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MENON DESIGNS WITH PARAMETERS (256,120,56)

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ABSTRACT. There are exactly 18 symmetric (256,120,56) designs admitting an automorphism group isomorphic to $Frob_{17\cdot 8} \times Z_3$ acting with orbit size distribution (1,17,17,17,51,51,51,51) for blocks and points. For sixteen of these designs the full automorphism group has order 408 and is isomorphic to $Frob_{17\cdot 8} \times Z_3$, and the remaining two designs have $Frob_{17\cdot 8} \times Frob_{7\cdot 3}$ as full automorphism group. All 18 designs are self-dual. The derived designs (with respect to the fixed block) of the designs with full automorphism group isomorphic to $Frob_{17\cdot 8} \times Frob_{7\cdot 3}$ are 1-rotational.

1. INTRODUCTION

A 2-(v, k, λ) design is a finite incidence structure $(\mathcal{P}, \mathcal{B}, I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1.: $|\mathcal{P}| = v$;
2.: every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$;
3.: every pair of distinct elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

The elements of the set $\mathcal{P}$ are called points and the elements of the set $\mathcal{B}$ are called blocks. If $|\mathcal{P}| = |\mathcal{B}| = v$ and $2 \leq k \leq v - 2$, then a 2-(v, k, λ) design is called a symmetric design.

Given two designs $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, I_1)$ and $\mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2, I_2)$, an isomorphism from $\mathcal{D}_1$ onto $\mathcal{D}_2$ is a bijection which maps points onto points and blocks onto blocks preserving the incidence relation. An isomorphism from a symmetric design $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$. The set of all automorphisms of the design $\mathcal{D}$ forms a group; it is called the full automorphism group of $\mathcal{D}$ and denoted by $Aut\mathcal{D}$.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric (v, k, λ) design and G a subgroup of $Aut\mathcal{D}$. The action of G produces the same number of point and block orbits (see [6, Theorem 3.3, p. 79]). We denote that number by t, the point orbits by $\mathcal{P}_1, \ldots, \mathcal{P}_t$, the block orbits by $\mathcal{B}_1, \ldots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. We shall

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denote the points of the orbit $\mathcal{P}_r$ by $r_0, \ldots, r_{\omega_r - 1}$, (i.e. $\mathcal{P}_r = \{r_0, \ldots, r_{\omega_r-1}\}$). Further, we denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with a representative of the block orbit $\mathcal{B}_i$. The numbers γ_{ir} are independent of the choice of the representative of the block orbit $\mathcal{B}_i$. For those numbers the following equalities hold (see [5]):

$$\sum_{r=1}^{t} \gamma_{ir} = k, \tag{1}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij} \cdot (k - \lambda). \tag{2}$$

Definition 1. *Let $\mathcal{D}$ be a symmetric (v,k,λ) design and $G \leq Aut\ \mathcal{D}$. Further, let $\mathcal{P}_1, \ldots, \mathcal{P}_t$ be the point orbits and $\mathcal{B}_1, \ldots, \mathcal{B}_t$ the block orbits with respect to G, and let $\omega_1, \ldots, \omega_t$ and $\Omega_1, \ldots, \Omega_t$ be the respective orbit lengths. We call $(\mathcal{P}_1, \ldots, \mathcal{P}_t)$ and $(\mathcal{B}_1, \ldots, \mathcal{B}_t)$ the orbit distributions, and $(\omega_1, \ldots, \omega_t)$ and $(\Omega_1, \ldots, \Omega_t)$ the orbit size distributions for the design $\mathcal{D}$ and the group G. A $(t \times t)$-matrix (γ_{ir}) with entries satisfying conditions (1) and (2) is called an orbit structure for the parameters (v,k,λ) and orbit distributions $(\mathcal{P}_1, \ldots, \mathcal{P}_t)$ and $(\mathcal{B}_1, \ldots, \mathcal{B}_t)$.*

The first step – when constructing designs for given parameters and orbit distributions – is to find all compatible orbit structures (γ_{ir}). The next step, called indexing, consists in determining exactly which points from the point orbit $\mathcal{P}_r$ are incident with a chosen representative of the block orbit $\mathcal{B}_i$ for each number γ_{ir}. Because of the large number of possibilities, it is often necessary to involve a computer in both steps of the construction.

Definition 2. *The set of all indices of points of the orbit $\mathcal{P}_r$ which are incident with a fixed representative of the block orbit $\mathcal{B}_i$ is called the index set for the position (i,r) of the orbit structure and the given representative.*

A Hadamard matrix of order m is an $(m \times m)$-matrix $H = (h_{i,j})$, $h_{i,j} \in \{-1, 1\}$, satisfying $HH^T = H^TH = mI$, where I is the unit matrix. A Hadamard matrix is regular if the row and column sums are constant. It is well known that the existence of a symmetric design with parameters $(4u^2, 2u^2 - u, u^2 - u)$ is equivalent to the existence of a regular Hadamard matrix of order $4u^2$ (see [10, Theorem 1.4 p. 280]). Such symmetric designs are called Menon designs.

It is known that using a regular Hadamard matrix of order $4u^2$ one can construct a regular Hadamard matrix of order $16u^2$ (see, e.g., [2]). So one constructs symmetric (256,120,56) designs from symmetric (64,28,12) designs. However, as far as we know, the two designs described in [7] are the only explicitly constructed Menon (256,120,56) designs.

2. Symmetric (256,120,56) Designs

Lemma 1. *Let ρ be an automorphism of a symmetric* $(256,120,56)$ *design* $\mathcal{D}$. *If* $|\langle\rho\rangle| = 17$, *then ρ fixes exactly one point and one block of* $\mathcal{D}$.

Proof. By [6, Theorem 3.1 p. 78], $\langle\rho\rangle$ fixes the same number of points and blocks. Denote that number by f. Obviously, $f \equiv 1\,(mod\,17)$. Using the formula $f \leq v-2(k-\lambda)$ (see [6, Corollary 3.7 p. 82]) we get $f \in \{1,18,35,52,69,86,103,120\}$. Suppose that $f=18$. Since a fixed block must be a union of $\langle\rho\rangle$-orbits of points, every fixed block contains 1 or 18 fixed points. Two fixed blocks must intersect in 5 fixed points, since $\lambda = 56$. This is impossible, so $f \neq 18$. In a similar way one can prove that $f \notin \{35,52,69,86,103,120\}$. □

We are not able to construct all symmetric (256,120,56) designs admitting an automorphism of order 17, because there are too many possibilities to be scrutinized for today's computer capabilities. Therefore, in this paper, we shall assume that an automorphism group isomorphic to $Frob_{17\cdot 8} \times Z_3$ acts on the symmetric (256,120,56) designs to be constructed with orbit size distribution (1,17,17,17,51,51,51,51) for blocks and points. That means that the permutation of order eight has precisely 16 fixed points and 16 fixed blocks, and the permutation of order three fixes precisely 52 points and blocks.

Lemma 2. *Let the group $Frob_{17\cdot 8}$ acts as an automorphism group of a symmetric* $(256,120,56)$ *design $\mathcal{D}$ in such a way that the permutation of order eight fixes exactly 18 points of $\mathcal{D}$. Then $Frob_{17\cdot 8}$ acts on the design $\mathcal{D}$ semistandardly with one fixed block and point and* 15 *orbits of length* 17, *with the orbit structure OS shown below:*

$$\begin{pmatrix}
1 & 17 & 17 & 17 & 17 & 17 & 17 & 17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 9 & 9 & 9 & 9 & 9 & 9 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
1 & 9 & 1 & 9 & 9 & 9 & 9 & 9 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
1 & 9 & 9 & 1 & 9 & 9 & 9 & 9 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
1 & 9 & 9 & 9 & 1 & 9 & 9 & 9 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
1 & 9 & 9 & 9 & 9 & 1 & 9 & 9 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
1 & 9 & 9 & 9 & 9 & 9 & 1 & 9 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
1 & 9 & 9 & 9 & 9 & 9 & 9 & 1 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 1 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 9 & 1 & 9 & 9 & 9 & 9 & 9 & 9 \\
0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 9 & 9 & 1 & 9 & 9 & 9 & 9 & 9 \\
0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 9 & 9 & 9 & 1 & 9 & 9 & 9 & 9 \\
0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 9 & 9 & 9 & 9 & 1 & 9 & 9 & 9 \\
0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 1 & 9 & 9 \\
0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 1 & 9 \\
0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 1
\end{pmatrix}$$

where the first row and column correspond to the fixed block and point, respectively.

Proof. Let the group G be isomorphic to the Frobenius group $Frob_{17\cdot 8}$, with a presentation

$$G = \langle \rho, \sigma | \ \rho^{17} = 1, \sigma^8 1, \ \rho^\sigma = \rho^2 \rangle.$$

The Frobenius kernel $\langle \rho \rangle$ of order 17 acts on $\mathcal{D}$ semistandardly with one fixed block and point and 15 orbits of length 17. Since $\langle \rho \rangle \lhd G$, the element σ of order 8 maps $\langle \rho \rangle$-orbits onto $\langle \rho \rangle$-orbits. The permutation σ fixes exactly 16 points, so G acts on $\mathcal{D}$ with one fixed block and point and 15 orbits of length 17 for blocks and points.

The stabilizer of each block from a block orbit of length 17 is conjugate to $\langle \sigma \rangle$. Therefore, the entries of the orbit structures corresponding to point and block orbits of length 17 must satisfy the condition $\gamma_{ir} \equiv 0, 1 \ (mod\ 8)$. Solving equations (1) and (2), we get – up to isomorphism – only one solution, the orbit structure OS. □

Let G_1 be isomorphic to the group $Frob_{17\cdot 8} \times Z_3$. We may write

$$G_1 = \langle \rho, \sigma, \tau | \ \rho^{17} = 1, \ \sigma^8 = 1, \ \tau^3 = 1, \ \rho^\sigma = \rho^2, \ \rho^\tau = \rho, \ \sigma^\tau = \sigma \rangle.$$

Theorem 1. *There are exactly* 18 *symmetric* $(256, 120, 56)$ *designs admitting an automorphism group isomorphic to* $Frob_{17\cdot 8} \times Z_3$ *acting with orbit size distribution* $(1, 17, 17, 17, 51, 51, 51, 51)$ *for blocks and points. For sixteen of these designs the full automorphism group has order* 408 *and is isomorphic to* $Frob_{17\cdot 8} \times Z_3$, *and the remaining two designs have* $Frob_{17\cdot 8} \times Frob_{7\cdot 3}$ *as full automorphism group. All* 18 *designs are self-dual.*

Proof. The designs have been constructed by the method described in [1] and [4]. We denote the points by $1_0, 2_i \ldots, 16_i$, $i = 0, 1, \ldots, 16$ and put $G_1 = \langle \rho, \sigma, \tau \rangle$ where the generators for G_1 are permutations defined as follows:

: $\rho = (1_0)(I_0, I_1, \ldots, I_{16})$, $I = 2, \ldots, 16$,
: $\sigma = (1_0)(K_0)(K_1K_2K_4K_8K_{16}K_{15}K_{13}K_9)(K_3K_6K_{12}K_7K_{14}K_{11}K_5K_{10})$, $K = 2, \ldots, 16$,
: $\tau = (1_0)(2_i)(3_i4_i5_i)(6_i7_i8_i)(9_i)(10_i)(11_i12_i13_i)(14_i15_i16_i)$, $i = 0, 1, \ldots, 16$.

Indexing the fixed part of an orbit structure is a trivial task. Therefore, we shall consider only the right-lower part of the orbit structure of order 15. To eliminate isomorphic structures during the indexing process we have used the permutation which – on each $\langle \rho \rangle$-point-orbit – acts as $x \mapsto 3x \ (mod\ 17)$, and those automorphisms of the orbit structure OS which commute with τ.

As representatives for the block orbits we chose blocks fixed by $\langle \sigma \rangle$. Therefore, the index sets – numbered from 0 to 4 – which could occur in the designs are

among the following:
$0 = \{0\}$, $1 = \{1, 2, 4, 8, 9, 13, 15, 16\}$, $2 = \{3, 5, 6, 7, 10, 11, 12, 14\}$,
$3 = \{0, 1, 2, 4, 8, 9, 13, 15, 16\}$, $4 = \{0, 3, 5, 6, 7, 10, 11, 12, 14\}$.

The indexing process of the orbit structure OS led to 18 designs, denoted by $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{18}$. These designs have mutually different statistics of intersections of any three blocks, hence they are mutually non-isomorphic. Statistics of intersections of any three blocks of dual designs tells us that all 18 designs are self-dual.

We have determined the automorphism groups of the designs constructed using GAP [8] and a program by V. D. Tonchev [9]. The designs $\mathcal{D}_{10}$ and $\mathcal{D}_{14}$ have the full automorphism group isomorphic to the group $Frob_{17\cdot 8} \times Frob_{7\cdot 3}$ of order 2856, and the full automorphism group of the other 16 designs is isomorphic to $Frob_{17\cdot 8} \times Z_3$.

The designs $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{18}$ are ordered lexicographically. We write down base blocks for the designs $\mathcal{D}_1$ and $\mathcal{D}_2$ in terms of the index sets defined above:

$\mathcal{D}_1$	$\mathcal{D}_2$
0 3 3 3 3 3 3 1 2 1 1 1 1 1 1	0 3 3 3 3 3 3 1 2 1 1 1 1 1 1
4 0 3 4 3 3 4 1 1 1 2 2 1 2 2	4 0 3 4 3 3 4 1 1 1 2 2 1 2 2
4 4 0 3 4 3 3 1 1 2 1 2 2 1 2	4 4 0 3 4 3 3 1 1 2 1 2 2 1 2
4 3 4 0 3 4 3 1 1 2 2 1 2 2 1	4 3 4 0 3 4 3 1 1 2 2 1 2 2 1
4 4 3 4 0 3 4 2 2 2 1 1 2 2 1	4 4 3 4 0 4 3 2 2 1 1 2 2 2 1
4 4 4 3 4 0 3 2 2 1 2 1 1 2 2	4 4 4 3 3 0 4 2 2 2 1 1 1 2 2
4 3 4 4 3 4 0 2 2 1 1 2 2 1 2	4 3 4 4 4 3 0 2 2 1 2 1 2 1 2
2 2 2 2 1 1 1 0 4 4 4 4 3 3 3	2 2 2 2 1 1 1 0 4 4 4 4 3 3 3
1 2 2 2 1 1 1 3 0 3 3 3 4 4 4	1 2 2 2 1 1 1 3 0 3 3 3 4 4 4
2 2 1 1 1 2 2 3 4 0 4 3 4 3 4	2 2 1 1 2 1 2 3 4 0 4 3 4 4 3
2 1 2 1 2 1 2 3 4 3 0 4 4 4 3	2 1 2 1 2 2 1 3 4 3 0 4 3 4 4
2 1 1 2 2 2 1 3 4 4 3 0 3 4 4	2 1 1 2 1 2 2 3 4 4 3 0 4 3 4
2 2 1 1 1 2 1 4 3 3 3 4 0 4 3	2 2 1 1 1 2 1 4 3 3 4 3 0 3 4
2 1 2 1 1 1 2 4 3 4 3 3 3 0 4	2 1 2 1 1 1 2 4 3 3 3 4 4 0 3
2 1 1 2 2 1 1 4 3 3 4 3 4 3 0	2 1 1 2 2 1 1 4 3 4 3 3 3 4 0

From these "small" incidence matrices it is easy to obtain incidence matrices in the ordinary form. □

The 2−rank of designs $\mathcal{D}_1, \ldots, \mathcal{D}_{18}$ equals 114. The designs $\mathcal{D}_{10}$ and $\mathcal{D}_{14}$ are isomorphic to the designs described in [7].

3. Derived and Residual Designs

Let $\mathcal{D}$ be a symmetric (v, k, λ) design and let x be a block of $\mathcal{D}$. Remove x and all points that do not belong to x from other blocks. The result is a 2-$(k, \lambda, \lambda - 1)$

design $\mathcal{D}_x$, a derived design of $\mathcal{D}$ with respect to the block x. Removing the block x and all the points belonging to that block from the design $\mathcal{D}$, one obtains its residual $2-(v-k, k-\lambda, \lambda)$ design $\mathcal{D}^x$ (see [6]).

Our aim is to investigate the full automorphism groups of all pairwise non-isomorphic derived and residual designs obtained from the symmetric designs $\mathcal{D}_1, \dots, \mathcal{D}_{18}$. Because of the following corollary, it suffice to consider the derived and the residual designs with respect to the block orbits representatives of the full automorphism groups of the constructed (256,120,56) designs.

Corollary 1. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric design, $x, x' \in \mathcal{B}$ and $G \leq Aut\mathcal{D}$. If $x' \in xG$, then $\mathcal{D}_x \cong \mathcal{D}_{x'}$ and $\mathcal{D}^x \cong \mathcal{D}^{x'}$.

Proof. [3, Corollary 1]. □

We have obtained 136 pairwise nonisomorphic derived 2-(120,56,55) designs, and 136 pairwise nonisomorphic residual 2-(136,64,56) designs. Further investigation of the constructed derived and residual designs leads to the following results:

TABLE 1. **Derived 2-(120,56,55) designs**

The order of the full automorphism group	*The full automorphism group structure*	*Number of designs*
2856	$Frob_{17\cdot 8} \times Frob_{7\cdot 3}$	2
408	$Frob_{17\cdot 8} \times Z_3$	16
168	$Frob_{7\cdot 3} \times Z_8$	2
24	Z_{24}	52
8	Z_8	64

TABLE 2. **Residual 2-(136,64,56) designs**

The order of the full automorphism group	*The full automorphism group structure*	*Number of designs*
2856	$Frob_{17\cdot 8} \times Frob_{7\cdot 3}$	2
408	$Frob_{17\cdot 8} \times Z_3$	16
168	$Frob_{7\cdot 3} \times Z_8$	2
24	Z_{24}	52
8	Z_8	64

A 2-(v,k,λ) design $\mathcal{D}$ is called $s-$rotational if some automorphism of $\mathcal{D}$ has one fixed point and s cycles each of length $\frac{v-1}{s}$. Obviously, the derived designs of $\mathcal{D}_{10}$ and $\mathcal{D}_{14}$ with respect to the first block are 1-rotational 2-(120,56,55) designs. These are the derived designs having the full automorphism group of order 2856.

Incidence matrices of the symmetric designs described in this article are available at ftp://polifem.ffri.hr/matematika/sanja/.

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