

**SOME INEQUALITIES FOR THE ČEBYŠEV FUNCTIONAL
AND GENERAL FOUR-POINT QUADRATURE FORMULAE OF
EULER TYPE**

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Abstract. We use inequalities for the Čebyšev functional in terms of the first derivative (see [5]), for some new bounds for the remainder of four-point quadrature formulae of Euler type and its generalizations for Euler Bullen-Simpson's 3/8 formula. As special cases, we consider some new bounds for Euler Simpson's 3/8 formula, Euler Simpson's formula and Euler Gauss 2-point formula.

1. INTRODUCTION

Let $x \in (0, 1/2]$ and $f : [0, 1] \rightarrow \mathbb{R}$ be such that $f^{(2n+1)}$ is continuous of bounded variation on $[0, 1]$ for some $n \geq 0$. In [6], the following formula is obtained:

$$\int_0^1 f(t)dt - (1/2 - A(x))[f(0) + f(1)] - A(x)[f(x) + f(1-x)] + T_{2n}(x) = \frac{1}{(2n+2)!} \int_0^1 F_{2n+2}(x, t)df^{(2n+1)}(t), \quad (1.1)$$

where, for $t \in \mathbb{R}$,

$$T_{2n}(x) = \sum_{k=2}^{2n} \frac{1}{k!} G_k(x, 0) [f^{(k-1)}(1) - f^{(k-1)}(0)],$$

$$G_k(x, t) = [1 - 2A(x)]B_k^*(1-t) + A(x) [B_k^*(x-t) + B_k^*(1-x-t)], \quad k \geq 1$$

$$F_k(x, t) = G_k(x, t) - G_k(x, 0), \quad k \geq 2.$$

The functions $B_k(t)$ are the Bernoulli polynomials, $B_k = B_k(0)$ are the Bernoulli numbers, and $B_k^*(t)$, $k \geq 0$, are periodic functions of period 1, related to the Bernoulli polynomials as

$$B_k^*(t) = B_k(t), \quad 0 \leq t < 1 \quad \text{and} \quad B_k^*(t+1) = B_k^*(t), \quad t \in \mathbb{R}.$$

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The Bernoulli polynomials $B_k(t)$, $k \geq 0$ are uniquely determined by the following identities

$$B'_k(t) = kB_{k-1}(t), \quad k \geq 1; \quad B_0(t) = 1, \quad B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0.$$

For some further details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [3]. We have that $B_0^*(t) = 1$ and $B_1^*(t)$ is a discontinuous function with a jump of -1 at each integer. It follows that $B_k(1) = B_k(0) = B_k$ for $k \geq 2$, so that $B_k^*(t)$ are continuous functions for $k \geq 2$. We get

$$B_k^{*'}(t) = kB_{k-1}^*(t), \quad k \geq 1 \quad (1.2)$$

for every $t \in \mathbb{R}$ when $k \geq 3$, and for every $t \in \mathbb{R} \setminus \mathbb{Z}$ when $k = 1, 2$.

If one wants to obtain from (1.1) the quadrature formula with the maximum degree of exactness (if values of derivatives at the end points are not to be included in the quadrature, then it is equal to 3), a condition $G_2(x, 0) = 0$ has to be imposed. In this way we get:

$$A(x) = -\frac{B_2}{2(B_2(x) - B_2)} = \frac{1}{12x(1-x)}. \quad (1.3)$$

Formula (1.1) now becomes:

$$\int_0^1 f(t)dt - Q(0, x, 1-x, 1) + T_{2n}^{Q_4}(x) = \frac{1}{(2n+2)!} \int_0^1 F_{2n+2}^{Q_4}(x, t) df^{(2n+1)}(t), \quad (1.4)$$

where

$$Q(0, x, 1-x, 1) = \frac{1}{12x(1-x)} [-6B_2(x)f(0) + f(x) + f(1-x) - 6B_2(x)f(1)], \quad (1.5)$$

$$T_{2n}^{Q_4}(x) = \sum_{k=2}^n \frac{1}{(2k)!} G_{2k}^{Q_4}(x, 0) [f^{(2k-1)}(1) - f^{(2k-1)}(0)], \quad (1.6)$$

$$G_k^{Q_4}(x, t) = \frac{1}{12x(1-x)} [B_k^*(x-t) - 12B_2(x) \cdot B_k^*(1-t) + B_k^*(1-x-t)], \quad (1.7)$$

$$F_k^{Q_4}(x, t) = G_k^{Q_4}(x, t) - G_k^{Q_4}(x, 0), \quad k \geq 2. \quad (1.8)$$

Assuming $f^{(2n-1)}$ is continuous of bounded variation on $[0, 1]$ for some $n \geq 1$, then we get:

$$\int_0^1 f(t)dt - Q(0, x, 1-x, 1) + T_{2n}^{Q_4}(x) = \frac{1}{(2n)!} \int_0^1 G_{2n}^{Q_4}(x, t) df^{(2n-1)}(t), \quad (1.9)$$

while assuming $f^{(2n)}$ fulfills the same condition for some $n \geq 0$, we get:

$$\int_0^1 f(t)dt - Q(0, x, 1-x, 1) + T_{2n}^{Q_4}(x) = \frac{1}{(2n+1)!} \int_0^1 G_{2n+1}^{Q_4}(x, t) df^{(2n)}(t). \quad (1.10)$$

More about quadrature formulae and error estimations (from the point of view of inequality theory) can be found in monographs [2] and [7].

For two Lebesgue integrable functions $f, g : [a, b] \rightarrow \mathbb{R}$, consider the Čebyšev functional:

$$T(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{b-a} \int_a^b f(t)dt \cdot \frac{1}{b-a} \int_a^b g(t)dt. \quad (1.11)$$

In [5] the authors proved the following theorems:

Theorem 1. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two absolutely continuous functions on $[a, b]$ with*

$$(\cdot - a)(b - \cdot)(f')^2, (\cdot - a)(b - \cdot)(g')^2 \in L[a, b].$$

Then we have the inequality

$$\begin{aligned} |T(f, g)| &\leq \frac{1}{\sqrt{2}} [T(f, f)]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left(\int_a^b (x-a)(b-x) [g'(x)]^2 dx \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2(b-a)} \left(\int_a^b (x-a)(b-x) [f'(x)]^2 dx \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_a^b (x-a)(b-x) [g'(x)]^2 dx \right)^{\frac{1}{2}}. \end{aligned} \quad (1.12)$$

The constant $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}$ are best possible in (1.12).

Theorem 2. *Assume that $g : [a, b] \rightarrow \mathbb{R}$ is monotonic nondecreasing on $[a, b]$ and $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[a, b]$. Then we have the inequality*

$$|T(f, g)| \leq \frac{1}{2(b-a)} \|f'\|_\infty \int_a^b (x-a)(b-x) dg(x). \quad (1.13)$$

The constant $\frac{1}{2}$ is best possible.

In this paper we will use the above theorems to get some new bounds for the remainders of general four-point formulae (1.9) and (1.10). Applications for Euler Bullen-Simpson's 3/8 formula are also proved. As special cases, some new bounds for Euler Simpson's 3/8 formula, Euler Simpson's formula and Euler Gauss 2-point formula are considered.

2. APPLICATIONS FOR THE GENERAL FOUR-POINT FORMULAE OF EULER TYPE

Using Theorem 1 for identities (1.9) and (1.10) we get the following Grüss type inequalities:

Theorem 3. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $f^{(2n)}$ is absolutely continuous for some $n \geq 1$ and $(f^{(2n+1)})^2 \in L_1[0, 1]$. Then for $x \in (0, 1/2]$ we have*

$$\int_0^1 f(t)dt - Q(0, x, 1-x, 1) + T_{2n}^{Q_4}(x) = TG_{2n}(f), \quad (2.1)$$

and the remainder $TG_{2n}(f)$ satisfies the estimations

$$\begin{aligned} |TG_{2n}(f)| &\leq \\ &\frac{1}{12x(1-x)} \left[\frac{1}{(4n)!} (-B_{4n} - B_{4n}(1-2x) + 24B_2(x)B_{4n}(x) \right. \\ &\quad \left. - 72B_2^2(x)B_{4n}) \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f^{(2n+1)}(t)]^2 dt \right)^{\frac{1}{2}}. \end{aligned} \quad (2.2)$$

For $f : [0, 1] \rightarrow \mathbb{R}$ such that $f^{(2n+1)}$ is absolutely continuous for some $n \geq 0$ and $(f^{(2n+2)})^2 \in L_1[0, 1]$ we have

$$\int_0^1 f(t)dt - Q(0, x, 1-x, 1) + T_{2n}^{Q_4}(x) = TG_{2n+1}(f), \quad (2.3)$$

and the remainder $TG_{2n+1}(f)$ satisfies the estimations

$$\begin{aligned} |TG_{2n+1}(f)| &\leq \\ &\frac{1}{12x(1-x)} \left[\frac{1}{(4n+2)!} (B_{4n+2} + B_{4n+2}(1-2x) - 24B_2(x)B_{4n+2}(x) \right. \\ &\quad \left. + 72B_2^2(x)B_{4n+2}) \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f^{(2n+2)}(t)]^2 dt \right)^{\frac{1}{2}}. \end{aligned} \quad (2.4)$$

Proof. If we apply Theorem 1 for $f \rightarrow G_k^{Q_4}$, $g \rightarrow f^{(k)}$, we deduce

$$\begin{aligned} &\left| \int_0^1 G_k^{Q_4}(x, t) f^{(k)}(t) dt - \int_0^1 G_k^{Q_4}(x, t) dt \cdot \int_0^1 f^{(k)}(t) dt \right| \\ &\leq \frac{1}{\sqrt{2}} \left[T \left(G_k^{Q_4}(x, \cdot), G_k^{Q_4}(x, \cdot) \right) \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f^{(k+1)}(t)]^2 dt \right)^{\frac{1}{2}} \end{aligned} \quad (2.5)$$

where

$$T \left(G_k^{Q_4}(x, \cdot), G_k^{Q_4}(x, \cdot) \right) = \int_0^1 [G_k^{Q_4}(x, t)]^2 dt - \left[\int_0^1 G_k^{Q_4}(x, t) dt \right]^2.$$

By easy calculation we get

$$\int_0^1 G_k^{Q_4}(x, t) dt = 0,$$

and using integration by part we have

$$\begin{aligned} &\int_0^1 \left(G_k^{Q_4}(x, t) \right)^2 dt \\ &= (-1)^{k-1} \frac{k(k-1) \dots 2}{(k+1)(k+2) \dots (2k-1)} \left[\int_0^1 G_1^{Q_4}(x, t) G_{2k-1}^{Q_4}(x, t) dt \right] \\ &= \frac{(-1)^{k-1} (k!)^2}{12x(1-x) (2k)!} \\ &\times \left[12x(1-x) \int_0^1 G_{2k}^{Q_4}(x, t) dt + G_{2k}^{Q_4}(x, x) + G_{2k}^{Q_4}(x, 1-x) - 12B_2(x)G_{2k}(x, 0) \right] \end{aligned}$$

$$= \frac{(-1)^{k-1}}{144x^2(1-x)^2} \frac{(k!)^2}{(2k)!} [2B_{2k} + 2B_{2k}(1-2x) - 48B_2(x)B_{2k}(x) + 144B_2^2(x)B_{2k}].$$

If we put $k = 2n$ using (1.9) and (2.5), we deduce the representation (2.1) and the bound (2.2). For $k = 2n + 1$ using (1.10) and (2.5), we deduce the representation (2.3) and the bound (2.4). \square

Remark 2.1: Because of (1.8) we get

$$\int_0^1 F_k^{Q4}(x, t) dt = \int_0^1 G_k^{Q4}(x, t) dt - \int_0^1 G_k^{Q4}(x, 0) dt = -G_k^{Q4}(x, 0),$$

and also

$$\begin{aligned} \int_0^1 [F_k^{Q4}(x, t)]^2 dt &= \int_0^1 [G_k^{Q4}(x, t)]^2 dt - 2G_k^{Q4}(x, 0) \int_0^1 G_k^{Q4}(x, t) dt \\ &\quad + [G_k^{Q4}(x, 0)]^2. \end{aligned}$$

So, if we put $k = 2n + 2$ in the proof of above theorem, using (1.4) similar as in (2.5) (with $n \leftrightarrow n + 1$), we deduce the representation (2.1) and the bound (2.2), too.

Corollary 3.1. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $f^{(2n+1)}$ is absolutely continuous for some $n \geq 0$, $(f^{(2n+2)})^2 \in L_1[0, 1]$ and $f^{(2n+1)} \geq 0$ on $[0, 1]$. Then for $x \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}]$ we have*

$$\begin{aligned} 0 \leq (-1)^n \left\{ \int_0^1 f(t) dt - Q(0, x, 1-x, 1) + T_{2n}^{Q4}(x) \right\} \leq & \quad (2.6) \\ \frac{1}{12x(1-x)} \left[\frac{1}{(4n+2)!} (B_{4n+2} + B_{4n+2}(1-2x) - 24B_2(x)B_{4n+2}(x) \right. \\ \left. + 72B_2^2(x)B_{4n+2}) \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f^{(2n+2)}(t)]^2 dt \right)^{\frac{1}{2}}, \end{aligned}$$

and for $x \in [\frac{1}{3}, \frac{1}{2}]$

$$\begin{aligned} 0 \leq (-1)^{n+1} \left\{ \int_0^1 f(t) dt - Q(0, x, 1-x, 1) + T_{2n}^{Q4}(x) \right\} \leq & \quad (2.7) \\ \frac{1}{12x(1-x)} \left[\frac{1}{(4n+2)!} (B_{4n+2} + B_{4n+2}(1-2x) - 24B_2(x)B_{4n+2}(x) \right. \\ \left. + 72B_2^2(x)B_{4n+2}) \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f^{(2n+2)}(t)]^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Proof. We are using Lemma 1 from [6]. \square

If in Theorem 3 we choose $x = 1/3, 1/2, 1/2 - \sqrt{3}/6$ we get inequality related to Euler Simpson's 3/8, Euler Simpson's and Euler Gauss 2-point formula:

Corollary 3.2. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $f^{(2n)}$ is absolutely continuous for some $n \geq 1$ and $(f^{(2n+1)})^2 \in L_1[0, 1]$. Then we have*

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \frac{1}{8} \left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1) \right] + T_{2n}^{Q4}\left(\frac{1}{3}\right) \right| \quad (2.8) \\ & \leq \frac{3}{8} \left[-\frac{1}{(4n)!} \left(\frac{1}{18} + \frac{7}{2} 3^{-4n} \right) B_{4n} \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f^{(2n+1)}(t)]^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

If $f^{(2n+1)}$ is absolutely continuous for some $n \geq 0$ and $(f^{(2n+2)})^2 \in L_1[0, 1]$ we have

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \frac{1}{8} \left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1) \right] + T_{2n}^{Q4}\left(\frac{1}{3}\right) \right| \quad (2.9) \\ & \leq \frac{3}{8} \left[\frac{1}{(4n+2)!} \left(\frac{1}{18} + \frac{7}{2} 3^{-4n-2} \right) B_{4n+2} \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f^{(2n+2)}(t)]^2 dt \right)^{\frac{1}{2}}, \end{aligned}$$

where $T_0^{Q4}\left(\frac{1}{3}\right) = T_2^{Q4}\left(\frac{1}{3}\right) = 0$ and

$$T_{2n}^{Q4}\left(\frac{1}{3}\right) = \frac{1}{8} \sum_{k=2}^n \frac{(3^{2-2k} - 1) B_{2k}}{(2k)!} [f^{(2k-1)}(1) - f^{(2k-1)}(0)].$$

Remark 2.2: For $n = 1$ in (2.8) if f'' is absolutely continuous and $(f''')^2 \in L_1[0, 1]$ we get:

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \frac{1}{8} \left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1) \right] \right| \\ & \leq \frac{1}{24\sqrt{90}} \times \left(\int_0^1 t(1-t) [f'''(t)]^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

If f' is absolutely continuous, $(f'')^2 \in L_1[0, 1]$ and $n = 0$ in (2.9) we have

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \frac{1}{8} \left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1) \right] \right| \\ & \leq \frac{1}{8\sqrt{3}} \times \left(\int_0^1 t(1-t) [f''(t)]^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Corollary 3.3. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $f^{(2n)}$ is absolutely continuous for some $n \geq 1$ and $(f^{(2n+1)})^2 \in L_1[0, 1]$. Then we have*

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \frac{1}{6} \left[f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right] + T_{2n}^{Q4}\left(\frac{1}{2}\right) \right| \quad (2.10) \\ & \leq \frac{1}{3} \left[-\frac{1}{(4n)!} \left(\frac{1}{2} + 2^{2-4n} \right) B_{4n} \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f^{(2n+1)}(t)]^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

If $f^{(2n+1)}$ is absolutely continuous for some $n \geq 0$ and $(f^{(2n+2)})^2 \in L_1[0, 1]$ we have

$$\begin{aligned} & \left| \int_0^1 f(t)dt - \frac{1}{6} \left[f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right] + T_{2n}^{Q4}\left(\frac{1}{2}\right) \right| \quad (2.11) \\ & \leq \frac{1}{3} \left[\frac{1}{(4n+2)!} \left(\frac{1}{2} + 2^{-4n}\right) B_{4n+2} \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f^{(2n+2)}(t)]^2 dt \right)^{\frac{1}{2}}, \end{aligned}$$

where $T_0^{Q4}\left(\frac{1}{2}\right) = T_2^{Q4}\left(\frac{1}{2}\right) = 0$ and

$$T_{2n}^{Q4}\left(\frac{1}{2}\right) = \frac{1}{3} \sum_{k=2}^n \frac{(1-2^{2-2k})B_{2k}}{(2k)!} [f^{(2k-1)}(1) - f^{(2k-1)}(0)].$$

Remark 2.3: For $n = 1$ in (2.10) if f'' is absolutely continuous and $(f''')^2 \in L_1[0, 1]$ we get:

$$\left| \int_0^1 f(t)dt - \frac{1}{6} \left[f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right] \right| \leq \frac{1}{12\sqrt{60}} \times \left(\int_0^1 t(1-t) [f'''(t)]^2 dt \right)^{\frac{1}{2}}.$$

If f' is absolutely continuous, $(f'')^2 \in L_1[0, 1]$ and $n = 0$ in (2.11) we have

$$\left| \int_0^1 f(t)dt - \frac{1}{6} \left[f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right] \right| \leq \frac{1}{6\sqrt{2}} \times \left(\int_0^1 t(1-t) [f''(t)]^2 dt \right)^{\frac{1}{2}}.$$

Corollary 3.4. Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $f^{(2n)}$ is absolutely continuous for some $n \geq 1$ and $(f^{(2n+1)})^2 \in L_1[0, 1]$. Then we have

$$\begin{aligned} & \left| \int_0^1 f(t)dt - \left[f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \right] + T_{2n}^{Q4}\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) \right| \quad (2.12) \\ & \leq \frac{1}{2} \left[-\frac{1}{(4n)!} \left(B_{4n} + B_{4n} \left(\frac{\sqrt{3}}{3}\right) \right) \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f^{(2n+1)}(t)]^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

If $f^{(2n+1)}$ is absolutely continuous for some $n \geq 0$ and $(f^{(2n+2)})^2 \in L_1[0, 1]$ we have

$$\begin{aligned} & \left| \int_0^1 f(t)dt - \left[f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \right] + T_{2n}^{Q4}\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) \right| \\ & \leq \frac{1}{2} \left[\frac{1}{(4n+2)!} \left(B_{4n+2} + B_{4n+2} \left(\frac{\sqrt{3}}{3}\right) \right) \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f^{(2n+2)}(t)]^2 dt \right)^{\frac{1}{2}}, \quad (2.13) \end{aligned}$$

where $T_0^{Q4}\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) = T_2^{Q4}\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) = 0$ and

$$T_{2n}^{Q4}\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) = \sum_{k=2}^n \frac{2^{2k}}{(2k)!} B_{2k} \left(\frac{3-\sqrt{3}}{6}\right) [f^{(2k-1)}(1) - f^{(2k-1)}(0)].$$

Remark 2.4: For $n = 1$ in (2.12) if f'' is absolutely continuous and $(f''')^2 \in L_1[0, 1]$ we get:

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \left[f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \right] \right| \\ & \leq \frac{1}{4} \left[-\frac{17}{270} + \frac{1}{9\sqrt{3}} \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f'''(t)]^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

If f' is absolutely continuous, $(f'')^2 \in L_1[0, 1]$ and $n = 0$ in (2.13) we have

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \left[f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \right] \right| \\ & \leq \frac{1}{8} \left[\frac{1}{3} - \frac{1}{2\sqrt{3}} \right]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) [f''(t)]^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

The following Grüss type inequality also holds.

Theorem 4. Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $f^{(2n)}$ is absolutely continuous and $f^{(2n+1)} \geq 0$ on $[0, 1]$, $n \geq 1$. Then we have the representation (2.1) and the remainder $TG_{2n}(f)$ satisfies the bound

$$\begin{aligned} |TG_{2n}(f)| & \leq \\ & \leq \frac{1}{2(2n-1)!} \left\| G_{2n-1}^{Q4}(x, t) \right\|_{\infty} \left\{ \frac{f^{(2n-1)}(0) + f^{(2n-1)}(1)}{2} - f^{(2n-2)}[0, 1] \right\}. \end{aligned} \quad (2.14)$$

Let $f^{(2n+1)}$ is absolutely continuous and $f^{(2n+2)} \geq 0$ on $[0, 1]$, $n \geq 0$. Then we have the representation (2.3) and the remainder $TG_{2n+1}(f)$ satisfies the bound

$$\begin{aligned} |TG_{2n+1}(f)| & \leq \\ & \leq \frac{1}{2(2n)!} \left\| G_{2n}^{Q4}(x, t) \right\|_{\infty} \left\{ \frac{f^{(2n)}(0) + f^{(2n)}(1)}{2} - f^{(2n-1)}[0, 1] \right\} \end{aligned} \quad (2.15)$$

for any $x \in (0, 1/2]$ and

$$f^{(k)}[0, 1] = f^{(k)}(1) - f^{(k)}(0).$$

Proof. If we apply Theorem 2 for $f \rightarrow G_{2n}^{Q4}$, $g \rightarrow f^{(2n)}$, we deduce

$$\begin{aligned} & \left| \int_0^1 G_{2n}^{Q4}(x, t) f^{(2n)}(t) dt - \int_0^1 G_{2n}^{Q4}(x, t) dt \cdot \int_0^1 f^{(2n)}(t) dt \right| \\ & \leq \frac{2n}{2} \left\| G_{2n-1}^{Q4}(x, t) \right\|_{\infty} \left(\int_0^1 t(1-t) f^{(2n+1)}(t) dt \right). \end{aligned} \quad (2.16)$$

Since

$$\begin{aligned} & \int_0^1 t(1-t) f^{(2n+1)}(t) dt = \int_0^1 f^{(2n)}(t) [2t-1] dt = \\ & = \left[f^{(2n-1)}(1) + f^{(2n-1)}(0) \right] - 2 \left(f^{(2n-2)}(1) - f^{(2n-2)}(0) \right), \end{aligned}$$

using the representation (2.1) and the inequality (2.16), we deduce (2.14).

Similarly, using the representation (2.3) we deduce (2.15). □

3. APPLICATIONS FOR EULER BULLEN-SIMPSON'S 3/8 FORMULA

For function $f : [0, 1] \rightarrow \mathbb{R}$, with continuous fourth derivative $f^{(4)}$ on $[0, 1]$ and $f^{(4)}(t) \geq 0, \forall t \in [0, 1]$, we have

$$\begin{aligned} \frac{1}{8} \left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right) \right] &\leq \int_0^1 f(t)dt \\ &\leq \frac{1}{8} \left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1) \right]. \end{aligned} \tag{3.1}$$

In the case when $f^{(4)}$ exists, the condition $f^{(4)}(t) \geq 0, \forall t \in [0, 1]$ is equivalent to the requirement that f is 4-convex function on $[0, 1]$. However, a function f may be 4-convex although $f^{(4)}$ does not exist.

P. S. Bullen in [4] proved that, if f is 4-convex, then (3.1) is valid. Moreover, he proved that the Maclaurin quadrature rule is more accurate than the Simpson's 3/8 quadrature rule, that is we have

$$\begin{aligned} 0 &\leq \int_0^1 f(t)dt - \frac{1}{8} \left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right) \right] \\ &\leq \frac{1}{8} \left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1) \right] - \int_0^1 f(t)dt, \end{aligned} \tag{3.2}$$

provided f is 4-convex. We shall call this inequality Bullen-Simpson's 3/8 inequality.

In [8] the authors established a generalizations of the inequalities (3.1) and (3.2) for a class of $(2r)$ -convex functions and also to obtain some estimates for the absolute value of difference between the absolute value of error in the Maclaurin quadrature rule and the absolute value of error in the Simpson's 3/8 quadrature rule. Let us define

$$\begin{aligned} D(0, 1) \\ : &= \frac{1}{16} \left[f(0) + 3f\left(\frac{1}{6}\right) + 3f\left(\frac{1}{3}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{2}{3}\right) + 3f\left(\frac{5}{6}\right) + f(1) \right]. \end{aligned}$$

We shall make use of the following seven-point quadrature formula

$$\int_0^1 f(t)dt \approx D(0, 1),$$

obtained by adding the Simpson 3/8 and the Maclaurin quadrature formulae. It is suitable for our purposes to rewrite the second inequality in (3.2) in the form

$$\int_0^1 f(t)dt \leq D(0, 1). \tag{3.3}$$

As we mentioned earlier, this inequality is valid for any 4-convex function f and we call it the Bullen-Simpson's 3/8 inequality.

We consider the sequences of functions $(G_k(t))_{k \geq 1}$ and $(F_k(t))_{k \geq 1}$ defined by

$$\begin{aligned} G_k(t) &= 2B_k^*(1-t) + 3B_k^*\left(\frac{1}{6}-t\right) + 3B_k^*\left(\frac{1}{3}-t\right) \\ &+ 2B_k^*\left(\frac{1}{2}-t\right) + 3B_k^*\left(\frac{2}{3}-t\right) + 3B_k^*\left(\frac{5}{6}-t\right), \quad t \in \mathbb{R} \end{aligned}$$

and

$$F_k(t) = G_k(t) - \tilde{B}_k, \quad t \in \mathbb{R} \quad (3.4)$$

where

$$\tilde{B}_k = B_k(0) + 3B_k\left(\frac{1}{6}\right) + 3B_k\left(\frac{1}{3}\right) + 2B_k\left(\frac{1}{2}\right) + 3B_k\left(\frac{2}{3}\right) + 3B_k\left(\frac{5}{6}\right) + B_k(1).$$

For any function $f : [0, 1] \rightarrow \mathbb{R}$ such that $f^{(n-1)}$ exists on $[0, 1]$ for some $n \geq 1$ we define $T_0(f) = T_1(f) := 0$ and

$$T_m(f) = \frac{1}{8} \sum_{k=2}^m \frac{1}{(2k)!} 2^{-2k} (1 - 3^{2-2k}) B_{2k} \left[f^{(2k-1)}(1) - f^{(2k-1)}(0) \right]. \quad (3.5)$$

In the next lemma the authors established the Euler Bullen-Simpson 3/8 formulae.

Lemma 1. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0, 1]$, for some $n \geq 1$. Then we have*

$$\int_0^1 f(t) dt = D(0, 1) + T_r(f) + \tau_n^1(f), \quad (3.6)$$

where $r = [n/2]$ and

$$\tau_n^1(f) = \frac{1}{16(n!)} \int_0^1 G_n(t) df^{(n-1)}(t).$$

Also,

$$\int_0^1 f(t) dt = D(0, 1) + T_s(f) + \tau_n^2(f), \quad (3.7)$$

where $s = [(n-1)/2]$ and

$$\tau_n^2(f) = \frac{1}{16(n!)} \int_0^1 F_n(t) df^{(n-1)}(t).$$

Using Theorem 1 for identity (3.6) we get the following Grüss type inequality:

Theorem 5. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n \geq 1$ and $(f^{(n+1)})^2 \in L_1[0, 1]$. Then, we have*

$$\int_0^1 f(t) dt - D(0, 1) - T_r(f) = TGB_n(f), \quad (3.8)$$

and the remainder $TGB_n(f)$ satisfies the estimations

$$|TGB_n(f)| \leq \frac{1}{16} \left[\frac{2}{(2n)!} (-21 + 43 \cdot 2^{-2n} + 63 \cdot 3^{-2n} - 63 \cdot 6^{-2n}) B_{2n} \right]^{\frac{1}{2}}$$

$$\times \left(\int_0^1 t(1-t) \left[f^{(n+1)}(t) \right]^2 dt \right)^{\frac{1}{2}}. \quad (3.9)$$

Proof. If we apply Theorem 1 for $f \rightarrow G_n$, $g \rightarrow f^{(n)}$, we deduce

$$\begin{aligned} & \left| \int_0^1 G_n(t) f^{(n)}(t) dt - \int_0^1 G_n(t) dt \cdot \int_0^1 f^{(n)}(t) dt \right| \\ & \leq \frac{1}{\sqrt{2}} [T(G_n(\cdot), G_n(\cdot))]^{\frac{1}{2}} \times \left(\int_0^1 t(1-t) \left[f^{(n+1)}(t) \right]^2 dt \right)^{\frac{1}{2}}, \end{aligned} \quad (3.10)$$

where

$$T(G_n(\cdot), G_n(\cdot)) = \int_0^1 [G_n(t)]^2 dt - \left[\int_0^1 G_n(t) dt \right]^2.$$

We get $\int_0^1 G_n(t) dt = 0$ and using integration by part we have

$$\begin{aligned} & \int_0^1 G_n^2(t) dt = (-1)^{n-1} \frac{n(n-1)\dots 2}{(n+1)(n+2)\dots(2n-1)} \left[\int_0^1 G_1(t) G_{2n-1}(t) dt \right] \\ & = (-1)^{n-1} \frac{(n!)^2}{(2n)!} \left[-16 \int_0^1 G_{2n}(t) dt + 4G_{2n}(0) + 6G_{2n}\left(\frac{1}{6}\right) + 6G_{2n}\left(\frac{1}{3}\right) \right] \\ & = (-1)^{n-1} \frac{(n!)^2}{(2n)!} \left[44B_{2n}(t) + 84B_{2n}\left(\frac{1}{6}\right) + 84B_{2n}\left(\frac{1}{3}\right) + 44B_{2n}\left(\frac{1}{2}\right) \right]. \end{aligned}$$

Using (3.6) and (3.10), we deduce the representation (3.8) and the bound (3.9). \square

Remark 3.1: Because of (3.4) we get

$$\int_0^1 F_k(t) dt = \int_0^1 G_k(t) dt - \int_0^1 \tilde{B}_k dt = -\tilde{B}_k,$$

and also

$$\int_0^1 [F_k(t)]^2 dt = \int_0^1 [G_k(t)]^2 dt - 2\tilde{B}_k \int_0^1 G_k(t) dt + \tilde{B}_k^2.$$

So, using (3.7) similar as in (3.10), we deduce the representation (3.8) and the bound (3.9), too.

The following Grüss type inequality also holds.

Theorem 6. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)} \geq 0$ on $[0, 1]$. Then we have the representation (3.8) and the remainder $TGB_n(f)$ satisfies the bound*

$$\begin{aligned} |TGB_n(f)| & \leq \\ & \frac{1}{2(n-1)!} \|G_{n-1}(t)\|_{\infty} \left\{ \frac{f^{(n-1)}(0) + f^{(n-1)}(1)}{2} - f^{(n-2)}[0, 1] \right\}. \end{aligned} \quad (3.11)$$

Proof. If we apply Theorem 2 for $f \rightarrow G_n$, $g \rightarrow f^{(n)}$, we deduce

$$\left| \int_0^1 G_n(t) f^{(n)}(t) dt - \int_0^1 G_n(t) dt \cdot \int_0^1 f^{(n)}(t) dt \right|$$

$$\leq \frac{n}{2} \|G_{n-1}(t)\|_{\infty} \left(\int_0^1 t(1-t)f^{(n+1)}(t)dt \right). \quad (3.12)$$

So, similarly as in Theorem 4, using the representation (3.8) and the inequality (3.12), we deduce (3.11). \square

Remark 3.2: From [8] we have that for $n - 1 = 2k$, $k \geq 2$

$$\|G_{n-1}(t)\|_{\infty} = \|G_{2k}(t)\|_{\infty} = 2^{1-2k}(1 - 3^{2-2k})|B_{2k}|.$$

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