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A NOTE ON SOLVING LINEAR ORDINARY DIFFERENTIAL EQUATION

Zdravko F. Starc

Abstract

A note on solving linear ordinary differential equation of the n-th order is given.

The object of this note is investigation of integrability of linear differential equation of the n-th order.

We start from known results. The substitution $y' = zy$ transforms the linear differential equation of the n-th order

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{1}$$

into the nonlinear differential eqation of the $(n-1)$st order, of the Riccati type,

$$a_0(x)F_n(z) + a_1(x)F_{n-1}(z) + \cdots + a_{n-1}(x)F_1(z) + a_n(x) = 0 \tag{2}$$

where

$$F_1(z) = z\,, \quad F_n(z) = zF_{n-1}(z) + \big(F_{n-1}(z)\big)' \quad (n = 2, 3, \ldots)\,. \tag{3}$$

According to K. A. Haseinov [2] the general solution of the eqation (1) is given by

$$y = C_1 \exp \int z_1 dx + \cdots + C_n \exp \int z_n dx$$

where z_i ($i = 1, 2, \ldots, n$) are particular solutions of the equation (2).

From (3) we immediately obtain

$$\begin{aligned}
F_2(z) &= z^2 + z' , \\
F_3(z) &= z^3 + 3zz' + z'' , \\
F_4(z) &= z^4 + 6z^2 z' + 4zz'' + 3z'^2 + z''' , \\
F_5(z) &= z^5 + 10z^3 z' + 15zz'^2 + 10z^2 z'' + 10z'z'' + 5zz''' + z^{(4)} \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \\
F_n(z) &= z^n + \cdots + z^{(n-1)}
\end{aligned} \tag{4}$$

and it is easy to see that the expression $F_n(z)$ contains the addend $z^{(n-1)}$.

Using (4) we can rewrite the equation (2) in the following form

$$\begin{aligned}
& a_0(x)z^{(n-1)} + a_1(x)z^{(n-2)} + \cdots + a_{n-3}(x)z'' + a_{n-2}(x)z' + \\
& a_{n-1}(x)z + a_0(x)(z^n + \cdots) + a_1(x)(z^{n-1} + \cdots) + \cdots + \\
& a_{n-3}(x)(z^3 + 3zz') + a_{n-2}(x)z^2 + a_n(x) = 0 .
\end{aligned}$$

Let we consider the following two equations, linear and nonlinear,

$$a_0(x)z^{(n-1)} + a_1(x)z^{(n-2)} + \cdots + a_{n-2}(x)z' + a_{n-1}(x)z = 0 , \tag{5}$$

$$a_0(x)(z^n + \cdots) + a_1(x)(z^{n-1} + \cdots) + \cdots + a_{n-2}(x)z^2 + a_n(x) = 0 . \tag{6}$$

Let $p(x)$ be a particular solution of the equation (5). Then we have from (6)

$$a_n(x) = -a_0(x)\big(p(x)^n + \cdots\big) - a_1(x)\big(p(x)^{n-1} + \cdots\big) - \cdots - a_{n-2}(x)p(x)^2 . \tag{7}$$

Substituting (7) into (1) we obtain the following equation

$$\begin{aligned} &a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' - \big(a_0(x)\big(p(x)^n + \cdots\big) + \\ &+ a_1(x)\big(p(x)^{n-1} + \cdots\big) + \cdots + a_{n-2}(x)p(x)^2\big)y = 0\,. \end{aligned} \tag{8}$$

In accordance with the above we can formulate the following result.

Theorem. Suppose that $a_i(x)$ $(i = 0, 1, \ldots, n)$ are continous functions, $a_0(x) \neq 0$ and $p(x)$ is a particular solution of the equation (5). Then a particular solution of the equation (8) is given by

$$y = \exp \int p(x)dx\,.$$

On the other hand, knowing one nontrivial particular solution of the equation of the $(n-1)st$ order implies knowing one nontrivial particular solution of the equation of the n-th order.

In particular, if $n = 3, 4, 5$ then (8) become

$$\begin{aligned} &a_0(x)y''' + a_1(x)y'' + a_2(x)y' - \\ &- \big[a_0(x)\big(p(x)^3 + 3p(x)p'(x)\big) + a_1(x)p(x)^2\big]y = 0\,, \end{aligned} \tag{9}$$

$$\begin{aligned} &a_0(x)y^{(4)} + a_1(x)y''' + a_2(x)y'' + a_3(x)y' - \\ &- \big[a_0(x)\big(p(x)^4 + 6p(x)^2p'(x) + 4p(x)p''(x) + 3p'(x)^2\big) + \\ &+ a_1(x)\big(p(x)^3 + 3p(x)p'(x)\big) + a_2(x)p(x)^2\big]y = 0\,, \end{aligned}$$

$$\begin{aligned} &a_0(x)y^{(5)} + a_1(x)y^{(4)} + a_2(x)y''' + a_3(x)y'' + a_4(x)y' - \\ &- \big[a_0(x)\big(p(x)^5 + 10p(x)^3p'(x) + 15p(x)p'(x)^2 + 10p(x)^2p''(x) + \\ &+ 10p'(x)p''(x) + 5p(x)p'''(x)\big) + a_1(x)\big(p(x)^4 + \\ &+ 6p(x)^2p'(x) + 4p(x)p''(x) + 3p'(x)^2\big) + \\ &+ a_2(x)\big(p(x)^3 + 3p(x)p'(x)\big) + a_3(x)p(x)^2\big]y = 0 \end{aligned} \tag{10}$$

respectively.

Two examples will now be considered.

Example 1. It is krown [1] that the general solution of the following equation

$$f(x)^2g(x)y'' - \big(2f(x)f'(x)g(x) + f(x)^2g'(x)\big)y' +$$
$$+ \big(2f'(x)^2g(x) + f(x)f'(x)g'(x) - f(x)f''(x)g(x)\big)y = 0$$

is given by

$$y = C_1\, f(x) + C_2\, f(x) \int g(x)dx\,.$$

With $p(x) = a\, f(x)\ \ (a \in R)$ the equation (9) becomes

$$f(x)^2g(x)y''' - \big(2f(x)f'(x)g(x) + f(x)^2g'(x)\big)y'' +$$
$$+ \big(2f'(x)^2g(x) + f(x)f'(x)g'(x) - f(x)f''(x)g(x)\big)y' +$$
$$+ a^2 f(x)^3\big(f(x)g'(x) - f'(x)g(x) - af(x)^2g(x)\big)y = 0$$

which a particular solution is given by

$$y = \exp a \int f(x)dx\,.$$

Example 2. The general solution of the equation

$$x^4y^{(4)} + x^3y''' - 2x^2y'' - 2xy' - 8y = 0$$

is given by

$$y = C_1\, x^2 + C_2\, x^2\, \ln x + C_3\, (x\, \ln x)^2 + C_4\, x^{-1}\,.$$

With $p(x) = ax^2\ \ (a \in R)$ the equation (10) becomes

$$x^4y^{(5)} + x^3y^{(4)} - 2x^2y''' - 2xy'' - 8y' -$$
$$- a^2(a^3x^{14} + 21a^2x^{11} + 70ax^8 + 20ax^7 + 8x^6 + 38x^5)y = 0$$

which a particular solution is given by

$$y = \exp \frac{a}{3} x^3 .$$

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ЗАБЕЛЕШКА ЗА РЕШАВАЊЕ НА ОБИЧНИ ЛИНЕАРНИ ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ

Здравко Ф. Старц

Р е з и м е

Во оваа работа е дадена забелешка за решавање на обични линеарни диференцијални равенки од n-ти ред.

Žarka Zrenjanina 93
26300 Vršac
YUGOSLAVIA