TRANSFORMATIONS OF BOOLEANS

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Abstract. We say that $(M;f)$ is a Boolean unar iff f is a mapping from a subset $\mathcal{D}_f$ of $\mathcal{B}(M)$[1] into $\mathcal{B}(M)$. Subunars, homomorphisms and free objects are defined in a usual way. The main subject of the paper is the problem of existence of free objects in several classes of Boolean unars. Almost everywhere one obtains the following "unusual" property: there are more than one endomorphisms in a free Boolean unar which induce the identity transformation of the basis (as it is well-known, free algebras have not this property ([1])).

§1. BOOLEAN UNARS AND SUBUNARS

Let M be a nonempty set, $\mathcal{D} \subseteq \mathcal{B}(M)$ and $f: \mathcal{D} \to \mathcal{B}(M)$ a mapping from $\mathcal{D}$ into $\mathcal{B}(M)$. Then we say that $\underline{M} = (M;f)$ is a Boolean unar with the carrier M, action f and domain $\mathcal{D} = \mathcal{D}_{\underline{M}}$. Further on, instead of "boolean unar" we will simply write "unar"[2].

We note (once more) that the carrier M of a unar $\underline{M}$ is not empty, but we allow the domain $\mathcal{D}_{\underline{M}}$ to be empty. In the case when $\mathcal{D}_{\underline{M}} = \emptyset$, we say that $\underline{M}$ is a zero unar.

If M is a given set, then the family $\mathcal{U}_M$ of all unars with the carrier M can be ordered in a natural way. Namely, if $\underline{M} = (M;f)$, $\underline{M}' = (M;f')$, then:

$$\underline{M} \leq \underline{M}' \iff \mathcal{D}_{\underline{M}} \subseteq \mathcal{D}_{\underline{M}'}, \text{ and } f = f'|\mathcal{D}_{\underline{M}}{}^{3)}. \tag{1.1}$$

It is clear that:

Proposition 1.1. The zero unar with the carrier M is the least member of $\mathcal{U}_M$. A unar $\underline{M} = (M;f)$ is a maximal member of $\mathcal{U}_M$ iff $\mathcal{D}_{\underline{M}} = \mathcal{B}(M)$. ∎

A unar $\underline{N} = (N;g)$ is called a subunar of a unar $\underline{M} = (M;f)$ iff the following conditions are satisfied:

$$N \subseteq M, \quad \mathcal{D}_{\underline{N}} = \mathcal{D}_{\underline{M}} \cap \mathcal{B}(N), \quad g = f|\mathcal{D}_{\underline{N}}. \tag{1.2}$$

The following proposition shows that the family of subunars of a given unar $(M;f)$ can be characterized by the family of their carriers (as in the case of the "usual" unars).

Proposition 1.2. If $\underline{M} = (M;f)$ is a unar and N a nonempty subset of M, then there exists at most one unar $\underline{N} = (N;g)$ which is a subunar of $\underline{M}$. Such a unar $\underline{N}$ do exists iff the following condition is satisfied:

$$X \in \mathcal{D}_{\underline{M}} \cap \mathcal{B}(N) \implies f(X) \in \mathcal{B}(N). \tag{1.3}$$

Proof. Let $\underline{N}' = (N;g')$ and $\underline{N}'' = (N;g'')$ be subunars of $\underline{M}$. By (1.2) we have:

$$\mathcal{D}_{\underline{N}'} = \mathcal{D}_{\underline{M}} \cap \mathcal{B}(N) = \mathcal{D}_{\underline{N}''} \text{ and } g' = g'',$$

i.e. $\underline{N}' = \underline{N}''$.

Now, let $\underline{N} = (N;g)$ be a subunar of $\underline{M}$ and $X \in \mathcal{D}_{\underline{M}} \cap \mathcal{B}(N)$. Then $f(X) = g(X) \in \mathcal{B}(N)$, i.e. the condition (1.3) is satisfied.

Conversely, if (1.3) holds, then putting

$$\mathcal{D}_{\underline{N}} = \mathcal{D}_{\underline{M}} \cap \mathcal{B}(N) \text{ and } (\forall X \in \mathcal{D}_{\underline{N}}) \quad g(X) = f(X),$$

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1) $\mathcal{B}(M)$ is the Boolean of the set M, i.e. the family of all subsets of M.

2) The term "unar" is usually used for an algebra with one unary operation ([2]). However this fact will not make any misunderstanding.

3) $f'|\mathcal{D}_{\underline{M}}$ is the restriction of f' on $\mathcal{D}_{\underline{M}}$, i.e. $(\forall X \in \mathcal{D}_{\underline{M}})\ f(X) = f'(X)$.

one obtains a subunar $\underline{N} = (N;g)$ of $\underline{M}$. ∎

According to this proposition, whenever we consider a *subunar* of a given unar $\underline{M} = (M;f)$, we will think of a nonempty subset N of M which satisfies the condition (1.3); in that case we will write $N \le \underline{M}$.

The following proposition gives a description of the unars whose all the nonempty subsets are subunars.

Proposition 1.3. Every nonempty subset N of a set M is a subunar of the unar $\underline{M} = (M;f)$ iff the following condition is satisfied:

$$(\forall X, Y \in \mathcal{B}(M))(X \in \mathcal{D}_{\underline{M}} \cap \mathcal{B}(Y) \Longrightarrow f(X) \in \mathcal{B}(Y)). \quad ∎ \tag{1.4}$$

It is clear that every zero unar and, more generally, every inclusive unar satisfies the condition (1.4). (We say that a unar $\underline{M} = (M;f)$ is *inclusive* iff $(\forall X \in \mathcal{D}_{\underline{M}})\ f(X) \subseteq X$.)

The following property will bring us to the concept of a subunar which is generated by a given nonempty subset.

Proposition 1.4. Let $\underline{M} = (M;f)$ be a unar and N_i, $i \in I$, a family of subunars of $\underline{M}$. If $N = \bigcap_{i \in I} N_i \neq \emptyset$, then $N \le \underline{M}$. ∎

Proposition 1.5. Let $\underline{M} = (M;f)$ be a unar and B a nonempty subset of M. Then there exists a unique subunar $\langle B \rangle$ of $\underline{M}$ with the following properties:

(i) $B \subseteq \langle B \rangle$;

(ii) $N \le \underline{M}$ and $B \subseteq N \Longrightarrow \langle B \rangle \subseteq N$.

Proof. Namely, $\langle B \rangle$ is the intersection of the family of subunars N of $\underline{M}$ such that $B \subseteq N$. ∎

We say that $\langle B \rangle$ is a *subunar generated by the set* B.

Remark. If the family of subunars of a unar has the least element, say P, then every subset of P generates P and so does the empty set $\emptyset$. Further on, the notation $\langle \emptyset \rangle$ will make sense only in this case.

Below we will give a more convenient description of $\langle B \rangle$, assuming that $\underline{M} = (M;f)$ is a given unar and B is a nonempty subset of M.

Let $B_0 = B$ and let

$$C_\alpha = \cup \{Y \in \mathcal{B}(M) \mid Y = f(X) \text{ for some } X \in \mathcal{D}_{\underline{M}} \cap \mathcal{B}(B_\alpha)\}. \tag{1.5}$$

Then we set

$$B_{\alpha+1} = B_\alpha \cup C_\alpha. \tag{1.6}$$

It is easy to show that

$$\langle B \rangle = \bigcup \{B_\alpha \mid \alpha \ge 0\}. \tag{1.7}$$

The B-*hierarchy* of an element $c \in \langle B \rangle$ is the least nonnegative integer α such that $c \in B_\alpha$. And, if $d \in M \setminus \langle B \rangle$, then we say that the B-hierarchy of d is infinite.

Now, by the above considerations, we obtain the following

Proposition 1.6. If $\underline{M} = (M;f)$ is a unar and $\emptyset \neq B \subseteq M$, then

$\langle B \rangle = \{x \in M \mid x \text{ has a finite B-hierarchy}\}$. ∎

§2. HOMOMORPHISMS

If $\underline{M} = (M;f)$ and $\underline{M}' = (M';f')$ are unars, then a mapping $\phi: M \to M'$ is called a homomorphism from $\underline{M}$ into $\underline{M}'$ iff the following conditions are fulfilled:

$$\phi(\mathcal{D}_{\underline{M}}) \subseteq \mathcal{D}_{\underline{M}'}, \text{ and } (\forall X \in \mathcal{D}_{\underline{M}})\ \phi(f(X)) = f'(\phi(X)). \tag{2.1}$$

Here, for $X \subseteq M$, $\phi(X) \stackrel{\text{df}}{=} \{\phi(x) \mid x \in X\}$, and, in the same sense:

$$\phi(\mathcal{D}_{\underline{M}}) = \{\phi(X) \mid X \in \mathcal{D}_{\underline{M}}\}.$$

The mapping ϕ is an *isomorphism* iff it is a bijective homomorphism such that ϕ^{-1} is also a homomorphism.

It is easy to show that:

Proposition 2.1. A bijective homomorphism $\phi:M \to M'$ is an isomorphism from $\underline{M}=(M;f)$ into $\underline{M}'=(M';f')$ iff $\phi(\mathcal{D}_{\underline{M}})=\mathcal{D}_{\underline{M}'}$. ■

We note that there are bijective homomorphisms which are not isomorphisms. For example, if $\underline{M}=(M;f)$ and $\underline{M}'=(M;f')$ are such that $\mathcal{D}_{\underline{M}} \subset \mathcal{D}_{\underline{M}'}$, then the identity transformation of M is a bijective homomorphism, but it is not an isomorphism.

Proposition 2.2. Let $\underline{M}_1=(M;f_1)$ and $\underline{M}_2=(M;f_2)$ be two unars with the same carrier M and let $\underline{M}_1 \le \underline{M}_2$. If $\phi:M \to M'$ is a homomorphism from $\underline{M}_2$ into $\underline{M}'=(M';f')$, then ϕ is a homomorphism from $\underline{M}_1$ into $\underline{M}'$ too. ■

Proposition 2.3. Let $\underline{M}=(M;f)$ be a unar, $\emptyset \ne N \subseteq M$, and let $\mathcal{V}_N$ be the family of all the unars $(N;g)$ with the carrier N such that the inclusion mapping[4] ι from N into M is a homomorphism from $(N;g)$ into $\underline{M}$. Then, the unar $\underline{N}=(N;h)$, defined by

$$\mathcal{D}_{\underline{N}} = \mathcal{D}_{\underline{M}} \cap \mathcal{B}(N) \text{ and } h = f|\mathcal{D}_{\underline{N}},$$

is the greatest member of $\mathcal{V}_N$. In the case when $N \le \underline{M}$, the unar $(N;h)$ coincides with the unar induced by N, by the agreement made afther P.1.2. ■

Proposition 2.4. Let ϕ be a homomorphism from a unar $\underline{M}=(M;f)$ into a unar $\underline{M}'=(M';f')$. If $N' \le \underline{M}'$ is such that $\phi^{-1}(N')=N\ne\emptyset$, then $N \le \underline{M}$.

Proof. Let $X\in\mathcal{D}_{\underline{M}} \cap \mathcal{B}(N)$. Then $\phi(X)\in\phi(\mathcal{D}_{\underline{M}}) \subseteq \mathcal{D}_{\underline{M}'}$ and $\phi(X)\subseteq N'$. Therefore, $\phi(X)\in\mathcal{D}_{\underline{M}'} \cap \mathcal{B}(N')$ which implies that $\phi(f(X))=f'(\phi(X))\subseteq N'$, i.e. $f(X)\subseteq \phi^{-1}(N') = N$. ■

Note that a homomorphic image of a subunar is not necessarily a subunar. For example, let $\underline{M}$ be the zero unar with the carrier M, and $\underline{M}'=(M;f)$ a unar with the same carrier, such that there exists at least one nonempty subset N of M which is not a subunar of $\underline{M}'$. Then $N \le \underline{M}$, $1_M:x \mapsto x$ is a homomorphism from $\underline{M}$ into $\underline{M}'$, $1_M(N)=N$, but N is not a subunar of $\underline{M}'$.

It is also natural to ask the following question: Is it possible to exist distinct homomorphisms $\phi,\psi:(M;f) \to (M';f')$ such that their restrictions on some generating set B of $(M;f)$ are equal? It is easy to show that the answer is yes, as the following example shows.

Example 2.5. Let $M=\{a,b,c,d\}=M'$,

$$\mathcal{D}_f = \{\{a,b\}\}, \quad f(\{a,b\}) = M = f'(\{a,b\}).$$

Then $B=\{a,b\}$ is a generating set of $\underline{M}=\underline{M}'$, and there are exactly 16 homomorphisms from $\underline{M}$ into $\underline{M}'$ (i.e. endomorphisms of $\underline{M}$) which are extensions of the inclusion $\iota=\binom{a\ b}{a\ b}$ from B into M. There are two distinct isomorphisms between them.

§3. FREE UNARS

Let $\mathcal{E}$ be a class of unars. We say that a unar $\underline{M}=(M;f)$ is a free object in $\mathcal{E}$ with a basis B iff the following conditions are fulfilled:

(i) B is a generating set of $\underline{M}$,

(ii) For every unar $\underline{M}'=(M';f')$ and every mapping $\lambda:B \to M'$ there exists a homomorphism $\phi:M \to M'$ which is an extension of λ, i.e. $(\forall b\in B)\ \phi(b) = \lambda(b)$.

The fact that every mapping from a zero unar into another one is a homomorphism implies that the following proposition is true.

[4] $(\forall x\in N)\ \iota(x)=x$.

Proposition 3.1. If $\underline{M} = (M;f)$ is the zero unar with the carrier M, then $\underline{M}$ is a free unar with the basis M, in every class of unars $\mathcal{C}$ such that $\underline{M} \in \mathcal{C}$. ✖

Now we will show that the zero unars are the only free objects in the class of all unars.

Proposition 3.2. If $\underline{M} = (M;f)$ is a nonzero unar, then $\underline{M}$ is not a free object in the class of all unars.

Proof. Let $\underline{M} = (M;f)$ be a nonzero unar and P be a set disjoint with M, but of greater cardinality than M. We set $M' = M \cup P$ and define a unar $\underline{M}' = (M';f')$ with:

$$\mathcal{D}_{\underline{M}'} = \mathcal{D}_{\underline{M}} \text{ and } (\forall X \in \mathcal{D}_{\underline{M}'})\ f'(X) = f(X) \cup P.$$

We will show that there is no homomorphism from $\underline{M}$ into $\underline{M}'$. Namely, if such a homomorphism $\phi : M \to M'$ exists, then for any $X \in \mathcal{D}_{\underline{M}}$ we would have

$$\phi(f(X)) = f'(\phi(X)) = f(\phi(X)) \cup P.$$

But this is not possible since the left hand side has strictly smaller cardinality than the right hand side. ✖

The proposition 3.2 suggests to consider "smaller" classes of unars and to search free objects there.

One of the "simpler" classes of unars in which free objects do not exist is the class of constant unars, which we denote by Con. Here, a unar $\underline{M} = (M;f)$ is said to be constant iff $\underline{M}$ is not a zero unar and there exists $A \subseteq M$, such that $(\forall X \in \mathcal{D}_{\underline{M}})\ f(X) = A$.

If $\underline{M}$ is a constant unar in the proof of the previous proposition, then the obtained unar $\underline{M}'$ will be also constant, and so the following is true:

Proposition 3.3. There are no free objects in Con. ✖

As a contrast of Con is the class $\mathcal{G}$ of unars defined in the following way:

$$\underline{M} = (M;f) \in \mathcal{G} \text{ iff } \mathcal{D}_{\underline{M}} = \mathcal{B}(M) \text{ and } (\forall X \subseteq M)\ f(X) = X.$$

Proposition 3.4. Every object $\underline{M} = (M;f) \in \mathcal{G}$ is free in $\mathcal{G}$ with the basis M. ✖

Now we will consider a family of subclasses of Con, everyone of which has free objects. Let α be a cardinal and let

$$\mathrm{Con}(\alpha) = \{\underline{M} \in \mathrm{Con} \mid \mathcal{D}_{\underline{M}} = \mathcal{B}(M) \wedge (\forall X \subseteq M)\ |f(X)| \le \alpha\}^{5)}.$$

The class $\mathrm{Con}(\alpha)$ can be described as a class of ordered pairs (M,A), where $A \subseteq M$ and $|A| \le \alpha$. Namely, the pair (M,A) will represent the unar $\underline{M} = (M;f) \in \mathrm{Con}(\alpha)$, such that

$$(\forall X \subseteq M)\ f(X) = A.$$

Specially,

$$\mathrm{Con}(0) = \{\underline{M} \in \mathrm{Con} \mid \mathcal{D}_{\underline{M}} = \mathcal{B}(M) \wedge (\forall X \subseteq M)\ f(X) = \emptyset\}.$$

Now we have:

Proposition 3.5. (i) $B \subseteq M$ is a generating set of (M,A) iff $M \setminus A \subseteq B$.

(ii) A mapping $\phi : M \to M'$ is a homomorphism from (M,A) into (M,A') iff $\phi(A) = A'$. ✖

By P.3.5, we easily come to the following description of the free objects in $\mathrm{Con}(\alpha)$.

Proposition 3.6. (M,A) is a free object in $\mathrm{Con}(\alpha)$ iff $\alpha = |A|$. In this case, $B = M \setminus A$ is a basis of (M,A). ✖

Assuming that $\alpha = |A|$, in the following statements we describe all the basis of the free object (M,A) in $\mathrm{Con}(\alpha)$.

Proposition 3.7. If A is a proper subset of M, then $M \setminus A$ is the unique basis of (M,A). ✖

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5) $|Y|$ is the cardinal number of the set Y.

Proposition 3.8. If $|M|=\alpha$, then $B\subseteq M$ is a basis of (M,M) iff $|M\setminus B|=\alpha$. ■

Proposition 3.9. If α is finite, then $\emptyset$ is the unique basis of (M,M). ■

Proposition 3.10. If α is infinite, then there exist infinitely many nonequivalent basis of (M,M). ■

At the end of the paper, we will consider one more class of unars without free objects and one subclass of it with free objects.

Let $\mathcal{F}$ be the class of unars $\underline{M}=(M;f)$ such that

$\mathcal{D}_{\underline{M}} = \{X\subseteq M: 1\leq |X| < \aleph_0\}$ and $(\forall X\in\mathcal{D}_{\underline{M}})\ 1\leq |f(X)| < \aleph_0$.

First we will show that:

Proposition 3.11. There are no free objects in $\mathcal{F}$.

Proof. Let $\underline{M}=(M;f)$, $\underline{M}'=(M',f')\in\mathcal{F}$ and $a\in M$, $a'\in M'$ be such that $|f(\{a\})| < |f'(\{a'\})|$. Then there is no homomorphism $\phi:M\to M'$ such that $\phi(a)=a'$, since if such a homomorphism ϕ would exist, then

$\phi(f(\{a\}))=f'(\{a'\})$ and this would imply $|f'(\{a'\})|\leq|f(\{a\})|$.

Now suppose that $\underline{M}=(M;f)\in\mathcal{F}$ with the basis B and $a\in B$ is such that $|f(\{a\})|=m$. Let M' be a finite set with $m+1$ elements and put $f'(X')=M'$ for every subset X' of M'. Then we obtain the unar $\underline{M}'=(M';f')\in\mathcal{F}$. By the above considerations it follows that there is no homomorphism $\phi:M\to M'$. ■

Now we will consider a subclass $\mathcal{F}_m$ of $\mathcal{F}$, defined in the following way:
$$\underline{M}\in\mathcal{F}_m \iff \underline{M}\in\mathcal{F} \text{ and } (\forall X\in\mathcal{D}_{\underline{M}})\ 1\leq |f(X)| \leq m$$
(where m is a finite cardinal).

Proposition 3.12. Every nonempty set B is a basis of a free unar $\underline{M}\in\mathcal{F}_m$.

Proof. Let us set $C_0=B$ and let
$$C_{\alpha+1} = C_\alpha \cup N_m\times\{X\subseteq C_\alpha:\ 1\leq |X| <\aleph_0\}$$
where $N_m = \{1,2,\dots,m\}$. Then we put $C = \cup\{C_\alpha \mid \alpha\geq 0\}$.

We will define a unar $\underline{C}=(C;f)\in\mathcal{F}_m$ as follows. If $X\subseteq C$ is a finite nonempty subset of C, then there exists an $\alpha\geq 0$ such that $X\subseteq C_\alpha$ and then $(1,X),(2,X),\dots,(m,X)\in C_{\alpha+1}\subset C$, i.e. $Y=\{(1,X),(2,X),\dots,(m,X)\}$ is a subset of C with m elements. Therefore we can define f by:
$$f(X) = \{(1,X),(2,X),\dots,(m,X)\}.$$

If is easy to show that B is a generating set of $\underline{C}$.

Clearly, the B-hierarchy of $u\in C$ is $\alpha+1$ iff $u\in C_{\alpha+1}\setminus C_\alpha=D_{\alpha+1}$. Also, $u\in D_{\alpha+1}$ iff u has the form $u=(i,X)$, where $i\in N_m$, $X\subseteq C_\alpha$ and $X\cap D_\alpha\neq\emptyset$. This implies that $D_{\alpha+1}$ is a disjoint union of the family of sets
$$\{(1,X),(2,X),\dots,(m,X)\}, \qquad (3.1)$$
where X is a finite nonempty subset of C_α such that $X\cap D_\alpha\neq\emptyset$.

Let $(M;g)\in\mathcal{F}_m$ and let λ be an arbitrary mapping from B into M. Suppose that, for every $\gamma\leq\alpha$, a mapping $\phi_\gamma:C_\gamma\to M$ is well defined with the following properties:

(i) $\phi_0=\lambda$,

(ii) $\phi_{\gamma+1}$ is an extension of ϕ_γ $(\gamma<\alpha)$,

(iii) if X is a finite nonempty subset of C_γ $(\gamma<\alpha)$, then
$$\phi_{\gamma+1}(\{(1,X),(2,X),\dots,(m,X)\}) = g(\phi_\gamma(X)). \qquad (3.2)$$

We will define $\alpha_{\alpha+1}:C_{\alpha+1}\to M$. First, we assume that $\phi_{\alpha+1}$ is an extension of ϕ_α. Upon that, we consider a subset of $D_{\alpha+1}$ of the form (3.1). Then $X'=\phi_\alpha(X)$ is a finite nonempty subset of M, and $1\leq|g(X')|\leq m$. By this it follows that there exists a surjective mapping
$$\psi_X:\{(1,X),\dots,(m,X)\}\to g(X') = g(\phi_\alpha(X)). \qquad (3.3)$$

Let $\psi_{\alpha+1}:D_{\alpha+1}\to M$ be the extension of all ψ_X, and $\phi_{\alpha+1}:C_{\alpha+1}\to M$ the extension of ϕ_α and $\psi_{\alpha+1}$. So we obtain a sequence of mappings

$\{\phi_\gamma : C_\gamma \to M \mid \gamma \geq 0\}$ which satisfies (i), (ii) and (iii) for every γ. Assuming ϕ to be the extension of this sequence on C, we obtain that ϕ is a homomorphism from $\underline{C}$ into $\underline{M}$ which is an extension of λ. ∎

Below we will denote by $\underline{C}=(C;f)$ the free object in $\mathcal{F}_m$ with the basis B, constructed in the proof of P.3.12.

Proposition 3.13. If ϕ is an endomorphism of $\underline{C}$ such that $(\forall b \in B)\ \phi(b)=b$, then ϕ is an automorphism of $\underline{C}$.

Proof. Suppose that $\phi(C_\alpha)=C_\alpha$ and that the corresponding transformation ϕ_α induced by ϕ on C_α is a permutation and also $\phi(D_\alpha)=D_\alpha$. Consider a subset of $D_{\alpha+1}$ of the form (3.1). Since X is a finite nonempty subset of C_α, we have:

$$\phi(\{(1,X),\dots,(m,X)\})=\phi(f(X))=f(\phi(X))=\{(1,\phi(X)),\dots,(m,\phi(X))\}.$$

The fact that $X \cap D_\alpha \neq \emptyset$ and the hypothesis $\phi(D_\alpha)=D_\alpha$ imply that $\phi(X) \cap D_\alpha \neq \emptyset$ and thus $\{(1,\phi(X)),\dots,(m,\phi(X))\} \subset D_{\alpha+1}$.
Therefore, ϕ induces a bijection from $\{(1,X),\dots,(m,X)\}$ into $\{(1,\phi(X)),\dots,(m,\phi(X))\}$.

By all this one easily comes to the asked final conclusion that ϕ is a permutation of C. ∎

Remark. We note that if α is infinite and if $\underline{M}$ is a free object in Con(α) with a basis B, then there exist infinitely many endomorphisms ϕ on $\underline{M}$, such that $(\forall b \in B)\ \phi(b)=b$, but they are not automorphisms.

Now we will complete the description of free objects in $\mathcal{F}_m$.

Proposition 3.14. Any two free objects in $\mathcal{F}_m$ with the same basis B are isomorphic.

Proof. It suffices to show that an arbitrary free object $\underline{M}=(M;g) \in \mathcal{F}_m$ with the basis B is isomorphic with the free object $\underline{C}$, constructed above.

Let $\phi : \underline{C} \to \underline{M}$, $\psi : \underline{M} \to \underline{C}$ be homomorphisms with the property $(\forall b \in B)$ $\phi(b)=\psi(b)=b$. Then $\xi=\psi\phi : \underline{C} \to \underline{C}$ is an endomorphism of $\underline{C}$ such that $(\forall b \in B)$ $\xi(b)=b$, and thus ξ is an automorphism. Therefore ϕ is an injective homomorphism, and thus it suffices to show that ϕ is surjective.

Namely, it can be easily shown that, in $\mathcal{F}_m$, homomorphic images of subunars are subunars. Therefore, $E=\phi(C)$ is a subunar of $\underline{M}$ such that $B \subseteq E$, and this implies that $E=M$, for B is a generating set of $\underline{M}$. ∎

R E F E R E N C E S

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ТРАНСФОРМАЦИИ НА БУЛЕАНИ

Р е з и м е

Нека M е непразно множество, $\mathcal{B}(M)$ е булеанот (т.е. партитивното множество) на M, $\mathcal{D} \subseteq \mathcal{B}(M)$ и $f : \mathcal{D} \to \mathcal{B}(M)$ е пресликување. Тогаш велиме дека (M;f) е булов унар со носител M, дејство f и домен $\mathcal{D}$. Поимите подунар, хомоморфизам и слободен објект, во трудот се воведуваат на вообичаен начин. Главниот предмет на работава е испитување на проблемот за егзистенција на слободни објекти во неколку класи од булови унари. Скоро во сите случаи се добива следново "необично" својство: постојат повеќе ендоморфизми во слободниот булов унар коишто ја индуцираат идентичната трансформација на базата (а добро е познато дека слободните алгебри го немаат тоа својство).