f)(\delta_1(f))'(\delta_2(f))'\delta_2(f)(\delta^1(f))'\delta^1(f)\delta^2(f)(\delta^2(f))']_2$
$(\partial_2(f))' = [(\delta_1(f))'\delta_1(f)\delta_2(f)(\delta_2(f))'\delta^1(f)(\delta^1(f))'(\delta^2(f))'\delta^2(f)]_1$
This implies that $(\partial_2(f))' = \partial_2(-f)$.
Now we prove
$\partial_1\partial_2(f) = \partial_1[\delta_1(f)(\delta_1(f))'(\delta_2(f))'\delta_2(f)(\delta^1(f))'\delta^1(f)\delta^2(f)(\delta^1(f))']_1$

$$
\begin{aligned}
&= [[\delta_1(\delta_1(f))(\delta_1(\delta_1(f)))'(\delta^1(\delta_1(f)))'\delta^1(\delta_1(f))]_1\\
&\quad [\delta_1(\delta_1(f))(\delta_1(\delta_1(f)))'(\delta^1(\delta_1(f)))'\delta^1(\delta_1(f))]_2\\
&\quad [\delta_1(\delta_2(f))(\delta_1(\delta_2(f)))'(\delta^1(\delta_2(f)))'\delta^1(\delta_2(f))]_2\\
&\quad [\delta_1(\delta_2(f))\delta_1(\delta_2(f))'(\delta^1(\delta_2(f)))'\delta^1(\delta_2(f))]_1\\
&\quad [\delta_1(\delta^1(f))'(\delta_1(\delta^1(f)))\delta^1(\delta^1(f))(\delta^1(\delta^1(f)))']_2\\
&\quad [\delta_1(\delta^1(f))'(\delta_1(\delta^1(f)))\delta^1(\delta^1(f))(\delta^1(\delta^1(f)))']_1\\
&\quad [\delta_1(\delta^2(f))'(\delta_1(\delta^2(f)))\delta^1(\delta^2(f))(\delta^1(\delta^2(f)))']_1\\
&\quad [\delta_1(\delta^2(f))'(\delta_1(\delta^2(f)))\delta^1(\delta^2(f))(\delta^1(\delta^2(f)))']_2]_1 =\\
&= [\delta_1(\delta_1(f))(\delta_1(\delta_1(f)))'(\delta^1(\delta_1(f)))'\delta^1(\delta_1(f))\\
&\quad (\delta_1(\delta_2(f)))'\delta_1(\delta_2(f))\delta^1(\delta_2(f))(\delta^1(\delta_2(f)))'\\
&\quad \delta_1(\delta^1(f))(\delta_1(\delta^1(f)))'(\delta^1(\delta^1(f)))'\delta^1(\delta^1(f))\\
&\quad (\delta_1(\delta^2(f)))'\delta_1(\delta^2(f))\delta^1(\delta^2(f))(\delta^1(\delta^2(f)))']_1
\end{aligned}
$$

Next, since

$\delta_1(\delta_2(f)) = \delta_1(\delta_1(f))$; $\delta_1(\delta^2(f)) = \delta_1(\delta^1(f))$;

$\delta^1(\delta_2(f)) = \delta^1(\delta^1(f))$; $\delta^1(\delta^2(f)) = \delta^1(\delta_1(f))$

$$
\begin{aligned}
\partial_1\partial_2(f) \;&= [\delta_1(\delta_1(f))(\delta_1(\delta_1(f)))'(\delta^1(\delta_1(f)))'\delta^1(\delta_1(f))\\
&\quad (\delta_1(\delta_1(f)))'\delta_1(\delta_1(f))\delta^1(\delta^1(f))(\delta^1(\delta^1(f)))'\\
&\quad \delta_1(\delta^1(f))(\delta_1(\delta^1(f)))'(\delta^1(\delta^1(f)))'\delta^1(\delta^1(f))\\
&\quad \delta_1(\delta^1(f))'\delta_1(\delta^1(f))\delta^1(\delta_1(f))(\delta^1(\delta_1(f)))']_1 = e
\end{aligned}
$$

In general, we first show that

$$
\begin{aligned}
\partial_{2k-1}(-f) &= [\delta_1(-f)(\delta_1(-f))'(\delta_2(-f))'\delta_2(-f)\dots(\delta_{2k-2}(-f))'\delta_{2k-2}(-f)\\
&\quad \delta_{2k-1}(-f)(\delta_{2k-1}(-f))'(\delta^1(-f))'\delta^1(-f)(\delta^2(-f))'\delta^2(-f)\dots\\
&\quad \dots\delta_{2k-2}(-f)(\delta_{2k-2}(-f))'(\delta_{2k-1}(-f))'\delta_{2k-1}(-f)]_1\\
&= [\delta^1(f)(\delta^1(f))'(\delta^2(f))'\delta^2(f)\dots(\delta^{2k-2}(f))'\delta^{2k-2}(f)\\
&\quad \delta^{2k-1}(f)(\delta^{2k-1}(f))'(\delta^1(f))'\delta^1(f)(\delta^2(f))'\delta^2(f)\dots\\
&\quad \dots\delta^{2k-2}(f)(\delta^{2k-2}(f))'(\delta^{2k-1}(f))'\delta^{2k-1}(f)]_1\\
&= [(\delta_1(f))'\delta_1(f)\delta_2(f)(\delta_2(f))'\dots\delta_{2k-2}(f)(\delta_{2k-2}(f))'\\
&\quad (\delta_{2k-1}(f))'\delta_{2k-1}(f)\delta^1(f)(\delta^1(f))'(\delta^2(f))'\delta^2(f)\dots\\
&\quad \dots(\delta^{2k-2}(f))'\delta^{2k-2}(f)\delta^{2k-1}(f)(\delta^{2k-1}(f))']_1\\
(\partial_{2k-1}(f))' &= [\delta_1(f)(\delta_1(f))'(\delta_2(f))'\delta_2(f)\dots(\delta_{2k-2}(f))'\delta_{2k-2}(f)\\
&\quad \delta_{2k-1}(f)(\delta_{2k-1}(f))'(\delta^1(f))'\delta^1(f)\delta^2(f)(\delta^2(f))'\dots\\
&\quad \dots\delta^{2k-2}(f)(\delta^{2k-2}(f))'(\delta^{2k-1}(f))'\delta^{2k-1}(f)]_1\\
&= [\delta_1(f)(\delta_1(f))'(\delta_2(f))'\delta_2(f)\dots(\delta_{2k-2}(f))'\delta_{2k-2}(f)\\
&\quad \delta_{2k-1}(f)(\delta_{2k-1}(f))'(\delta^1(f))'\delta^1(f)(\delta^2(f))'\delta^2(f)\dots\\
&\quad \dots\delta^{2k-2}(f)(\delta^{2k-2}(f))'(\delta^{2k-1}(f))'\delta^{2k-1}(f)]_2\\
&= [(\delta_1(f))'\delta_1(f)\delta_2(f)(\delta_2(f))'\dots\delta_{2k-2}(f)(\delta_{2k-2}(f))'\\
&\quad (\delta_{2k-1}(f))'\delta_{2k-1}(f)\delta^1(f)(\delta^1(f))'(\delta^2(f))'\delta^2(f)\dots\\
&\quad \dots(\delta^{2k-2}(f))'\delta^{2k-2}(f)\delta^{2k-1}(f)(\delta^{2k-1}(f))']_1
\end{aligned}
$$

i.e.$(\partial_{2k-1}(f))' = \partial_{2k-1}(-f)$.

Similarly $(\partial_{2k}(f))' = \partial_{2k}(-f),\ k \geq 1$.

This implies $(\partial_n(f))' = \partial_n(-f),\ n \geq 0$.

Now we prove $\partial_n\partial_{n+1} = 0$.

$\partial_{2k-1}(\delta_s) = [\delta_1(\delta_s(f))(\delta_1(\delta_s(f)))' \dots (\delta_{2k-2}(\delta_s(f)))'\delta_{2k-2}(\delta_s(f))\delta_{2k-1}(\delta_s(f))$
$(\delta_{2k-1}(\delta_s(f)))'(\delta^1(\delta_s(f)))'\delta^1(\delta_s(f))\dots(\delta^{2k-2}(\delta_s(f)))'\delta^{2k-2}(\delta_s(f))$
$\delta^{2k-1}(\delta_s(f))\delta^{2k-1}(\delta_s(f))]_1$

$\partial_{2k-1}(\delta^s) = [(\delta_1(\delta^s(f)))'\delta_1(\delta^s(f))\dots\delta_{2k-2}(\delta^s(f))(\delta_{2k-2}(\delta^s(f)))'(\delta_{2k-1}(\delta_s(f)))'$
$\delta_{2k-1}(\delta^s(f))\delta^1(\delta^s(f))(\delta^1(\delta^s(f)))'\dots(\delta^{2k-2}(\delta_s(f)))'\delta^{2k-2}(\delta_s(f))$
$\delta^{2k-1}(\delta_s(f))(\delta^{2k-1}(\delta_s(f)))']_1$

$\partial_{2k}(\delta_s) = [\delta_1(\delta_s(f))(\delta_1(\delta_s(f)))'\dots\delta_{2k-1}(\delta_s(f))(\delta_{2k-1}(\delta_s(f)))'(\delta_{2k}(\delta_s(f)))'$
$\delta_{2k}(\delta_s(f))(\delta^1(\delta_s(f)))'\delta^1(\delta_s(f))\dots(\delta^{2k-1}(\delta_s(f)))'\delta^{2k-1}(\delta_s(f))$
$\delta^{2k}(\delta_s(f))(\delta^{2k}(\delta_s(f)))']_1$

$\partial_{2k}(\delta^s) = [(\delta_1(\delta^s(f)))'\delta_1(\delta^s(f))\dots(\delta_{2k-1}(\delta^s(f)))'\delta_{2k-1}(\delta^s(f))\delta_{2k}(\delta^s(f))$
$(\delta_{2k}(\delta^s(f)))'\delta^1(\delta^s(f))(\delta^1(\delta^s(f)))'\dots\delta^{2k-1}(\delta^s(f))(\delta^{2k-1}(\delta^s(f)))'$
$(\delta^{2k}(\delta^s(f)))'\delta^{2k}(\delta^s(f))]_1.$

First we prove that $\partial_{2k-1}\partial_{2k}(f) = e$

$\partial_{2k-1}\partial_{2k}(f) = \partial_{2k-1}[\delta_1(f)(\delta_1(f))'\dots\delta_{2k-1}(f)(\delta_{2k-1}(f))'(\delta_{2k}(f))'\delta_{2k}(f)$
$(\delta^1(f))'\delta^1(f)\dots(\delta^{2k-1}(f))'\delta^{2k-1}(f)\delta^{2k}(f)(\delta^{2k}(f))']_1 =$
$= [[\delta_1(\delta_1(f))(\delta_1(\delta_1(f)))'\dots(\delta_{2k-2}(\delta_1(f)))'\delta_{2k-2}(\delta_1(f))$
$\delta_{2k-1}(\delta_1(f))(\delta_{2k-1}(\delta_1(f)))'(\delta^1(\delta_1(f)))'\delta^1(\delta_1(f))\dots$
$\dots\delta^{2k-2}(\delta_1(f))(\delta^{2k-2}(\delta_1(f)))'(\delta^{2k-1}(\delta_1(f)))'\delta^{2k-1}(\delta_1(f))]_1$
$[\delta_1(\delta_1(f))(\delta_1(\delta_1(f)))'\dots(\delta_{2k-2}(\delta_1(f)))'\delta_{2k-2}(\delta_1(f))$
$\delta_{2k-1}(\delta_1(f))(\delta_{2k-1}(\delta_1(f)))'(\delta^1(\delta_1(f)))'\delta^1(\delta_1(f))\dots$
$\dots\delta^{2k-2}(\delta_1(f))(\delta^{2k-2}(\delta_1(f)))'(\delta^{2k-1}(\delta_1(f)))'\delta^{2k-1}(\delta_1(f))]_2$
$[\delta_1(\delta_2(f))(\delta_1(\delta_2(f)))'\dots(\delta_{2k-2}(\delta_2(f)))'\delta_{2k-2}(\delta_2(f))$
$\delta_{2k-1}(\delta_2(f))(\delta_{2k-1}(\delta_2(f)))'(\delta^1(\delta_2(f)))'\delta^1(\delta_2(f))\dots$
$\dots\delta^{2k-2}(\delta_2(f))(\delta^{2k-2}(\delta_2(f)))'(\delta^{2k-1}(\delta_2(f)))'\delta^{2k-1}(\delta_2(f))]_2$
$[\delta_1(\delta_2(f))(\delta_1(\delta_2(f)))'\dots\delta_{2k-2}(\delta_2(f)))'\delta_{2k-2}(\delta_2(f))$
$\delta_{2k-1}(\delta_2(f))(\delta_{2k-1}(\delta_2(f)))'(\delta^1(\delta_2(f)))'\delta^1(\delta_2(f))\dots$
$\dots\delta^{2k-2}(\delta_2(f))(\delta^{2k-2}(\delta_2(f)))'(\delta^{2k-1}(\delta_2(f)))'\delta^{2k-1}(\delta_2(f))]_1$

....

$[(\delta_1(\delta^{2k-1}(f)))'\delta_1(\delta^{2k-1}(f))\dots\delta_{2k-2}(\delta^{2k-1}(f))(\delta_{2k-2}(\delta^{2k-1}(f)))'(\delta_{2k-1}(\delta^{2k-1}(f)))'$
$\delta_{2k-1}(\delta^{2k-1}(f))\delta^1(\delta^{2k-1}(f))(\delta^1(\delta^{2k-1}(f)))'\dots(\delta^{2k-2}(\delta^{2k-1}(f)))'\delta^{2k-2}(\delta^{2k-1}(f))]$
$\delta^{2k-1}(\delta^{2k-1}(f))(\delta^{2k-1}(\delta^{2k-1}(f)))']_2$
$[(\delta_1(\delta^{2k-1}(f)))'\delta_1(\delta^{2k-1}(f))\dots\delta_{2k-2}(\delta^{2k-1}(f))(\delta_{2k-2}(\delta^{2k-1}(f)))'(\delta_{2k-1}(\delta^{2k-1}(f)))'$
$\delta_{2k-1}(\delta^{2k-1}(f))\delta^1(\delta^{2k-1}(f))(\delta^1(\delta^{2k-1}(f)))'\dots(\delta^{2k-2}(\delta^{2k-1}(f)))'\delta^{2k-2}(\delta^{2k-1}(f))$
$\delta^{2k-1}(\delta^{2k-1}(f))(\delta^{2k-1}(\delta^{2k-1}(f)))']_1$
$[(\delta_1(\delta^{2k}(f)))'\delta_1(\delta^{2k}(f))\dots\delta_{2k-2}(\delta^{2k}(f))(\delta_{2k-2}(\delta^{2k}(f)))'(\delta_{2k-1}(\delta^{2k}(f)))'$
$\delta_{2k-1}(\delta^{2k}(f))\delta^1(\delta^{2k}(f))(\delta^1(\delta^{2k}(f)))'\dots(\delta^{2k-2}(\delta^{2k}(f)))'\delta^{2k-2}(\delta^{2k}(f))$
$\delta^{2k}(\delta^{2k}(f))(\delta^{2k}(\delta^{2k}(f)))']_1$
$[(\delta_1(\delta^{2k}(f)))'\delta_1(\delta^{2k}(f))\dots\delta_{2k-2}(\delta^{2k}(f))(\delta_{2k-2}(\delta^{2k}(f)))'(\delta_{2k-1}(\delta^{2k}(f)))'$
$\delta_{2k-1}(\delta^{2k}(f))\delta^1(\delta^{2k}(f))(\delta^1(\delta^{2k}(f)))'\dots(\delta^{2k-2}(\delta^{2k}(f)))'\delta^{2k-2}(\delta^{2k}(f))$
$\delta^{2k}(\delta^{2k}(f))(\delta^{2k}(\delta^{2k}(f)))']_2]_1 =$
$= [\delta_1(\delta_1(f))(\delta_1(\delta_1(f)))'\dots(\delta_{2k-2}(\delta_1(f)))'\delta_{2k-2}(\delta_1(f))\delta_{2k-1}(\delta_1(f))(\delta_{2k-1}(\delta_1(f)))'$

$(\delta^1(\delta_1(f)))'\delta^1(\delta_1(f))\dots\delta^{2k-2}(\delta_1(f))(\delta^{2k-2}(\delta_1(f)))'(\delta^{2k-1}(\delta_1(f)))'\delta^{2k-1}(\delta_1(f))$
$(\delta_1(\delta_2(f)))'\delta_1(\delta_2(f))\dots\delta_{2k-2}(\delta_2(f))(\delta_{2k-2}(\delta_2(f)))'(\delta_{2k-1}(\delta_2(f)))'\delta_{2k-1}(\delta_2(f))$
$(\delta^1(\delta_2(f)))(\delta^1(\delta_2(f)))'\dots(\delta^{2k-2}(\delta_2(f)))'\delta^{2k-2}(\delta_2(f))\delta^{2k-1}(\delta_2(f))(\delta^{2k-1}(\delta_2(f)))'$

.....

$\delta_1(\delta_{2k-1}(f))(\delta_1(\delta_{2k-1}(f)))'\dots(\delta_{2k-2}(\delta_{2k-1}(f)))'\delta_{2k-2}(\delta_{2k-1}(f))\delta_{2k-1}(\delta_{2k-1}(f))$
$(\delta_{2k-1}(\delta_{2k-1}(f)))'(\delta^1(\delta_{2k-1}(f)))'\delta^1(\delta_{2k-1}(f))\dots\delta^{2k-2}(\delta_{2k-1}(f))(\delta^{2k-2}(\delta_{2k-1}(f)))'$
$(\delta^{2k-1}(\delta_{2k-1}(f)))'\delta^{2k-1}(\delta_{2k-1}(f))$

$(\delta_1(\delta_{2k}(f)))'\delta_1(\delta_{2k}(f))\dots\delta_{2k-2}(\delta_{2k}(f))(\delta_{2k-2}(\delta_{2k}(f)))'(\delta_{2k-1}(\delta_{2k}(f)))'$
$\delta_{2k-1}(\delta_{2k}(f))\delta^1(\delta_{2k}(f))(\delta^1(\delta_{2k}(f)))'\dots(\delta^{2k-2}(\delta_{2k}(f)))'\delta^{2k-2}(\delta_{2k}(f))$
$\delta^{2k-1}(\delta_{2k}(f))(\delta^{2k-1}(\delta_{2k}(f)))'$

$\delta_1(\delta^1(f))(\delta_1(\delta^1(f)))'\dots(\delta_{2k-2}(\delta^1(f)))'\delta_{2k-2}(\delta^1(f))\delta_{2k-1}(\delta^1(f))(\delta_{2k-1}(\delta^1(f)))'$
$(\delta^1(\delta^1(f)))'\delta^1(\delta^1(f))\dots\delta^{2k-2}(\delta^1(f))(\delta^{2k-2}(\delta^1(f)))'(\delta^{2k-1}(\delta^1(f)))'\delta^{2k-1}(\delta^1(f))$

$(\delta_1(\delta^2(f)))'\delta_1(\delta^2(f))\dots\delta_{2k-2}(\delta^2(f))(\delta_{2k-2}(\delta^2(f)))'(\delta_{2k-1}(\delta^2(f)))'\delta_{2k-1}(\delta^2(f))$
$\delta^1(\delta^2(f))(\delta^1(\delta^2(f)))'\dots(\delta^{2k-2}(\delta^2(f)))'\delta^{2k-2}(\delta^2(f))\delta^{2k-1}(\delta^2(f))(\delta^{2k-1}(\delta^2(f)))'$
.....
$\delta_1(\delta^{2k-1}(f))(\delta_1(\delta^{2k-1}(f)))'\dots(\delta_{2k-2}(\delta^{2k-1}(f)))'\delta_{2k-2}(\delta^{2k-1}(f))$
$\delta_{2k-1}(\delta^{2k-1}(\delta_{2k-1}(\delta^{2k-1}(f))))'$
$(\delta^1(\delta^{2k-1}(f)))'\delta^1(\delta^{2k-1}(f))\dots\delta^{2k-2}(\delta^{2k-1}(f))(\delta^{2k-2}(\delta^{2k-1}(f)))'$
$(\delta^{2k-1}(\delta^{2k-1}(f)))'\delta^{2k-1}(\delta^{2k-1}(f))$
$(\delta_1(\delta^{2k}(f)))'\delta_1(\delta^{2k}(f))\dots\delta_{2k-2}(\delta^{2k}(f))(\delta_{2k-2}(\delta^{2k}(f)))'$
$(\delta_{2k-1}(\delta^{2k}(f)))'\delta_{2k}(\delta^{2k-1}(f))$
$\delta^1(\delta^{2k}(f))(\delta^1(\delta^{2k}(f)))'\dots(\delta^{2k-2}(\delta^{2k}(f)))'\delta^{2k-2}(\delta^{2k}(f))$
$\delta^{2k-1}(\delta^{2k}(\delta^{2k-1}(\delta^{2k}(f))))' =$

(Using (3.1))

$= [\delta_1(\delta_1(f))(\delta_1(\delta_1(f)))'\dots(\delta_{2k-2}(\delta_1(f)))'\delta_{2k-2}(\delta_1(f))\delta_{2k-1}(\delta_1(f))(\delta_{2k-1}(\delta_1(f)))'$
$(\delta^1(\delta_1(f)))'\delta^1(\delta_1(f))\dots\delta^{2k-2}(\delta_1(f))(\delta^{2k-2}(\delta_1(f)))'(\delta^{2k-1}(\delta_1(f)))'\delta^{2k-1}(\delta_1(f))$
$(\delta_1(\delta_1(f)))'\delta_1(\delta_1(f))\dots\delta_{2k-2}(\delta_2(f))(\delta_{2k-2}(\delta_2(f)))'(\delta_{2k-1}(\delta_2(f)))'\delta_{2k-1}(\delta_2(f))$
$(\delta^1(\delta^1(f)))(\delta^1(\delta^1(f)))'\dots(\delta^{2k-2}(\delta_2(f)))'\delta^{2k-2}(\delta_2(f))\delta^{2k-1}(\delta_2(f))(\delta^{2k-1}(\delta_2(f)))'$
.....
$\delta_{2k-2}(\delta_1(f))(\delta_{2k-2}(\delta_1(f)))'\dots(\delta_{2k-2}(\delta_{2k-2}(f)))'\delta_{2k-2}(\delta_{2k-2}(f))\delta_{2k-1}(\delta_{2k-1}(f))$
$(\delta_{2k-1}(\delta_{2k-1}(f)))'$
$(\delta^{2k-2}(\delta^1(f)))'\delta^{2k-2}(\delta^1(f))\dots\delta^{2k-2}(\delta^{2k-2}(f))(\delta^{2k-2}(\delta_{2k-1}(f)))'(\delta^{2k-1}(\delta_{2k-1}(f)))'$
$\delta^{2k-1}(\delta_{2k-1}(f))$
$(\delta_{2k-1}(\delta_1(f)))'\delta_{2k-1}(\delta_1(f))\dots\delta_{2k-1}(\delta_{2k-2}(f))(\delta_{2k-1}(\delta_{2k-2}(f)))'(\delta_{2k-1}(\delta_{2k-1}(f)))'$
$\delta_{2k-1}(\delta_{2k-1}(f))$
$\delta^{2k-1}(\delta^1(f))(\delta^{2k-1}(\delta^1(f)))'\dots(\delta^{2k-2}(\delta^{2k-2}(f)))'\delta^{2k-2}(\delta^{2k-2}(f))\delta^{2k-1}(\delta^{2k-1}(f))$
$(\delta^{2k-1}(\delta^{2k-1}(f)))'$
$\delta_1(\delta^1(f))(\delta_1(\delta^1(f)))'\dots(\delta_{2k-2}(\delta^1(f)))'\delta_{2k-2}(\delta^1(f))\delta_{2k-1}(\delta^1)(\delta_{2k-1}(\delta^1(f)))'$
$(\delta^1(\delta^1(f)))'\delta^1(\delta^1(f))...\delta^{2k-2}(\delta^1(f))(\delta^{2k-2}(\delta^1(f)))'(\delta^{2k-1}(\delta^1(f)))'\delta^{2k-1}(\delta^1(f))$
$(\delta_1(\delta^1(f)))'\delta_1(\delta^1(f))\dots\delta_{2k-2}(\delta^2(f))(\delta_{2k-2}(\delta^2(f)))'(\delta_{2k-1}(\delta^2(f)))'\delta_{2k-1}(\delta^2(f))$

$\delta^1(\delta_1(f))(\delta^1(\delta_1(f)))' \dots (\delta^{2k-2}(\delta^2(f)))'(\delta^{2k-2}(\delta^2(f)))\delta^{2k-1}(\delta^2(f))(\delta^{2k-1}(\delta^2(f)))'$

.....

$\delta_{2k-2}(\delta^1(f))(\delta_{2k-2}(\delta^1(f)))' \dots (\delta_{2k-2}(\delta^{2k-2}(f)))'\delta_{2k-2}(\delta^{2k-2}(f))\delta_{2k-1}(\delta^{2k-2}(f))$
$(\delta_{2k-1}(\delta^{2k-2}(f)))'$
$(\delta^{2k-2}(\delta_1(f)))'\delta^{2k-2}(\delta_1(f)) \dots \delta^{2k-2}(\delta_{2k-2}(f))(\delta^{2k-2}(\delta_{2k-2}(f)))'(\delta^{2k-1}(\delta^{2k-1}(f)))'$
$\delta^{2k-1}(\delta^{2k-1}(f))$
$(\delta_{2k-1}(\delta^1(f)))'\delta_{2k-1}(\delta^1(f)) \dots \delta_{2k-1}(\delta^{2k-2}(f))(\delta_{2k-2}(\delta^{2k}(f)))'(\delta_{2k-1}(\delta^{2k-1}(f)))'$
$\delta_{2k-1}(\delta^{2k-1}(f))$
$\delta^{2k-1}(\delta_1(f))(\delta^{2k-1}(\delta_1(f)))' \dots \delta^{2k-1}(\delta_{2k-2}(f))(\delta^{2k-1}(\delta_{2k-2}(f)))'(\delta^{2k-1}(\delta_{2k-1}(f)))'$
$\delta^{2k-1}(\delta_{2k-1}(f)) = e$

Similary $\partial_{2k}\partial_{2k+1}(f) = e$, hence $\partial_n\partial_{n+1}(f) = e$ for each $n \geq 1$.

If a function $\varphi : X \to Y$ is continuous and function $f : I^n \to X$ is continuous, then $\varphi \circ f : I^n \to Y$ is continuous. Denote by $(C_n(X), [\,])$ the free commutative (4,2)-group generated by the set $\Sigma_n = \{f : I^n \to X, f - continuous\}$. Also by $(C_n(Y), [\,])$ denote free the commutative (4,2)-group generated by the set $\Sigma_n^{'} = \{f : I^n \to Y, f - continuous\}$. Then the sequence of (4,2)-homomorphisms $C_n\varphi : (C_n(X), [\,]) \to (C_n(Y), [\,])$ defined by

$$C_n\varphi[f_1f_2f_3f_4] = [(\varphi \circ f_1)(\varphi \circ f_2)(\varphi \circ f_3)(\varphi \circ f_4)]$$

define (4,2)-chain map denoted by $S\varphi : wC(X) \to wC(Y)$, where $wC(X)$ and $wC(Y)$ denote the cubic weak (4,2)-chain complexes. Indeed it is enough to prove, that for every $n \in N$ the following diagram is commutative

$$\begin{array}{ccc} (C_n(X), [\,]) & \overset{\partial_{n+1}}{\longleftarrow} & ((C_{n+1}(X), [\,]) \\ \downarrow C_n\varphi & & \downarrow C_{n+1}\varphi \\ (C_n(Y), [\,]) & \overset{\partial_{n+1}}{\longleftarrow} & ((C_{n+1}(Y), [\,]) \end{array}$$

First we note that
$\delta_i(C_{n+1}\varphi(f)(x_1^n) = \delta_i(\varphi \circ f)(x_1^n) = (\varphi \circ f)(x_1^{i-1}, 0, x_i^n) = (\varphi(f(x_1^{i-1}, 0, x_i^n))$
$C_n\varphi(\delta_i(f))(x_1^n) = C_n\varphi(f(x_1^{i-1}, 0, x_i^n) = (\varphi \circ f)(x_1^{i-1}, 0, x_i^n) = \varphi(f(x_1^{i-1}, 0, x_i^n))$
i.e. $C_n\varphi\,(\delta_i(f)) = \delta_i(C_{n+1}\varphi\,(f))$
$\delta^i(C_{n+1}\varphi(f)(x_1^n) = \delta^i(\varphi \circ f)(x_1^n) = (\varphi \circ f)\left((1 - x_k)_{k=1}^{i-1}, 1, (1 - x_k)_{k=i}^n\right)$
$C_n\varphi(\delta^i f)(x_1^n) = C_n\varphi(f\left((1 - x_k)_{k=1}^{i-1}, 1, (1 - x_k)_{k=i}^n\right) =$
$= (\varphi \circ f)\left((1 - x_k)_{k=1}^{i-1}, 1, (1 - x_k)_{k=i}^n\right)$
$C_n\varphi\,(\delta_i^i(f)) = \delta^i(C_{n+1}\varphi\,(f))$
So,
$\partial_{n+1} \circ C_{n+1}\varphi(f) = \partial_{n+1}(C_{n+1}\varphi(f)) = \partial_{n+1}(\varphi \circ f) =$
$= [\delta_1(\varphi \circ f)\delta_2(\varphi \circ f) \cdots \delta_{n+1}(\varphi \circ f)\delta_{n+1}^{'}(\varphi \circ f) \cdots \delta_2^{'}(\varphi \circ f)\delta_1^{'}(\varphi \circ f)(\delta^1(\varphi \circ f))'$
$\cdots (\delta^{n+1}(\varphi \circ f))'\delta^{n+1}(\varphi \circ f) \cdots \delta^2(\varphi \circ f)\delta^1(\varphi \circ f)]_1 =$
$= [(\delta_1(C_{n+1}(f))(\delta_2(C_{n+1}(f)) \cdots (\delta_{n+1}(C_{n+1}(f))(\delta_{n+1}^{'}(C_{n+1}(f)) \cdots$
$\cdots (\delta_1^{'}(C_{n+1}(f)((\delta^1 C_{n+1}(f)))' \cdots ((\delta^{n+1}(C_{n+1}(f)))'(\delta^{n+1}(C_{n+1}(f)) \cdots$
$\cdots (\delta^2(C_{n+1}(f))(\delta^1(C_{n+1}(f))]_1 =$
$= [C_n\varphi(\delta_1(f))C_n\varphi(\delta_2(f)) \cdots C_n\varphi(\delta_{n+1})(f)C_n\varphi(\delta_{n+1}^{'}(f)) \cdots C_n\varphi(\delta_{n+1}^{'}(f))$

$C_n\varphi(\delta^1(f))' \cdots C_n\varphi(\delta^{n+1}(f))' C_n\varphi(\delta^{n+1}(f)) \cdots C_n\varphi(\delta^2(f)) C_n\varphi(\delta^1(f))]_1 =$
$= C_n\varphi[\delta_1(f)\delta_2(f) \cdots \delta_{n+1}(f)(\delta_{n+1}(f))' \cdots (\delta_1(f))'(\delta^1(f))' \cdots (\delta^{n+1}(f))'(\delta^{n+1}(f))$
$\cdots (\delta^1(f))]_1$
$= C_n\varphi(\partial_{n+1}(f)) = C_n\varphi \circ \partial_{n+1}(f)$

Proposition 3.2. *$C : Top \to (4,2) - w\partial K$ is a covariant functor*
Indeed, if $\varphi : X \to Y$; $\psi : Y \to Z$, then
$C_n(\varphi \circ \psi)(f) = (\varphi \circ \psi) \circ f = \varphi \circ (\psi \circ f) = (C_n\varphi)(\psi \circ f) = (C_n\varphi)((C_n\psi)) =$
$= C_n\varphi \circ C_n\psi(f)$
If, $1_X : X \to X$, then
$C_n(1_X)(f) = 1_X \circ f = f = 1_C(f)$ □

Since $C : Top \to (4,2) - w\partial K$; $F_2 : (4,2) - w\partial K \to \partial K$; $H_n : \partial K \to Ab$; $Z_n : K \to Ab$; $B_n : \partial K \to Ab$ are covariants functor then:
$Z_n \circ F_2 \circ C : Top \to Ab$ is a covariant functor denoted by $ZC_{n,2}$;
$B_n \circ F_2 \circ C : Top \to Ab$ is a covariant functor denoted by $BC_{n,2}$;
$H_n \circ F_2 \circ C : Top \to Ab$ is a covariant functor denoted by $HC_{n,2}$.
$n, 2$ **Cubic homology groups of a spaces** X is called homology groups of cubic weak (4,2)-chain complexes, denoted by $HC_{n,2}(X)$. So

$$HC_{n,2}(X) = ZC_{n,2}(X)/BC_{n,2}(X),\ n \geq 0$$

Since $C : Top \to (4,2) - w\partial K$; $F_+ : (4,2) - w\partial K \to \partial K$; $H_n : \partial K \to Ab$; $Z_n : \partial K \to Ab$; $B_n : \partial K \to Ab$ are covariants functor then:
$Z_n \circ F_+ \circ C : Top \to Ab$ is a covariant functor denoted by $ZC_{n,+}$;
$B_n \circ F_+ \circ C : Top \to Ab$ is a covariant functor denoted by $BC_{n,+}$;
$H_n \circ F_+ \circ C : Top \to Ab$ is a covariant functor denoted by $HC_{n,+}$.
$n, +$ **Cubic homology groups of a spaces** X is called homology groups of cubic weak (4,2)-chain complexes, denoted by $HC_{n,+}(X)$. So

$$HC_{n,+}(X) = ZC_{n,+}(X)/BC_{n,+}(X),\ n \geq 0$$

Since $C : Top \to (4,2) - w\partial K$; $F_* : (4,2) - w\partial K \to \partial K$; $H_n : \partial K \to Ab$; $Z_n : \partial K \to Ab$; $B_n : \partial K \to Ab$ are covariants functor then:
$Z_n \circ F_* \circ C : Top \to Ab$ is a covariant functor denoted by $ZC_{n,*}$;
$B_n \circ F_* \circ C : Top \to Ab$ is a covariant functor denoted by $BC_{n,*}$;
$H_n \circ F_* \circ C : Top \to Ab$ is a covariant functor denoted by $HC_{n,*}$.
$n, *$ **Cubic homology groups of a spaces** X is called homology groups of cubic weak (4,2)-chain complexes , denoted by $HC_{n,*}(X)$. So

$$HC_{n,*}(X) = ZC_{n,*}(X)/BC_{n,*}(X), n \geq 0.$$

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