Математички Билтен
22 (XLVIII)
1998 (19–30)
Скопје, Македонија

ISSN 0351–336X

ON THE QUASI-INNER PRODUCT SPACES

Pavle M. Miličić

Abstract

A quasi-inner product space X ($q.\ i.\ p.$ space) is strictly convex. If a sequence (x_n) converges weakly to x_0 ($x_n, x_0 \in X$, $x_n \rightharpoonup x_0$) and $\|x_n\| \to \|x_0\|$, then $\|x_n - x_0\| \to 0$. The orthogonality relation $\perp^g$ defined by (7), is uniquely resolvable, i.e. there exists a unique $a \in \mathbf{R}$ such that $x \perp^g (ax + y)$ ($\|x\| \cdot \|y\| \neq 0$). Under certain conditions the vector $-ax$ is the best approximation of the vector y with the vectors from $[x]{:}=\text{span}\{x\}$. In regard to the relation $\perp^g$ in a q.i.p. space, the lengths of the diagonals of the parallelogram are equal and the diagonals are perpendicular iff this parallelogram is a rectangle. A q.i.p. space i an inner product space iff (27) holds.

0. Introduction and definitions

Let X be a real normed space, $S(X)$ the unit sphere in X, and X^* the dual space of X. On X^2 always exist the functionals

$$\tau_{\pm}(x,y){:}= \lim_{t \to \pm 0} t^{-1}(\|x+ty\| - \|x\|)$$

$$g(x,y){:}= \frac{1}{2}\|x\|\big(\tau_-(x,y) + \tau_+(x,y)\big)\,.$$

The functional g is natural generalization of the inner product and reduces to it in the inner product space (cf. [6]). In any normed space, it has following properties:

$$g(x,x) = \|x\|^2\,, \tag{1}$$

$$g(\alpha x, \beta y) = \alpha\beta g(x,y)\,, \tag{2}$$

$$g(x, x+y) = \|x\|^2 + g(x,y)\,, \tag{3}$$

$$|g(x,y) \le \|x\| \cdot \|y\|\,, \tag{4}$$

$$\|x\| \frac{\|x+\lambda y\| - \|x\|}{\lambda} \le g(x,y) \le \le \|x\| \frac{\|x+ty\| - \|x\|}{t} \qquad (\lambda < 0,\ t > 0) \quad (\text{cf. [3],[6]})\,. \tag{5}$$

If X is smooth, then g is linear in the second argument, and in this case

$$[y,x] := g(x,y)$$

defines a semi-inner product in the sense of Lumer.

The orthogonality of the vector $x \neq 0$ to vector $y \neq 0$ in X may be defined in several ways. We mention some kinds of orthogonality and their denotations:

$x \perp_B y \Leftrightarrow (\forall \lambda \in \mathbf{R})\, \|x\| \le \|x + \lambda y\|$ (x is orthogonal to y in the sense of Birkhoff),

$x \perp_J y \Leftrightarrow \|x - y\| = \|x + y\|$ (James isosceles orthogonality),

$x \perp_S y \Leftrightarrow \left\| \frac{x}{\|x\|} - \frac{y}{\|x\|} \right\| = \left\| \frac{x}{\|x\|} + \frac{y}{\|x\|} \right\|$ (Singer orthogonality).

By use of the functional g, the orthogonality relations $\overset{g}{\perp}$ and $\perp^g$ are defined in the paper [8] as follows:

$$x \overset{g}{\perp} y \Leftrightarrow g(x,y) + g(y,x) = 0\,, \tag{6}$$

$$x \perp^g y \Leftrightarrow \|x\|^2 g(x,y) + \|y\|^2 g(y,x) = 0\,. \tag{7}$$

Alongside these relations we shall use the orthogonality relation $\underset{g}{\perp}$ defined by

$$x \underset{g}{\perp} y \Leftrightarrow g(x,y) = g(y,x)\,. \tag{8}$$

We shall also use the angle between the vectors x and y defined as

$$\cos(x,y) := \frac{g(x,y) + g(y,x)}{2\|x\| \cdot \|y\|} \qquad (x \neq 0\ \ y \neq 0) \quad (\text{cf. [7]}). \tag{9}$$

If X is an inner product space with inner product $(\cdot\,,\cdot)$, then the conditions (6), (7) and (8) are reduced to $(x,y) = 0$. Additionally, we remark that $\underset{g}{\perp} \subset \overset{g}{\perp} \cap \perp^g$.

According to the definition of the functional g, in the space l^p $(p \geq 1)$, we get

$$g(x,y) = \|x\|^{2-p} \sum_k |x_k|^{p-1}(\operatorname{sgn} x_k) y_k \qquad (x = (x_1, x_2, \dots) \in l^p \setminus \{0\}).$$

Consequently, the equality

$$\|x+y\|^4 - \|x-y\|^4 = 8(\|y\|^2 g(y,x) + \|x\|^2 g(x,y)) \tag{10}$$

holds in l^4, but doesn't hold in l^1.

The equality (10) is a generalization of the parallelogram equality ([9]). According to this, we say that *a space X is a quasi-inner product space if the equality* (10) *holds for all $x, y \in X$* ([9]).

1. Strictly convexity of quasi-inner product spaces

It is proved in [9] that a q.i.p. space X is smooth, very smooth, uniformly smooth and, in the case of a Banach space, reflexive. The convexity is not considered.

Theorem 1.1. *Let X is smooth. Then X is strictly convex if and only if*

$$\text{whenever} \quad \cos(x,y) = 1, \ x \neq 0, \ y \neq 0, \ \text{then } y = \lambda x \text{ for some } \lambda > 0. \tag{11}$$

Proof. Suppose that X is smooth and strictly convex. Then by (11) it follows

$$\frac{g(x,y)+g(y,x)}{2\|x\|\cdot\|y\|} = 1 \Leftrightarrow g(x,y) + g(x,y) = 2\|x\|\cdot\|y\| \Leftrightarrow$$

$$\Leftrightarrow g(x,y) = g(y,x) = \|x\|\cdot\|y\|$$

$$(|g(x,y)| \leq \|x\|\cdot\|y\| \quad \text{and} \quad |g(y,x)| \leq \|x\|\cdot\|y\|).$$

Since X is smooth, $g(x,y)$ is a semi-inner product. Hence, from $g(x,y) = \|x\|\cdot\|y\|$ we have $y = \lambda x$ for some $\lambda > 0$ (cf. Lemma 5, [3]).

Conversely, assume X is smoth and

$$\cos(x,y) = 1 \Rightarrow y = \lambda x \qquad (\lambda > 0). \tag{12}$$

Let us suppose

$$g(x,y) = \|x\|\cdot\|y\| \qquad (x \neq 0, \ y \neq 0). \tag{13}$$

Then from Theorem 3, p.25 [1], we get

$$y \perp_B g(x,\cdot).$$

It follows from Theorem 2 [5] $g(y,h)=0$ for all $h \in g(x,\cdot)$. Suppose that

$$y=\lambda x+h \quad \big(\lambda \in \mathbf{R},\ h \in g(x,\cdot)\big).$$

Then

$$g(y,y)=\|y\|^2=\lambda g(y,x)+g(y,h)$$

i.e.

$$\|y\|^2=\lambda g(y,x). \tag{14}$$

But also $g(x,y)=\lambda\|x\|^2$. By (13) we obtain $\|y\|=\lambda\|x\|$ $(\lambda>0)$. Thus, from (14) we get $g(y,x)=\|x\|\cdot\|y\|$. So, $\cos(x,y)=1$ and hence $y=\lambda x$ $(\lambda>0)$ by (12).

We conclude that the assumption $g(x,y)=\|x\|\cdot\|y\|$ $(x\neq 0,\ y\neq 0)$ implies $y=\lambda x$ for some $\lambda>0$. By Lemma 5 [3] it follows that X is strictly convex. □

Corollary 1.1. *A q.i.p. space X is strictly convex.*

Proof. In q.i.p. space X, for all $x,y \in X\setminus\{0\}$ we have

$$16-\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^4 \geq \left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|^4-\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^4=16\cos(x,y). \tag{15}$$

If $\cos(x,y)=1$ then from (15) we get $\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^4 \leq 0$. Hence $y=\frac{\|y\|}{\|x\|}x$. Since X is smoth, from Theorem 1.1 we obtain that X is strictly convex. □

2. Weak convergence in quasi inner product spaces

It is well known that in an inner-product space $x_n \rightharpoonup x_0$ and $\|x_n\|\to\|x_0\|$ implies $\|x_n-x_0\|\to 0$. This is not true in every normed space.

Theorem 2.1. *Let X is smooth and $x_n \in X$ $(n\in\mathbf{N})$, $x_0\in X$ such that*

(i) $x_n \rightharpoonup x_0$ $(n\to\infty)$

(ii) $\|x_n\|\to\|x_0\|$ $(n\to\infty)$.

Then $g(x_n,x_0)\to\|x_0\|^2$ $(n\to\infty)$.

Proof. Since X is smooth, $g(x,\cdot) \in X^*$ for all $x \in X$. (ii) implies that

$$(\forall \varepsilon > 0)(\exists n_1 \in \mathbf{N}) n \geq n_1 \Rightarrow \left|g(x_n, x_n) - \|x_0\|^2\right| < \frac{\varepsilon}{2}. \tag{16}$$

(i) implies that

$$(\forall n \geq n_1)(\forall \varepsilon > 0)(\exists n_2 \in \mathbf{N}) m \geq n_2 \Rightarrow |g(x_n, x_m) - g(x_n, x_0)| < \frac{\varepsilon}{2}. \tag{17}$$

Thus, for $n_0 = \max\{n_1, n_2\}$ and $m = n \geq n_0$, (17) implies

$$|g(x_n, x_n) - g(x_n, x_0)| < \frac{\varepsilon}{2}.$$

It follows, for $n \geq n_0$

$$|g(x_n, x_0) - \|x_0\|^2| - |g(x_n, x_n) - \|x_0\|^2| \leq |g(x_n, x_n) - g(x_n, x_0)| < \frac{\varepsilon}{2}.$$

Using (16) it follows, for $n \geq n_0$

$$\left|g(x_n x_0) - \|x_0\|^2\right| < \varepsilon.$$

So, $\lim\limits_{n\to\infty} g(x_n x_0) = \|x_0\|^2$.

Corollary 2.1. *Let X a q.i.p. space and $x_n \in X$ $(n \in \mathbf{N})$, $x_0 \in X$ such that*

(i) $x_n \to x_0 \quad (n \to \infty)$.

(ii) $\|x_n\| \to \|x_0\| \quad (n \to \infty)$.

Then $\|x_n - x_0\| \to 0 \quad (n \to \infty)$.

Proof. By (10) we have

$$\|x_n + x_0\|^4 - \|x_n - x_0\|^4 = 8\left(\|x_n\|^2 g(x_n, x_0) + \|x_0\|^2 g(x_0, x_n)\right). \tag{18}$$

Using this, we obtain

$$8\left(\|x_n\|^2 g(x_n, x_0) + \|x_0\|^2 g(x_0, x_n)\right) \leq (\|x_n\| + \|x_0\|)^4 - \|x_n - x_0\|^4. \tag{19}$$

Since X is smooth, (i) implies $g(x_0, x_n) \to \|x_0\|^2 \ (n \to \infty)$ and Theorem 2.1 implies $g(x_n, x_0) \to \|x_0\|^2 \ (n \to \infty)$. Hence (19) and (ii) implies

$$16\|x_0\|^4 \leq 16\|x_0\|^4 - \lim_{n\to\infty} \|x_n - x_0\|^4,$$

so

$$\lim_{n\to\infty} \|x_n - x_0\| = 0.$$

3. Orthogonalities in quasi-inner product spaces

From now on we assume that points $0, x, y, x+y$ are the vertices of a parallelogram, and $\|x-y\|$, $\|x+y\|$ are the lengths of its diagonals. If $\|x\| = \|y\|$, we say that this parallelogram is a rhomb, and if $x \perp y$ we say that the parallelogram is a rectangle with respect to orthogonality $\perp$, $(\perp \in \{\overset{g}{\perp}, \perp^g, \underset{g}{\perp}\})$.

From the next theorem we see the similarity of q.i.p. spaces with inner-product spaces.

Theorem 3.1.

1° *The lengths of the diagonals of the parallelogram in a q.i.p. space are equal if and only if this parallelogram is a $\perp^g$-rectangle, i.e. $x \perp^g y$;*

2° *The diagonals of the rhomb in a q.i.p. space are $\perp^g$-orthogonal, i.e. $(x-y) \perp^g (x+y)$;*

3° *The parallelogram is a $\perp^g$-quadrangle if and only if its lengths of the diagonals are equal and the diagonals are perpendicular;*

4° *For $x, y \in X \setminus \{0\}$ we have* $x + \frac{\|x\|}{\|y\|}y \perp^g \left(x - \frac{\|x\|}{\|y\|}y\right)$.

Proof.

1° The assertion we immediately get from (10).

2° Replacing x and y respectively by $x+y$ and $x-y$ in (10), we have

$$\|2x\|^4 - \|2y\|^4 = 8\big(\|x+y\|^2 g(x+y,\ x-y) + \|x-y\|^2 g(x-y,\ x+y)\big).$$

If $\|x\| = \|y\|$, then $(x+y) \perp^g (x-y)$.

3° If $(0, x, y, x+y)$ is a $\perp^g$-quadrangle, then $x \perp^g y$. From (10) we then get $\|x+y\| = \|x-y\|$. Moreover, by 2°, $\|x\| = \|y\|$ implies $(x-y) \perp^g (x+y)$. Conversely, from (10) and $\|x+y\| = \|x-y\|$ it follows $x \perp^g y$, and from (10) and $(x+y) \perp^g (x-y)$ we get $\|x\| = \|y\|$.

4° Replacing x and y respectively by $x + \frac{\|x\|}{\|y\|}y$ and $x - \frac{\|x\|}{\|y\|}y$ in (10), we get

$$0 = \left\|x + \frac{\|x\|}{\|y\|}y\right\|^2 g\left(x + \frac{\|x\|}{\|y\|}y, x - \frac{\|x\|}{\|y\|}y\right) +$$

$$+\left\|x-\frac{\|x\|}{\|y\|}y\right\|^2 g\Big(x-\frac{\|x\|}{\|y\|}y, x+\frac{\|x\|}{\|y\|}y\Big). \qquad \square$$

Observe that in a q.i.p. space the following relations are always true

$$x \overset{g}{\perp} y \Leftrightarrow \left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\| = \left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|, \tag{20}$$

$$x \perp^g y \Leftrightarrow \|x+y\| = \|x-y\|, \tag{21}$$

i.e. the $\overset{g}{\perp}$-orthogonality is equivalent with Singer orthogonality and the $\perp^g$-orthogonality is equivalent with James isosceles orthogonality. Since in the case of Singer orthogonality a cathet may be greater than the hypotenuse (cf. [4]) (if $x \perp_s y$ then $\|x\| > \|x+y\|$ is possible), then a cathet in the $\overset{g}{\perp}$-orthogonality may be greater than the hypotenuse. So the following assertion is an interesting one.

Theorem 3.2. *Let X be a normed space and $x, y \in X$. If $x \overset{g}{\perp} y$, then:*

1° $\min\{\|x\|, \|y\|\} \le \|x+y\|$;

2° $\max\{\|x\|, \|y\|\} \le 2\|x+y\|$;

3° $\|y\| > \|x+y\| \Rightarrow g(x,y) > 0 \wedge \|x+ty\| \ge \|x\| + \frac{t}{\|x\|} g(x,y) \ (t > 0)$.

Proof.

$$1° \quad x \overset{g}{\perp} y \Leftrightarrow g(x,y)+\|x\|^2+\|y\|^2+g(y,x) =$$

$$= \|x\|^2+\|y\|^2(3) g(x,x+y)+g(g(y,x+y) = \|x\|^2+\|y\|^2(4)$$

$$\|x\|^2+\|y\|^2 \le \|x+y\|(\|x\|+\|y\|). \tag{22}$$

Let $\|x\| \le \|y\|$ and $\|x+y\| \le \|x\|$. Then. by (22),

$$\|x\|^2+\|y\|^2 < \|x\|^2+\|x\|\cdot\|y\| \quad \text{i.e.} \quad \|y\| < \|x\|,$$

and we obtain a contradiction.

2° Suppose $\|x\| \le \|y\|$. Then, by (22) we have

$$\|y\|^2 \le \|x\|^2+\|y\|^2 \le \|x+y\|(\|x\|+\|y\|) \le 2\|y\|\cdot\|x+y\|.$$

Hence we get $\|y\| \le 2\|x+y\|$.

3° $\|y\| > \|x+y\| \Rightarrow \dfrac{\|x+y\|-\|y\|}{1} < 0(5)g(x,y) < 0(6)g(x,y) > 0.$
Besides this, from (5) we get

$$\|x+ty\| \geq \|x\| + \frac{t}{\|x\|}g(x,y)$$

with $t > 0$.

Theorem 3.3. *Let X be a q.i.p. space and $x, y \in X$. Then $x \perp^g y$ implies $\max\{\|x\|, \|y\| \leq \|x \pm y\|$ i.e. a cathet is not greater than the hypotenuse with respect to $\perp^g$-orthogonality.*

Proof. The function $f(t) = \|x+ty\|$ is convex on $\mathbb{R}$ with fixed $x, y \in X$. Assume $x \perp^g y$ and $\|x+y\| < \|x\|$. Now, $f(-1) = f(1) < f(0)$ follows from (21). So we get a contradiction, because the function f is a convex one.

It is proved in [8] that $\overset{g}{\perp}$-orthogonality is uniquely resolvable in smooth spaces, i.e. there exists $a \in \mathbb{R}$ such that $x \overset{g}{\perp}(ax+y)$ for all $x \neq 0$ and $y \in X$. Now we resolve the problem in connection with $\perp^g$-orthogonality and $\underset{g}{\perp}$-ortogonality.

Theorem 3.4. *Let X be a q.i.p. space and $x \neq 0$, $y \in X$. Then $\perp^g$-orthogonality is uniquely resolvable, i.e. there exists a unique $a \in \mathbb{R}$ such that $x \perp^g (ax+y)$.*

Proof. If $x \perp^g y$ then $a = 0$. Assume that not $x \perp^g y$. For fixed $x, y \in X \setminus \{0\}$ we consider the function $f\colon \mathbb{R} \mapsto \mathbb{R}$ defined by

$$f(t) = \|tx+y\|^2 g(tx+y, x) + \|x\|^2 g(x, tx+y)\,.$$

By (10) we have

$$f(t) = \frac{1}{8}(\|(t+1)x\|^4 - \|(t-1)x+y\|^4)\,.$$

Hence, the function f is continuous on $\mathbb{R}$. Since

$$g\left(x+\frac{y}{t}, x\right) = g\left(x+\frac{y}{t}, x+\frac{y}{t}-\frac{y}{t}\right) = \left\|x+\frac{y}{t}\right\|^2 - \frac{1}{t}g\left(x+\frac{y}{t}, y\right), (t \neq 0)$$

from (2) and (3) we then have

$$f(t) = t^3\left\|x+\frac{y}{t}\right\|^4 - t^2\|x+\frac{y}{t}\|^2 g\left(x+\frac{y}{t}, y\right) + t\|x\|^4 + \|x\|^2 g(x,y) \quad (t \neq 0)\,. \tag{23}$$

Since X is smooth space we have

$$\lim_{t\to\pm\infty} g\left(x+\frac{y}{t}, y\right) = g(x,y) \quad (cf.\ [3])\,.$$

Hence by (23) we have

$$\lim_{t\to\pm\infty} f(t) = \pm\infty\,.$$

So there exists $a \in \mathbb{R}$ such that $f(a) = 0$ or $x \perp^g (ax + y)$.

If $x \perp^g (a'x + y)$ for some $a' \neq a$, in view of (10) we have

$$\|(a+1)x + y\| = \|(a-1)x + y\| \wedge \|ax + y\|^2 g(ax + y, x) + \\ + \|x\|^2 g(x, ax + y) = 0\,, \tag{24}$$

$$\|(a'+1)x + y\| = \|(a'-1)x + y\| \wedge \|a'x + y\|^2 g(a'x + y, x) + \\ + \|x\|^2 g(x, a'x + y) = 0\,. \tag{25}$$

Redarding a and a' we may have one of the possibilities

$$[a' - 1,\ a' + 1] \subset [a - 1,\ a + 1]\ \ ([a - 1,\ a + 1] \subset [a' - 1,\ a' + 1])\,.$$

In this case $a = a'$ because $(a + 1) - (a - 1) = 2$. If these possibilities do not hold, denote by

$$\alpha = \min\{a' - 1,\ a - 1\}\,, \qquad \beta = \max\{a' + 1,\ a + 1\}\,.$$

Then, from the convexity of the function $t \mapsto f(t) = \|y + tx\|$ it follows that the function f is constant on the interval $[\alpha, \beta]$, and therefore f achieves its minimum at every $t \in [\alpha, \beta]$. Since the space X is smooth, there exists $f'(t)$ for all $t \in \mathbb{R}$ and $f'(t) = 0$ with $t \in (\alpha, \beta)$. Besides this, $(tx + y) \perp_B x$ for all $t \in (\alpha, \beta)$, and hence $g(tx + y, x) = 0$ with $t \in (\alpha, \beta)$. Since $a, a' \in (\alpha, \beta)$ from (24) and (25) we obtain

$$\alpha = a' = -\frac{g(x, y)}{\|x\|}\,. \qquad \square$$

Now, denote by $P_{[x]}y$ the set of the best approximations of y with vectors from $[x]$. Besides this, if $x \perp (ax + y)$ we say that $-ax$ is a projection of vector y on vector x in the sense of the $\perp$-orthogonality. In the next theorem it is given the relation between this projection and the best approximation of the vector y with vectors from $[x]$.

Theorem 3.5. *Let the space X be smooth and $x \perp (ax + y)$, where $\perp \in \{\overset{g}{\perp}, \perp^g, \underset{g}{\perp}\}$. Then $-ax \in P_{[x]}y$ if and only if*

$$a = -\frac{g(x, y)}{\|x\|^2}\,. \tag{26}$$

Proof. Suppose $x \perp^g (ax+y)$ and $-ax \in P_{[x]}y$. Then we get $\|y+ax\| \le \|y-\lambda x\|$ for all $\lambda \in \mathbb{R}$ and therefore $\|y+ax\| \le \|y+ax+tx\|$ for every $t \in \mathbb{R}$, i.e. $(y+ax) \perp_B x$. Hence $g(ax+y,x)=0$. Using $x \perp^g (ax+y)$, i.e.

$$\|ax+y\|^2 g(ax+y,x) + \|x\|^2 g(x,ax+y) = 0\,,$$

we get (26).

Conversely, assume non $x \perp^g (ax+y)$ and (26). Then from (26) we get $g(x,ax+y)$, and therefore in view of $x \perp^g (ax+y)$, we obtain

$$g(ax+y,x)=0 \Leftrightarrow (ax+y) \perp_B x \Leftrightarrow \|ax+y\| \le \|ax+y+tx\| \quad (\forall t \in \mathbb{R}) \Leftrightarrow$$

$$\Leftrightarrow \|y-(-ax)\| \le \|y-\lambda x\| \quad (\forall \lambda \in \mathbb{R})\,.$$

But this signifies that $-ax \in P_{[x]}y$.

The proof of the $\overset{g}{\perp}$-orthogonality is as above (in the case of the $\perp^g$-orthogonality). Besides this we have $\underset{g}{\perp} \subset \perp^g$. □

Corollary 3.1. *If $y_x := -ax$, then in a q.i.p. space holds*

$$y_x \in P_{[x]}y \Leftrightarrow \|y-y_x+x\| = \|y-y_x-x\|\,.$$

4. Quasi-inner product spaces and inner product spaces

Finally, we give the answer of the question when a q.i.p. space is an inner-product space.

Theorem 4.1. *A q.i.p. space X is an inner-product space if and only if the equivalence*

$$\|x+y\| = \|x-y\| \Leftrightarrow g(x,y)=0 \tag{27}$$

holds on X.

Proof. If the space X is an inner-product space with the inner-product $(\cdot,\cdot)$, then $g(x,y)=g(y,x)=(x,y)$ for all $x,y \in X$. Therefore we have (10) and (27).

Assume that (10) and (27) are true. Then from (10) we get

$$\|x+ty\|^4 - \|x-ty\|^4 = 8t\big(\|x\|^2 g(x,y) + t^2\|y\|^2 g(x,y)\big) \quad (x,y \in X;\ t \in \mathbb{R}). \tag{28}$$

Now assume $\|x+y\| = \|x-y\|$. Then, from (27) and (28) we obtain $g(y,x) = 0$, and so $g(x,y) = g(y,x) = 0$. Hence, by (28), $\|x+ty\| = \|x-ty\|$, for all $t \in \mathbb{R}$. That is, a q.i.p. space X

$$\|x+y\| = \|x-y\| \Rightarrow \|x+ty\| = \|x-ty\| \quad (t \in \mathbb{R})$$

is valid. But this is necessary and sufficient Ficken's condition for X to be an inner-product space (cf. [2], p. 193). □

References

[1] Diestel J.: *Geometry of Banach space – Selected Topics*, Springer-Verlag Berlin, Heidelberg, New York, 1975.

[2] Dej M. M.: *Normirovanie Linejnie Prostranstva*, Izdatelstvo Inostranoj Literaturi, Moskva, 1961.

[3] Giles R. J.: *Classes of semi-inner product space*, Trans. Amer. Math. Soc. V. 129, 436-446, 1967.

[4] Lin Zhen and Zhuang Yadong: *Singer orthogonality and chasacterization of inner-product spaces*, Arch. Math., V. 55, 588-594, 1990.

[5] Miličić M. P.: *Sur la g-orthogonalité danns un espace normé*, Mat. Vesnik 39, 325-334, 1987.

[6] Miličić M. P.: *Une géneéralization naturelle du product scalaire dans des espace normé et son utilisation*, Publ. Inst. Math. Beograd, 42(56), 63-70, 1987.

[7] Miličić M. P.: *Sur la g-angle dans un espace normé*, Mat. Vesnik 45, 43-48, 1993.

[8] Miličić M. P.: *Resolvability of g-orthogonality in normed spaces*, Mathematica Balkanica, New Series V. 12, 1998.

[9] Miličić M. P.: *A generalization of the parallelogram equality in normed spaces*, Jour. of Math. Kyoto University, V. 38 N°1, 1997.

ЗА ПРОСТОРИ СО ПОЛУСКАЛАРЕН ПРОИЗВОД

Павле М. Миличиќ

Р е з и м е

Простор X со полускаларен производ е стриктно конвексен. Ако низата (x_n) конвергира слабо кон (x_0) $(x_n, x_0 \in X,\ x_n \rightharpoonup x_0)$ и $\|x_n\| \to \|x_0\|$, тогаш $\|x_n - x_0\| \to 0$. Релацијата за ортогоналност $\perp^g$, дефинирана со (7) е единствено решлива, т.е. постои единствено $a \in \mathbb{R}$ така што $x \perp^g (ax + y)$ $(\|x\| \cdot \|y\| \neq 0)$. Под некои услови, векторот $-ax$ е најдобрата апроксимација на векторот y со вектори од $[x]{:}=\text{span}\{x\}$. Поради релацијата $\perp^g$ во простор со полускаларен производ должините на дијагоналите на паралелограм се еднакви и дијагоналите се нормални ако и само ако паралелограмот е правоаголник. Простор со полускаларен производ е простор со скаларен производ ако и само ако важи (27).

Faculty of Mathematics,
University of Belgrade,
YU - 11000,
Yugoslavia