Primitive varieties of algebras

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Abstract. A variety $\mathcal{V}$ of Ω-algebras is said to be primitive if it is defined by a system of primitive identities, i.e. formulas of form (1.1). The main results are descriptions of closed set of primitive identities and of free objects in primitive varieties.

1. Introduction

Let Ω be a set of (finitary) functional symbols, i.e. a type of algebras. By Ω_n we denote the set of n-ary symbols in Ω $(n \geq 0)$. A formula of form

$$f(x_{i_1}, \ldots, x_{i_n}) = g(x_{j_1}, \ldots, x_{j_m}) \tag{1.1}$$

where $f \in \Omega_n$, $g \in \Omega_m$ and $x_1, x_2, \ldots$ are variables, is said to be a primitive Ω-identity. (Further on we will usually omit the prefix "Ω", and thus by an algebra we mean an Ω-algebra, by an identity we mean an Ω-identity, ...)

Let $\mathbf{A}$ be an algebra and let Σ, Σ' be sets of primitive identities. Then $\mathbf{A} \models \Sigma$ means that each identity from Σ holds on $\mathbf{A}$, and $\Sigma \models \Sigma'$ means that for every algebra $\mathbf{A}$ we have $\mathbf{A} \models \Sigma \Rightarrow \mathbf{A} \models \Sigma'$. We say that Σ is closed iff $\Sigma \models \Sigma' \Rightarrow \Sigma' \subseteq \Sigma$.

We can certainly assume that the set of variables coincides with the set $\mathbf{N}$ of positive integers, and so we can interpret the formula (1.1) as an ordered quadruple (f, α, β, g), where $\alpha \in \mathbf{N}^n$, $\beta \in \mathbf{N}^m$ are such that $\alpha(\nu) = i_\nu$, $\beta(\lambda) = j_\lambda$ for each $\nu \in \{1, 2, \ldots, n\} = \mathbf{n}$, $\lambda \in \{1, 2, \ldots, m\} = \mathbf{m}$. If $n = 0$ then we take $\mathbf{n} = \varnothing$ and $\mathbf{N}^0 = \{\varnothing\}$. We say that (f, α, β, g) is an equation.

The main result of Section 2 is a convenient description of closed sets of equations and, assuming that Σ is a closed set of equations, we give a special interpretation of Σ-algebras, i.e. of algebras in the primitive variety of algebras defined by Σ. By using this interpretation, in Section 3 we give a convenient description of free Σ-algebras. (Corresponding results for primitive varieties of n-groupoids are considered in [2] and [3].)

In what follows we will use the following convention about the notations. If S, T, U are sets and $\eta: S \to T$, $\xi: T \to U$ are mappings, then by $\xi \circ \eta$ we denote their superposition, i.e. $\xi \circ \eta(s) = \xi(\eta(s))$ for each $s \in S$. The image of η is denoted by $im\, \eta$, i.e. $im\, \eta = \{\eta(s) \mid s \in S\}$. Let A be a set and $\alpha \in A^n$, $\beta \in A^m$, where n, m are nonnegative integers. Then by $\alpha\beta$ we denote the concatenation of α and β, i.e. $\alpha\beta \in A^{n+m}$ is defined by $(\alpha\beta)(i) = \alpha(i)$, $(\alpha\beta)(n+j) = \beta(j)$ for each $i \in \mathbf{n}$, $j \in \mathbf{m}$. If $m = 0$ then $\beta = \varnothing$ and $\alpha\varnothing = \varnothing\alpha = \alpha$. If $e = (f, \alpha, \beta, g)$ is an equation then its kernel $ker\, e$ is defined by $ker\, e = ker\, \alpha\beta$.

The set of all equations will be denoted by $\Pi(\Omega)$.

2. Closed sets of equations

Let $e = (f, \alpha, \beta, g)$ be an equation, $\mathbf{A}$ an algebra with carrier $\mathbf{A}$ and $\xi: \mathbf{N} \to \mathbf{N}$ a mapping. Define $R(e)$, $L(e)$, e^{-1}, $\xi(e)$ as follows:

$$L(e) = (f, \alpha), \qquad R(e) = (g, \beta), \qquad e^{-1} = (g, \beta, \alpha, f),$$
$$\xi(e) = (f, \xi \circ \alpha, \xi \circ \beta, g).$$

We say that e holds on $\mathbf{A}$ if $f_\mathbf{A}(\eta(\alpha(1)), \ldots, \eta(\alpha(n))) = g_\mathbf{A}(\eta(\beta(1)), \ldots, \eta(\beta(m)))$ for each mapping $\eta: \mathbf{N} \to A$, and then we write $\mathbf{A} \models e$. Thus, $\mathbf{A} \models \Sigma \Leftrightarrow (\forall e \in \Sigma)\mathbf{A} \models e$.

If $e, e', e'' \in \Pi(\Omega)$ and $\xi: \mathbf{N} \to \mathbf{N}$ is a mapping then:

$$R(e^{-1}) = L(e), \qquad (e^{-1})^{-1} = e, \qquad e \models e^{-1} \tag{2.1}$$
$$L(e) = R(e) \quad \Rightarrow \quad \varnothing \models e, \qquad e \models \xi(e) \tag{2.2}$$
$$R(e) = L(e'), L(e) = L(e''), R(e') = R(e'') \quad \Rightarrow \quad e, e' \models e'' \tag{2.3}$$
$$e = (f, \alpha, \beta, g), e' = (f, \alpha', \beta', g), ker\, e \subseteq ker\, e' \quad \Rightarrow \quad e \models e'. \tag{2.4}$$

A description of the family of closed sets of equations is given by

THEOREM 2.1. *A set of equations $\Sigma \subseteq \Pi(\Omega)$ is closed iff the following conditions hold*:

(i) $L(e) = R(e) \Rightarrow e \in \Sigma$
(ii) $e \in \Sigma \Rightarrow e^{-1} \in \Sigma$
(iii) $R(e) = L(e')$, $L(e) = L(e'')$, $R(e') = R(e'') \Rightarrow (e, e' \in \Sigma \Rightarrow e'' \in \Sigma)$
(iv) $e \in \Sigma \Rightarrow \xi(e) \in \Sigma$, *for each mapping* $\xi: \mathbf{N} \to \mathbf{N}$.

Proof. It follows from (2.1)–(2.4) that if Σ is closed then (i)–(iv) are satisfied.

Assume now that Σ satisfies the above conditions. If $\Sigma = \Pi(\Omega)$ then Σ is certainly closed, and thus we can assume that there is an equation $e = (f, \alpha, \beta, g) \in \Pi(\Omega) \backslash \Sigma$. Define an algebra **A** with a carrier **N** as follows:

$$h_{\mathbf{A}}(\gamma) = \begin{cases} 1 & \text{if } (f, \alpha, \gamma, h) \in \Sigma \\ 2 & \text{otherwise} \end{cases}$$

where $h \in \Omega_p$, $\gamma \in \mathbf{N}^p$ $(p \geq 0)$. Then $f_{\mathbf{A}}(\alpha) = 1$ by (i), and $e \notin \Sigma$ imply $g_{\mathbf{A}}(\beta) = 2$, which means that $\mathbf{A} \nvDash e$.

Let $e' = (f', \alpha', \beta', g') \in \Sigma$. If $f'_{\mathbf{A}}(\alpha') = 1$, then $(f, \alpha, \alpha', f') \in \Sigma$ and by (iii) we have $(f, \alpha, \beta', g') \in \Sigma$, i.e. $g'_{\mathbf{A}}(\beta') = 1$. If $g'_{\mathbf{A}}(\beta') = 1$ then (iii) implies $f'_{\mathbf{A}}(\alpha') = 1$, since by (ii) we have $(g', \beta', \alpha', f') = e'^{-1} \in \Sigma$. So, $f'_{\mathbf{A}}(\alpha') = g'_{\mathbf{A}}(\beta')$. If $\xi: \mathbf{N} \to \mathbf{N}$ is any mapping then by (iv) we get $\xi(e') = (f', \xi \circ \alpha', \xi \circ \beta', g') \in \Sigma$, and thus we have shown that $\mathbf{A} \vDash e'$. □

Further on we will assume that Σ is a given closed set of equations.

Let A be a set such that $A \cup \Omega_0 \neq \varnothing$. Define a set $\Omega(A) = \bigcup \{\Omega_n \times A^n \mid n \geq 0\}$ and consider $\Omega_0 \times A^0$ as another notation for Ω_0 (i.e. we take $\Omega_0 \subseteq \Omega(A)$). Further, define a relation $\approx_{A,\Sigma}$ on the set $\Omega(A)$ as follows:

$$(f, \mathbf{a}) \approx_{A,\Sigma} (g, \mathbf{b}) \quad \Leftrightarrow \quad (\exists (f, \alpha, \beta, g) \in \Sigma)\ ker\ \mathbf{ab} = ker\ \alpha\beta,$$

where $(f, \mathbf{a}) \in \Omega_n \times A^n$, $(g, \mathbf{b}) \in \Omega_m \times A^m$ and $\mathbf{ab} \in A^{m+n}$ is the concatenation of **a** and **b**. In what follows we will write $\approx$ instead of $\approx_{A,\Sigma}$.

PROPOSITION 2.2. *$\approx$ is an equivalence relation on $\Omega(A)$. (We denote the quotient set $\Omega(A)/_{\approx}$ by $\Sigma(A)$.)*

Proof. We need to show the transitivity of $\approx$ only. Let $(f, \mathbf{a}) \in \Omega_n \times A^n$, $(g, \mathbf{b}) \in \Omega_m \times A^m$, $(h, \mathbf{c}) \in \Omega_p \times A^p$ and $(f, \mathbf{a}) \approx (g, \mathbf{b}) \approx (h, \mathbf{c})$. Then there are $(f, \alpha, \beta, g), (g, \gamma, \delta, h) \in \Sigma$ such that $ker\ \mathbf{ab} = ker\ \alpha\beta$, $ker\ \mathbf{bc} = ker\ \gamma\delta$, and this implies

$$ker\ \mathbf{a} = ker\ \alpha, \qquad ker\ \mathbf{b} = ker\ \beta = ker\ \gamma, \qquad ker\ \mathbf{c} = ker\ \delta.$$

We can assume that $im\ \alpha\beta \cap im\ \gamma\delta = \varnothing$ since, in contrary, we can take an injection $\eta: \mathbf{N} \to \mathbf{N}$ such that $\alpha' = \eta \circ \alpha$, $\beta' = \eta \circ \beta$, $(f, \alpha, \beta, g) \vDash (f, \alpha', \beta', g) \vDash (f, \alpha, \beta, g)$ and $im\ \alpha'\beta' \cap im\ \gamma\delta = \varnothing$. So we have

$$im\ \alpha \cap im\ \gamma = im\ \alpha \cap im\ \delta = im\ \beta \cap im\ \gamma = im\ \beta \cap im\ \delta = \varnothing \tag{2.5}$$

Define a mapping $\xi: \mathbf{N} \to \mathbf{N}$ such that ξ is identical over $\mathbf{N} \backslash im\ \gamma$ and $\xi \circ \gamma = \beta$. The equality $ker\ \beta = ker\ \gamma$ implies that ξ is well defined and, by (2.5), ξ is injective over $\mathbf{N} \backslash im\ \beta$. Then $ker\ \xi \circ \delta = ker\ \delta = ker\ \mathbf{c}$ and, by Theorem 2.1, we have $(g, \beta, \xi \circ \delta, h), (f, \alpha, \xi \circ \delta, h) \in \Sigma$. We remark that, as a consequence of (2.5), one can show that

$$ker\ \alpha(\xi \circ \delta) \subseteq ker\ \mathbf{ac} \tag{2.6}$$

and this complete the proof, since (2.6) implies the existence of a mapping $\psi: \mathbf{N} \to \mathbf{N}$ such that $ker\ \psi \circ (\alpha(\xi \circ \delta)) = ker\ \mathbf{ac}$ and $(f, \psi \circ \alpha, \psi \circ \xi \circ \delta, h) \in \Sigma$. □

Let **A** be an algebra with carrier A. We can interpret **A** as a mapping $\mathbf{A}: \Omega(A) \to A$, such that $\mathbf{A}(f, \mathbf{a}) = f_{\mathbf{A}}(\mathbf{a})$ for each $f \in \Omega_n$, $\mathbf{a} \in A^n$. Having in mind such an interpretation of the algebras, we obtain the following characterization of the Σ-algebras (i.e. algebras in the variety of algebras defined by Σ):

PROPOSITION 2.3. *An algebra $\mathbf{A}: \Omega(A) \to A$ is a Σ-algebra iff $\approx \subseteq ker\ \mathbf{A}$.* □

So, any Σ-algebra with carrier A can be viewed as a mapping $\mathbf{A}: \Sigma(A) \to A$, and further on such an interpretation will be assumed.

PROPOSITION 2.4. *If A, G are sets and $\varphi: \Sigma(A) \to G$ and $\mathbf{G}: G \to A$ are mappings, then $\mathbf{A} = \mathbf{G} \circ \varphi$ is a Σ-algebra with carrier A. (If φ is bijective, then usually we do not make any distinction between* **A** *and* **G**.) □

EXAMPLE 2.5. Let $\mathscr{V}$ be the variety of commutative groupoids which satisfy the identity $x^2 = y^2$, and let Σ be the corresponding closed set of equations. We may assume that $\Omega(A) = A^2$, where $A \neq \varnothing$, and then $\Sigma(A)$ has the following description: $u \in \Sigma(A)$ iff $u = \{(a, a) \mid a \in A\}$ or $u = \{(a, b), (b, a)\}$, where $a, b \in A$ and $a \neq b$. This suggests that we consider the set $G = \{e\} \cup \{\{a, b\} \mid a, b \in A, a \neq b\}$ and define a bijection $\varphi: \Sigma(A) \to G$ by

$$\varphi: \{(a, a) \mid a \in A\} \mapsto e, \qquad \{(a, b), (b, a)\} \mapsto \{a, b\} \quad (a \neq b)$$

Moreover, having in mind this bijection, we could replace $\Sigma(A)$ by G. □

A partial Σ-algebra with carrier A and domain D is said to be any mapping $\mathbf{A}: D \to A$, where $D \subseteq \Sigma(A)$. Given a partial Σ-algebra $\mathbf{A}: D \to A$, there exists a Σ-algebra $\mathbf{A}^*: \Sigma(A) \to A$ such that **A** is the restriction of $\mathbf{A}^*$ on D. Also, we note that if Ω is finite, then there are only finitely many nonequivalent primitive

Ω-identities. Consequently, as a corollary of Evans' Theorem ([4], p. 68), we have the following.

PROPOSITION 2.6. *If Ω is finite then the word problem is solvable in any primitive variety of Ω-algebras.* $\square$

Let A, A' be sets and consider a mapping $\tau: A \to A'$. Then τ induces two uniquely determined natural mappings $\tau^{(n)}: A^n \to A'^n$, where $n \geq 0$, and $\tau^{\Omega}: \Omega(A) \to \Omega(A')$. We will usually omit the upperscripts, i.e. τ will be a common notation for each of the mappings τ, $\tau^{(n)}$, τ^{Ω}. Then we have $(f, \mathbf{a}) \approx (g, \mathbf{b}) \Rightarrow \tau(f, \mathbf{a}) = (f, \tau \circ \mathbf{a}) \approx (g, \tau \circ \mathbf{b}) = \tau(g, \mathbf{b})$ and, moreover, if $\tau: A \to A'$ is injective, then $\tau(f, \mathbf{a}) \approx \tau(g, \mathbf{b}) \Rightarrow (f, \mathbf{a}) \approx (g, \mathbf{b})$. Therefore, τ also induces a mapping $\tau^{\Sigma}: \Sigma(A) \to \Sigma(A')$ and if τ is injective then τ^{Σ} is injective too. Hence, if $A \subseteq A'$ we can assume $\Sigma(A) \subseteq \Sigma(A')$ as well.

Homomorphisms, congruences and subalgebras are characterized as usual. Thus, a homomorphism from a Σ-algebra $\mathbf{A}$ into a Σ-algebra $\mathbf{A}'$ is a mapping $\tau: A \to A'$ such that $\tau \circ \mathbf{A} = \mathbf{A}' \circ \tau^{\Sigma}$.

3. Free Σ-algebras

Here we assume again that Σ is a given closed set of equations. If $f \in \Omega$ is such that there exists an equation $(f, \alpha, \beta, g) \in \Sigma$, where $im\, \alpha \cap im\, \beta = \varnothing$, then we say that f is a Σ-constant. (Thus, if $f \in \Omega_0$, then f is a Σ-constant for any Σ.) If $f \in \Omega_n$ and $k \leq n$ is the largest nonnegative integer such that

$$\alpha, \beta \in \mathbf{N}^n, \qquad |im\, \alpha| \leq k, \qquad |im\, \beta| \leq k \quad \Rightarrow \quad (f, \alpha, \beta, f) \in \Sigma,$$

then we say that k is the order of Σ-singularity of f.

For technical reasons only, the following conditions will be also supposed in this section:

(I) If $f, g \in \Omega_n$ and $(f, \epsilon, \epsilon, g) \in \Sigma$, where $\epsilon(i) = i$ for each $i \in \mathbf{n}$, then $f = g$.

(II) If $f \in \Omega_n$ is a Σ-constant and $n \geq 1$, then there are $g \in \Omega_0$ and $\alpha \in \mathbf{N}^n$ such that $(f, \alpha, \varnothing, g) \in \Sigma$.

(III) If $f \in \Omega_n$, $n \geq 2$ and $\alpha \in \mathbf{N}^n$ are such that $\alpha(i) = i$ for each $i \in \mathbf{n}$ and $(f, \alpha, \beta, g) \in \Sigma$, then $im\, \beta = \mathbf{n}$.[1]

As a consequence of (I)–(III) we can make the following assumptions:

(a) If $A \cup \Omega_0 \neq \varnothing$ then $\Omega_0 \subseteq \Sigma(A)$.

(b) If $\mathbf{A}: \Sigma(A) \to A$ is injective, then $\Omega_0 \subseteq A$.

(c) If $f \in \Omega_1$ then f is not a Σ-constant.

(d) If k is the order of Σ-singularity of $f \in \Omega_n$, $n \geq 1$, then $k < n$.

Let A be a set, $u \in \Sigma(A)$ and denote by $[u]$ the following collection of subsets of A: $[u] = \{im\, \mathbf{a} \mid (f, \mathbf{a}) \in u\}$.

We say that $(f, \mathbf{a}) \in u$ is a minimal element of u iff $im\, \mathbf{a}$ is a minimal member in $[u]$. (The existence of at least one minimal member in u is obvious.)

[1] If the pair (Ω, Σ) does not satisfy the conditions (I)–(III), then we can define a new type of algebras Ω' and a closed set of equations Σ' such that the pair (Ω', Σ') does satisfy the mentioned conditions and the corresponding primitive variety defined by Σ' is not essentially distinct from that defined by Σ (see [6], 29–34).

By an application of the condition (iv) of Theorem 2.1, one can show the following statement:

PROPOSITION 3.1. *If $u \in \Sigma(A)$, then there exists a unique minimal member $im\, \mathbf{a}$ in $[u]$, and it is the least member in $[u]$. (We say that $im\, \mathbf{a}$ is the content of u and denote it by $Cont(u)$. Thus, $Cont(u) = \varnothing$ iff $u \in \Omega_0$.)* $\square$

EXAMPLE 3.2. If condition (II) does not hold, then the conclusion of the last proposition could not be true. Such a situation occurs in Example 2.5. Namely, if $u_0 = \{(a, a) \mid a \in A\}$ and if $|A| \geq 2$, then $[u_0] = \{\{a\} \mid a \in A\}$, i.e. each one-element subset of A is a minimal member in $[u_0]$. In this case we add a new 0-ary symbol e to $\Omega = \{\cdot\}$ and obtain $\Omega' = \{e, \cdot\}$. If Σ' is the closed set of Ω'-equations generated by $x^2 = e$, $xy = yx$, then we obtain a pair (Ω', Σ') which satisfies all three conditions. $\square$

As a corollary of Proposition 3.1 we have

PROPOSITION 3.3. *If $A \subseteq A'$ and $u \in \Sigma(A')$, then $u \in \Sigma(A)$ iff $Cont(u) \subseteq A$.* $\square$

Now we are ready to "build" a free Σ-algebra with a given basis. Given a set B such that $B \cap \Omega_0 = \varnothing$ and $B \cup \Omega_0 \neq \varnothing$, define a sequence of sets $\{B_p \mid p \geq 1\}$ and a set U $(= U(B, \Sigma))$ as follows:

$$B_1 = B \cup \Omega_0, \qquad B_{p+1} = B_p \cup \Sigma(B_p), \qquad U = \bigcup \{B_p \mid p \geq 1\}.$$

Define a mapping χ (called hierarchy) from U into $\mathbf{N}$ by: $\chi(u) = k$ iff k is the least positive integer such that $u \in B_k$.

We note that $U = B_1$ iff $\Omega = \Omega_0$, and if $\Omega \neq \Omega_0$ then $B_p \subsetneq B_{p+1}$ for each $p \geq 1$. Also, if $u \in U$ then $\chi(u) = 1 \Leftrightarrow u \in B_1$, $\chi(u) = p + 1 (p \geq 1) \Leftrightarrow Cont(u) \subseteq B_p$ & $(\exists v \in Cont(u))\chi(v) = p$.

PROPOSITION 3.4. $\Sigma(U) = U \backslash B$. □

The following statement gives a description of free Σ-algebras.

THEOREM 3.5. *If* $\mathbf{U}$ *is the embedding of* $\Sigma(U)$ *into* U, *i.e.* $\mathbf{U}(u) = u$ *for each* $u \in \Sigma(U)$, *then* $\mathbf{U}$ *is a free* Σ*-algebra with unique basis* B.

Proof. $\mathbf{U}$ is well defined by Proposition 3.4, and by an induction on hierarchy one can show that B is a generating subset of $\mathbf{U}$.

Let $\mathbf{A}: \Sigma(A) \to A$ be a Σ-algebra and let $\tau: B \to A$ be a mapping. Define a chain of mappings $\{\tau_p \mid p \geq 1\}$ as follows. $\tau_1(b) = \tau(b)$ for each $b \in B$, and $\tau_1(f) = \mathbf{A}(f)$ for each $f \in \Omega_0$. Further, if $\tau_p: B_p \to A$ is defined, let $\tau_{p+1}: B_{p+1} \to A$ be the extension of τ_p such that it coincides with $\mathbf{A} \circ \tau_p^{\Sigma}$ on $\Sigma(B_p) \backslash B_p$. Then $\bar{\tau} = \bigcup(\tau_p \mid p \geq 1)$ is an extension of τ and a homomorphism from $\mathbf{U}$ into $\mathbf{A}$ as well. □

Below we will give another description of free Σ-algebras, and for that purpose we need a few more definitions. Let $\mathbf{A}: \Sigma(A) \to A$ be a Σ-algebra. If $\mathbf{A}(u) = a$ and $b \in Cont(u)$ then we say that b is a divisor of a. A sequence $a_1, a_2, \ldots,$ of elements of A is said to be a divisor chain in $\mathbf{A}$ if a_{i+1} is a divisor of a_i for each i. An element $a \in A$ is said to be prime in $\mathbf{A}$ iff $a \notin im\, \mathbf{A}$.

THEOREM 3.6. *A* Σ*-algebra* $\mathbf{F}: \Sigma(F) \to F$ *is free* (*in the primitive variety of* Σ*-algebras*) *iff it satisfies the following conditions*:

(i) $\mathbf{F}$ *is injective.*

(ii) *Every divisor chain in* $\mathbf{F}$ *is finite.*

Then, the set of prime elements in $\mathbf{F}$ *is the unique basis of* $\mathbf{F}$.

Proof. Clearly, the free Σ-algebra defined in Theorem 3.5 satisfies (i) and (ii), and B is the set of primes in $\mathbf{U}$.

Assume now that (i) and (ii) hold in a Σ-algebra $\mathbf{F}: \Sigma(F) \to F$. By (i) and Proposition 3.1, the set D of divisors of an element $a \in F$ is finite, and $D = \varnothing$ iff a is prime or $a \in \mathbf{F}(\Omega_0)$. By (ii) we obtain that the set of lengths of the divisor chains with a common first member is bounded. This can be shown, for example, by an application of König's lemma ([5], 381). If $a \in F$, then we denote by $\delta(a)$ the largest possible length of a divisor chain with the first member a. Let B be the set of primes in $\mathbf{F}$, and let $\mathbf{U}: \Sigma(U) \to U$ be the free Σ-algebra defined in Theorem 3.5. Then there is a unique isomorphism $\varphi: \mathbf{U} \to \mathbf{F}$ such that $\chi(a) = \delta(\varphi(a))$, $\varphi(c) = c$, $\varphi(f) = \mathbf{F}(f)$, for any $a \in U$, $c \in B$, $f \in \Omega_0$. □

Since the conditions (i) and (ii) of Theorem 3.6 are hereditary we have

COROLLARY 3.7. *Every subalgebra of a free* Σ*-algebra is free too.* □

The following statement is a generalization of well known results concerning relations between the ranks of free Ω-algebra and their subalgebras, in the variety of Ω-algebras.

THEOREM 3.8. *A free* Σ*-algebra contains subalgebras with an infinite rank iff at least one of the following conditions is satisfied*:

(i) $|\Omega_1| \geq 2$.

(ii) *There exists an* $f \in \Omega \backslash (\Omega_0 \cup \Omega_1)$ *which is not a* Σ*-constant.*

(iii) $|\Omega_1| \geq 1$ *and* $\Omega \backslash (\Omega_0 \cup \Omega_1) \neq \varnothing$.

(iv) $\Omega \backslash (\Omega_0 \cup \Omega_1) \neq \varnothing$ *and* $|\Omega_0| > k$, *where k is the least order of* Σ*-singularity of some functional symbol in* $\Omega \backslash (\Omega_0 \cup \Omega_1)$. □

EXAMPLE 3.9. Let $\Omega = \Omega_3 = \{f\}$ and denote by Σ the closed set of equations induced by the following set of identities:

$x^3 = y^2 z, \qquad xyz = xzy = yxz.$

Note that the order of Σ-singularity of f is 2. Define a new signature $\Omega' = \{e, f\}$, where e is a 0-ary symbol, and a new set of equations Σ' generated by $x^3 = x^2 y = e$, $xyz = xzy = yxz$. Then (Ω', Σ') satisfies the conditions (I)–(III), and we can use the pair (Ω', Σ') to apply the construction of free objects given in this section. If $B = \{a, b\}$ then by (iv) of Theorem 3.8 the free Σ-grouped $\mathbf{U}$ with basis B contains subgroupoids with infinite ranks. To get such a subgroupoid, we give firstly a more detailed description of $\mathbf{U}$. Namely, $U = \bigcup\{B_k \mid k \geq 1\}$, where

$B_1 = \{e, a, b\}, \qquad B_2 = \{e, a, b, \{e, a, b\}\},$

$B_{k+1} = B_k \cup \{\{x, y, z\} \mid x, y, z \in B_k,\ x \neq y \neq z \neq x\}$

and the ternary operation on U is defined by

$$xyz = \begin{cases} e & \text{if } |\{x, y, z\}| \le 2 \\ \{x, y, z\} & \text{otherwise} \end{cases}$$

If we define an infinite subset $C = \{c_k \mid k \ge 1\}$ of U by $c_1 = \{e, a, b\}$, $c_{k+1} = \{e, a, c_k\}$, then the subgroupoid of $\mathbf{U}$ generated by C has infinite rank.

Note that for each $u \in \mathbf{U}$, $\{u\}$ is the basis of the subgroupoid $\{e, u\}$ of $\mathbf{U}$. $\square$

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