VECTOR VALUED SEMIGROUPS

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Dedicated to Prof. L.M. Gluskin
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An (n,m)-semigroup $A[\ ]$ is an associative mapping $[\ ]:A^n \to A^m$. Thus, an $(n,1)$-semigroup is an n-semigroup, and more specially a $(2,1)$-semigroup is a semigroup. We show that each (n,m)-semigroup $A[\ ]$ can be embedded in a semigroup S generated by A such that $[a_1\ldots a_n]=a_1\cdots a_n$ for any $a_1,\ldots,a_n \in A$. The classes of cancellative (n,m)-semigroups, (n,m)-groups and commutative (n,m)-semigroups are also considered. A number of known results for n-semigroups hold in the class of (n,m)-semigroups as well, but some different situations arise in the case $m \geq 2$. For example, any semigroup is a covering semigroup of an $(n,1)$-semigroup but this is not true if $m \geq 2$; it is also well known that any nonempty set is the carrier of an n-group, but it is not known whether this is true for (n,m)-groups when m is not a divisor of n.

1. Let A be a nonempty set, n,m two positive integers and

$$[\ ]: (x_1,\ldots,x_n) \mapsto [x_1\ldots x_n]$$

a mapping from the n-th cartesian power A^n of A into A^m. Throughout the paper it will be assumed that $k = n-m > 0$. We say that A is an (n,m)-groupoid.

If $0 \leq i \leq k$ and $[x_{i+1}\ldots x_{i+n}] = (y_1,\ldots,y_m)$ then the "product"

$$[x_1\ldots x_i\ y_1\ldots y_m\ x_{i+n+1}\ldots x_{2k+m}]$$

will be denoted by

$$[x_1\ldots x_i[x_{i+1}\ldots x_{i+n}]x_{i+n+1}\ldots x_{2k+m}].$$

$A[\ ]$ is called an (n,m)-semigroup if the following equation is an identity on $A[\ ]$

$$[[x_1\ldots x_n]x_{n+1}\ldots x_{2k+m}] = [x_1\ldots x_i[x_{i+1}\ldots x_{i+n}]x_{i+n+1}\ldots x_{2k+m}] \quad (1.1)$$

for each $i \in \{1,\ldots,k\}$.

It can be easily seen that in an (n,m)-groupoid there exist "continued products" $\Pi(x_1,\ldots,x_{sk+m})$ of a length $sk+m$ for each $s \geq 1$. If $A[\ ]$ is an (n,m)-semigroup, then two continued products $\Pi'(x_1,\ldots,x_{sk+m})$, $\Pi''(x_1,\ldots,x_{sk+m})$ of the same length $sk+m$ are equal, i.e. "the general associative law" holds. Therefore we can omit "inside brackets", i.e. a continued product $\Pi(x_1,\ldots,x_{sk+m})$ will be written in the following form $[x_1\ldots x_{sk+m}]_s$ or $[x_1\ldots x_{sk+m}]$. Thus we can state the following proposition.

1.1. *If $A[\ ]$ is an (n,m)-semigroup, then $A[\ ]_s$ is an $(sk+m,m)$-semigroup for each $s \geq 1$.*

Examples. 1) An $(n,1)$-semigroup is an n-semigroup, and a $(2,1)$-semigroup is a usual binary semigroup.

2) Let A and B be two nonempty sets.

a) If $a=(a_1,\ldots,a_m)$ is a fixed element of A^m, then the identity $[x_1\ldots x_n]=a$ defines a constant (n,m)-semigroup.

b) $[x_1\ldots x_n]=(x_1,\ldots,x_m)$ defines a left zero (n,m)-semigroup on A.

c) An (n,m)-semigroup can be built on $A\times B$ by:

$$[(x_1,y_1)\dots(x_n,y_n)] = ((x_1,y_{k+1}),\dots,(x_m,y_n)).$$

3) Let $(A,\cdot)$ be a (binary) semigroup and $n=tm$. Then the t-th cartesian power of the m-th cartesian power of the given semigroup determines an (n,m)-semigroup. Namely, in this case we have:

$$[x_1\dots x_n] = (x_1x_{m+1}\cdots x_{(t-1)m+1},\ x_2\cdots x_{(t-1)m+2},\dots,x_mx_{2m}\cdots x_{tm}).$$

2. *Here we will show that each (n,m)-semigroup is embeddable in a semigroup.*

Let S be a semigroup and A a subset of S (i.e. of the carrier of the semigroup) with the following properties:

$$\text{(I)}\quad (\forall a_1,\cdots,a_n \in A)(\exists b_1,\cdots,b_m \in A)\ a_1\cdots a_n = b_1\cdots b_m$$

$$\text{(II)}\quad (\forall a_\nu,b_\nu \in A)(a_1\cdots a_m = b_1\cdots b_m \Rightarrow a_1 = b_1,\dots,a_m = b_m)$$

$$\text{(III)}\quad (\forall a_\nu,b_\lambda \in A)(i < m\ \&\ a_1\cdots a_i = b_1\cdots b_p \Rightarrow$$
$$\Rightarrow i = p\ \&\ a_1 = b_1,\dots,a_i = b_i).$$

Then, an (n,m)-semigroup $A[\ ]$ can be obtained by:

$$\text{(IV)}\quad a_1\dots a_n = (b_1,\dots,b_m) \Leftrightarrow a_1\cdots a_n = b_1\cdots b_m.$$

We will say that $A[\ ]$ is an (n,m)-*subsemigroup* of S, and if, moreover, S is *generated* by A, then S is called a *covering semigroup of* $A[\ ]$.

It is assumed in the following statements that S is a *covering semigroup* of a given (n,m)-semigroup $A[\ ]$, and S_* is the subset of S defined by:

$$S_* = \{a_1\cdots a_i \mid i \ge m,\ a_\nu \in A\}. \qquad (2.1)$$

We will say that S_* is a *strong covering* of $A[\ ]$.

2.1. *S_* is an ideal in S, and $S = S_*$ iff $m = 1$.*

2.2. *S is a group iff $m = 1$ and $A[\ ]$ is an n-group.*

2.3. *If $m \ge 2$, then S is commutative only if $|A| = 1$, and then S is a cyclic semigroup with an index m and a period which is a divisor of k.*

2.4. *If S and T are covering semigroups of $A[\ ]$, then $S \cong T$ iff $S_* \cong T_*$.*

We note that the propositions 2.1, 2.3, and 2.4 are obvious; the same is true for 2.4 if $m \ge 2$, and the case $m = 1$ is considered in [10; Th. 2] and [1; Th. 2.2]. We also note that the assumption $m \ge 2$ is essential in 2.3, for *there exist* n-*semigroups admitting only commutative coverings* [11; §2].

We will show now that *each (n,m)-semigroup* $A[\ ]$ *admits a covering semigroup*. Namely, let $A\hat{}$ be the semigroup with the following presentation:

$$\langle A; \{a_1\cdots a_m = b_1\cdots b_n \mid (a_1,\dots,a_m) = b_1\dots b_n \text{ in } A[\ ]\}\rangle. \qquad (2.2)$$

Clearly,

$$a,b \in A \Rightarrow (a = b \text{ in } A\hat{} \Rightarrow a = b \text{ in } A),$$

and thus it can be assumed that A is a generating subset of $A\hat{}$. It is also clear that the conditions (I), (III) and (IV) are satisfied. And, it is easy to see that if $a_\nu,b_\lambda \in A$, $t \ne m$ and $a_1\cdots a_m = b_1\cdots b_t$ in $A\hat{}$, then there is an $s \ge 1$ such that $t = sk+m$ and $(a_1,\dots,a_m) = [b_1\dots b_t]$ in the given (n,m)-semigroup; this implies that condition (II) is also satisfied. Thus we can state the following propositions.

2.5. *If* $A[\ ]$ *is an* (n,m)-*semigroup, then* $A^{\wedge}$ *is a covering semigroup of* $A[\ ]$.

2.6. *If* S *is a covering semigroup of an* (n,m)-*semigroup* $A[\ ]$, *then there exists a unique homomorphism* $f{:}A \to S$, *such that* $f(a) = a$ *for each* $a \in A$.

The last proposition suggests that $A^{\wedge}$ is the *universal covering* of $A[\ ]$.

It is not difficult to give explicit descriptions of universal semigroups of the (n,m)-semigroups given in the example 1.2).

3. Assuming that $\underline{C}$ is a class of semigroups, it is desirable to find a set of classes $\{\underline{C}(n,m) \mid n > m \geq 1\}$ such that $\underline{C}(2,1) = \underline{C}$ and $\underline{C}(n,m)$ is a class of (n,m)-semigroups for each pair (n,m). The class $\underline{C}(n,m)$ can be defined by an axiom system $\Phi(n,m)$ in such a way that $\Phi(2,1)$ is an axiom system of $\underline{C}$. Also, $\underline{C}(n,m)$ can be the class of (n,m)-semigroups $A[\ ]$ such that either the (strong) universal covering $A^{\wedge}(A^{\wedge}_{*})$ is in $\underline{C}$, or some (strong) covering S (S_{*}) of A is in $\underline{C}$, or every (strong) covering of $A[\ ]$ is in $\underline{C}$. Here we will consider only the cases when $\underline{C}$ is the class of all cancellative semigroups, groups, commutative semigroups - respectively.

An (n,m)-semigroup $A[\ ]$ is called *cancellative* if it satisfies the following condition:

$$(\forall a \in A^{k}, x,y \in A^{m})\ [a\ x] = [a\ y] \text{ or } [x\ a] = [y\ a] \Rightarrow x = y. \quad (3.1)$$

Almost all the results on cancellative n-semigroups proved in [11; §3] and [9; III.1] hold for cancellative (n,m)-semigroups as well. Let us state one of them.

3.1. *An* (n,m)-*semigroup* $A[\ ]$ *is cancellative iff some covering semigroup of* $A[\ ]$ *is cancellative.*

Proof. It is clear that if some covering semigroup of $A[\ ]$ is cancellative, then $A[\ ]$ is also cancellative.

Assume now that $A[\ ]$ is a cancellative (n,m)-semigroup. Denote by F the semigroup freely generated by A, i.e. $F = A^{+}$, and define a relation $\equiv$ on F by:

$$u \equiv v \Leftrightarrow (\exists a \in F)\ [au] = [av]. \quad (3.2)$$

Then, $\equiv$ is a congruence on F, and $F/{\equiv}$ is a cancellative covering of $A[\ ]$. Namely, $F/{\equiv}$ is the universal cancellative covering of $A[\ ]$. We are not giving a detailed proof, for it would be an obvious generalization of the proof of the corresponding special result for cancellative n-semigroups given in [9; III.1].

An (n,m)-semigroup $A[\ ]$ is called an (n,m)-*group* if the following statement is satisfied:

$$(\forall a \in A^{k}, b \in A^{m})(\exists x,y \in A^{m})\,[a\ x] = b,\ \ [y\,a] = b. \quad (3.3)$$

We have a similar situation in the class of (n,m)-groups as in the class of cancellative (n,m)-semigroups, i.e. a lot of known results on n-groups which can be found in 4 , 11 and 1 also hold for (n,m)-groups. Below we will only state some of them.

3.2. *If* $A[\ ]$ *is an* (n,m)-*semigroup then the following statements are equivalent:*

(a) $A[\ ]$ *is an* (n,m)-*group*;

(b) *there is an* $s \geq 1$ *such that* $A[\ ]_s$ *is an* $(sk+m,m)$-*group*;

(c) $A[\ ]_s$ *is an* $(sk+m,m)$-*group for each* $s \geq 1$;

(d) $A^{\wedge}_{*}$ *is a group*.

3.3. *An* (n,m)-*semigroup* $A[\ ]$ *is an* (n,m)-*group iff there exist* (n,m)-*operations*

$$[/] : (x_1,\ldots,x_n) \mapsto [x_1\ldots x_m/x_{m+1}\ldots x_n]$$

$$[\backslash] : (x_1,\ldots,x_n) \mapsto [x_{m+1}\ldots x_n\backslash x_1\ldots x_m]$$

on A *such that*:

$$(\forall x \in A^k, y \in A^m)\ [x[x\backslash y]] = y,\ [[y/x]x] = y. \qquad (3.4)$$

3.4. *If the semigroup in the example* 1.3) *is cancellative (a group) then* $A[\ ]$ *is a cancellative* (n,m)-*semigroup (an* (n,m)-*group)*.

From 3.4 it follows that if m is a divisor of n, then any nonempty set is the carrier of an (n,m)-group. The situation is not so simple if m is not a divisor of n. To ilustrate this statement, let us prove the following proposition.

3.5. *If a* $(3,2)$-*semigroup* $A[\ ]$ *satisfies the following condition*:

$$(\forall a,b \in A^2)(\exists x,y \in A)\quad [a\ x] = b,\ [y\ a] = b,$$

then $|A| = 1$.

Proof. Let $a \in A^2$ and $e,f \in A$ be such that $[e\ a] = a$, $[a\ f] = a$. Then, in a usual way, we can show that:

$$(\forall b \in A^2)\ [e\ b] = b,\ [b\ f] = b,$$

which will imply that $(\forall x \in A)\,(x,f) = [e\ x\ f] = (e,x)$.

The existence of $(k+m,m)$-groups for any pair of positive integers k,m will be shown in 4.3.

An (n,m)-semigroup $A[\ ]$ is called *commutative* if the following identity is satisfied

$$[x_1\ldots x_n] = [x_{i_1}\ldots x_{i_n}],$$

for any permutation $\nu \mapsto i_\nu$ of $N_n = \{1,2,\ldots,n\}$. In a usual way we obtain:

3.6. *If* $A[\ ]$ *is a commutative* (n,m)-*semigroup, then for each* $s \geq 1$ $A[\ ]_s$ *is also commutative*.

By 2.3, if $m \geq 2$ and $|A| \geq 2$, no covering semigroup of an (n,m)-semigroup $A[\ ]$ is commutative. But, if $A[\ ]$ is a commutative (n,m)-semigroup, then some covering semigroups of $A[\ ]$ do have corresponding properties of commutativity. First, let us say that a semigroup S is t-commutative if products with at least t factors are invariant under permutations.

Assume now $A[\ ]$ to be a commutative (n,m)-semigroup, and consider the semigroup S with the presentation $\langle A;\ \Lambda' \cup \Lambda''\rangle$, where

$$\Lambda' = \{a_1\ldots a_m = b_1\ldots b_n \mid (a_1,\ldots,a_m) = b_1\ldots b_n \text{ in } A[\ ]\},$$

$$\Lambda'' = \{a_0\ldots a_m = a_0\ldots a_{i-1}a_{i+1}a_i a_{i+2}\ldots a_m \mid a_\nu \in A,\ 0 \leq i \leq m-1\}.$$

It is clear that S is an $m+1$-commutative semigroup and that it satisfies the conditions 2(I), 2(III), 2(IV); and it is easy to show that 2(II) also holds. Therefore:

3.7. *If* $A[\ ]$ *is a commutative* (n,m)-*semigroup, then there exists an* $m+1$-*commutative covering semigroup of* $A[\ ]$.

The following proposition is also clear.

3.8. *If* S *is an* m+1-*commutative covering of an* (n,m)-*semigroup* A[ ], *then* S_* *and* A[ ] *are commutative, and* A^m *is in the centre of* S.

4. An (n,m)-groupoid A[ ] induces an algebra $(A;[\,]_1,\dots,[\,]_m)$ with m n-ary operations defined in the following way:

$$[x_1\dots x_n] = (y_1,\dots,y_m) \Leftrightarrow (\forall i \in N_m)[x_1\dots x_n]_i = y_i. \qquad (4.1)$$

We say that the algebra obtained is the *component algebra* of the given (n,m)-groupoid. In an obvious way, *axiom systems* for the classes of *component algebras* of (n,m)-*semigroups*, *cancellative* (n,m)-*semigroups*, (n,m)-*groups*, *commutative* (n,m)-*semigroups* respectively can be formulated. We will state the corresponding proposition only for (n,m) - semigroups.

4.1. *An* (n,m)-*groupoid* A[ ] *is an* (n,m)-*semigroup iff the corresponding component algebra satisfies the following identity:*

$$[[x_1\dots x_n]_1\dots[x_1\dots x_n]_m\, x_{n+1}\dots x_{2k+m}]_i = \qquad (4.2)$$
$$= [x_1\dots x_j[x_{j+1}\dots x_{j+n}]_1\dots[x_{j+1}\dots x_{j+n}]_m\dots x_{2k+m}]_i$$

for each $i \in N_m$, $j \in N_k$.

Thus, the class of (n,m)-semigroups can be considered as a variety of algebras with m n-ary operations. The same is also true for the class of commutative (n,m)-semigroups, and the class of cancellative (n,m)-semigroups is equivalent with a quasivariety of algebras. The axiom system of component algebras corresponding to (n,m)-groups involves existential quantifiers, but (as in the binary and the n-ary case) the class of (n,m)-groups can be described by a variety of algebras. Namely, by 3.3 an (n,m)-group can be defined as a "vector valued algebra" (A;[ ], [\], [/]) such that the identities (1.1) and (3.4) are satisfied. Therefore we can state the following proposition.

4.2. *An* (n,m)-*semigroup* A[ ] *is an* (n,m)-*group iff there is an algebra*

$$(A;[\,]_1,\dots,[\,]_m,{}^1[\backslash],\dots,{}^m[\backslash],[/]^1,\dots,[/]^m) \qquad (4.3)$$

with 3m *n-ary operations such that* $(A;[\,]_1,\dots,[\,]_m)$ *is the component algebra of* A[ ], *and the following identities are satisfied:*

$$x_i = [x_{m+1}\dots x_n\,{}^1[x_{m+1}\dots x_n\backslash x_1\dots x_m]\dots{}^m[x_{m+1}\dots\backslash x_1\dots x_m]]_i, \qquad (4.4)$$

$$x_i = [[x_1\dots x_m/x_{m+1}\dots x_n]^1\dots[x_1\dots x_m/x_{m+1}\dots x_n]^m x_{m+1}\dots x_n]_i, \qquad (4.5)$$

for each $i \in N_m$.

The algebra (4.3) is called the *component algebra of the* (n,m)-*group* A[ ], or better of the (n,m)-group (A;[ ],[\],[/]). Therefore any algebra (4.3) which satisfies all the identities (4.2), (4.4) and (4.5) induces an (n,m)-group, and conversely.

The usual notions of *homomorphisms*, *direct products*, *free* (n,m)-*semigroups*, and others, can be defined directly, but we do not give explicit definitions, for they are meaningful in the class of component algebras.

Proposition 4.2 can be used in proving the existence of notrivial (n,m)-groups.

4.3. *If* $n > m \geq 1$, *then there exists a notrivial* (n,m)-*group*.

Proof. By 3.2 we can assume that $n = m+1$.

Consider the variety of algebras (4.3) which satisfy all the identities (4.2), (4.4) and (4.5). The existence of nontrivial algebras in this variety is equivalent to the statement that an identity $x_1 = x_2$ is not a consequence of the defining set of identities of the variety.

Assume that $x_1 = p$ is a consequence of the set of identities

(4.2), (4.4) and (4.5), where p is a term in the corresponding first order language. By induction it can be shown that p must have one of the following forms:

(a) p is x_1

(b) p is $[[p_{11}\dots p_{1m}/q_1]^1\dots[p_{m1}\dots p_{mm}/q_m]^m q]_i$

(c) p is $[q\, {}^1[q_1\backslash p_{11}\dots p_{1m}]\dots{}^m[q_m\backslash p_{m1}\dots p_{mm}]]_i$,

where $x_1 = p_{ii}$, $q_\nu = q_\lambda$, $q_\nu = q$ are consequences of the mentioned set of identities, for any ν,λ. This completes the proof.

5. We will state some problems concerning the class of (n,m)-semigroups, assuming that $m \geq 2$.

5.1. Although we gave some examples of (n,m)-semigroups, the problem of the existence of "good" examples of (n,m)-semigroups is open. For example, we know only one kind of nontrivial (n,m)-groups (the class of free (n,m)-groups) when m is not a divisor of n.

5.2. It is not true that each semigroup is a covering semigroup of an (n,m)-semigroup, and thus we have the problem of giving corresponding description of (strong) covering semigroups for (n,m)-semigroups belonging to convenient classes.

5.3. We do not know any convenient description of the complete system of identities that hold in the class of component algebras of (n,m)-semigroups.

5.4. Is it true that any commutative and cancellative (n,m)-semigroup is an (n,m)-subsemigroup of a commutative (n,m)-group?

5.5. Is it true that every (n,m)-semigroup is an (n,m)-subsemigroup of an (m+1,m)-semigroup?

5.6. Cohn-Rebane's theorem ([2; IV.4] or [8; 12]), and Gluskin-Hosszu's theorem ([4], [7] or [6; II.5]) are well known. Are the corresponding generalizations true?

5.7. Does there exist nontrivial (n,m)-quasigroups ([3]) which are also (n,m)-groups?

5.8. If $i_\nu, j_\lambda > 0$, $s, t \geq 0$, then we have an (n,m)-semigroup identity of the following form

$$[x_{i_1}\dots x_{i_{sk+m}}] = [x_{j_1}\dots x_{j_{tk+m}}], \tag{5.1}$$

and a variety of (n,m)-semigroups can be defined by a set of (n,m)-semigroups identities. Any identity (5.1) induces a semigroup identity, and thus each variety of (n,m)-semigroups induces a variety of semigroups. What are the connections between the corresponding varieties?

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